Journal of Technical Physics, J. Tech. Phys., **44**, 3, 235–241, 2003.
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ELECTRON PLASMA UPPER HYBRID KINETIC SYMMETRIES

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Symmetry group of integro-differential equations describing nonlinear upper hybrid waves in magnetized electron plasma is found. It is shown that the extension of the symmetry in the cold plasma limit allows us to build the general solution in this case.

Key words: symmetry, integro-differential equations, magnetized plasma, nonlinear waves.

1. Introduction

In recent decades the Lie group method has been applied to explore many physically interesting nonlinear problems in gas dynamics, plasma physics etc. [1–4]. Furthermore, extensions of the classical Lie algorithm to the integro-differential systems of equations of kinetic theory were proposed [5–8]. In this work, we generalize the results obtained previously [5] for the electron plasma high frequency longitudinal waves in absence of an external magnetic field to the case when the external magnetic field is present, i.e. for the upper hybrid waves. In Sec. 2, the corresponding nonlinear integro-differential model equations are derived. In the Sec. 3, the symmetry transformations for upper hybrid waves are presented. Specification to the cold electron plasma case, for which it is possible to obtain a general solution, is considered in Sec. 4. For the sake of completeness, similar previous results for Langmuir waves are presented in the Sec. 5. In the Sec. 6, the obtained results are discussed and conclusions are drawn.

2. Model

Let us consider the high frequency plasma motion with constant ion background density n_0. In this case, the Vlasov–Maxwell system of integro-differential equations holds:

$$
\begin{gathered}
\partial_t f + \mathbf{v}\cdot\nabla f - \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f = 0,\\
\nabla\times\mathbf{E} + \frac{1}{c}\,\partial_t\mathbf{B} = 0, \qquad \nabla\cdot\mathbf{E} = 4\pi e\left(n_0 - \int d\mathbf{v} f\right),\\
\nabla\times\mathbf{B} = \frac{1}{c}\,\partial_t\mathbf{E} - \frac{4\pi e}{c}\int d\mathbf{v}\,\mathbf{v} f, \qquad \nabla\cdot\mathbf{B} = 0,
\end{gathered}
\tag{2.1}
$$

where $f(t,\mathbf{r},\mathbf{v})$ is electron distribution function. We assume that the plasma is subjected to a constant external magnetic field

$$
\mathbf{B} = B_0\mathbf{e}_z, \qquad B_0 = \mathrm{const},
$$

so the electron cyclotron frequency is equal to $\omega_c = eB_0/mc$. We consider the case of transverse plasma motion (see, e.g. [9]):

$$
f = f(t,x,v_x,v_y), \qquad \mathbf{E} = E(t,x)\mathbf{e}_x.
$$

Electron distribution function f is already integrated over v_z and no longitudinal current is present, $\int dv_z\, v_z f = 0$. In this way we obtain the simplified system of model equations describing *upper hybrid waves* in the electron plasma

$$
\begin{gathered}
\partial_t f + v_x\partial_x f - \frac{e}{m}\,E\,\partial_{v_x} f + \omega_c\left(v_x\partial_{v_y} f - v_y\partial_{v_x} f\right) = 0,\\
\partial_x E = 4\pi e\left(n_0 - \int dv_x\,dv_y\, f\right), \qquad \partial_t E = 4\pi e\int dv_x dv_y\, v_x\, f,\\
j_0 = e\int dv_x dv_y\, v_y\, f,
\end{gathered}
\tag{2.2}
$$

where *external* current density $\mathbf{j}_0 = j_0\mathbf{e}_y$ is needed for compensation of a time-dependent magnetic field generated due to plasma dynamics, since our model is restricted to the constant magnetic field only. This problem is sometimes overlooked in the literature (compare Sec. 3.4.1 of [10]).

3. Symmetries of the Model

Lie group of point transformations admitted by the integro-differential equations (2.2) can be found by the use of the indirect method [5], exploring the symmetry of an infinite set of equations for the moments of the distribution function, as well as by the direct method developed recently in [6–8]. After some cumbersome but straightforward algebra, we obtain the following six generators which span the Lie algebra of the symmetry group of the Eqs. (2.2):

$$
\begin{aligned}
&X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \partial_{v_y} - \frac{m}{e}\,\omega_c\,\partial_E,\\
&X_4 = x\partial_x + v_x\partial_{v_x} + v_y\partial_{v_y} - 2f\partial_f + E\partial_E,\\
&X_5 = \cos(\omega t)\partial_x - \omega\sin(\omega t)\partial_{v_x} + \omega_c\cos(\omega t)\partial_{v_y} + \frac{m}{e}\,\omega^2\cos(\omega t)\partial_E,\\
&X_6 = \sin(\omega t)\partial_x + \omega\cos(\omega t)\partial_{v_x} + \omega_c\sin(\omega t)\partial_{v_y} + \frac{m}{e}\,\omega^2\sin(\omega t)\partial_E,
\end{aligned}
\tag{3.1}
$$

where $\omega \equiv \omega_{uh}$ is the frequency of upper hybrid oscillations ($\omega_{uh}^2 = \omega_p^2 + \omega_c^2$) and $\omega_p^2 = 4\pi e^2 n_0/m$, ω_p is the electron plasma (Langmuir) frequency.

The non-vanishing commutators between generators (3.2) are as follows:

$$\begin{aligned} &[X_1, X_5] = -\omega X_6, && [X_1, X_6] = \omega X_5, && [X_2, X_4] = X_2, \\ &[X_3, X_4] = X_3, && [X_4, X_5] = -X_5, && [X_4, X_6] = -X_6. \end{aligned} \tag{3.2}$$

The algebra is solvable. Using the adjoint representation we find the *optimal system* of one-dimensional subalgebras (see [1, 3]). They are generated by the following operators:

$$\begin{aligned} &\pm X_1 + a_2 X_2 + a_3 X_3, && a_2 X_2 + a_3 X_3, \\ &\tau X_1 + X_4, && a_2 X_2 + a_3 X_3 + X_6, \end{aligned} \qquad \text{where} \quad \tau, a_2, a_3 \in \mathbf{R}^1. \tag{3.3}$$

Particularly interesting cases correspond to vanishing parameters $\tau = 0, a_2 = 0, a_3 = 0$. For example, the first operator with $a_2 = 0, a_3 = 0$ is equal to X_1 and the corresponding invariant solutions are stationary solutions. For $a_3 = 0$ we have $X_1 \pm X_2$ and BGK waves. In general, we find invariant solutions corresponding to the operators (3.3) by the method of invariants. The invariants of transformations generated by the operators (3.3) can be constructed in the simplest way from explicit form of these transformations of variables. They are easily found as a superposition of transformations generated by generators X_1, X_2, X_3, X_4, X_6 since each of them commutes with the other ones. To illustrate this procedure we consider the case of $\tau X_1 + X_4$. Under finite transformation $\exp[\epsilon(\tau X_1 + X_4)]$, the variables change as follows:

$$\widetilde{t} = t + \epsilon\tau, \qquad \widetilde{x} = x e^{\epsilon}, \qquad \widetilde{v_x} = v_x e^{\epsilon}, \qquad \widetilde{v_y} = v_y e^{\epsilon}, \qquad \widetilde{f} = f e^{-2\epsilon}, \qquad \widetilde{E} = E e^{\epsilon}.$$

It is convenient to introduce a new variable $t' := \exp(t/\tau)$ which transforms like the other variables: $\widetilde{t'} = t' \exp(\epsilon)$. Invariants have the form

$$\begin{aligned} &\frac{x}{t'} = x e^{-t/\tau}, \quad \frac{v_x}{t'} = v_x e^{-t/\tau}, \quad \frac{v_y}{t'} = v_y e^{-t/\tau}, \quad \frac{v_x}{x}, \quad \frac{v_y}{x}, \\ &t'^2 f = f e^{2t/\tau}, \quad x^2 f, \quad v_x^2 f, \quad v_y^2 f, \quad f E^2, \quad \frac{E}{t'} = E e^{-t/\tau}, \ldots . \end{aligned}$$

We choose five independent invariants, three of them built from independent variables, as follows (see [4]):

$$x e^{-t/\tau}, \qquad v_x e^{-t/\tau}, \qquad v_y e^{-t/\tau}, \qquad f e^{2t/\tau}, \qquad E e^{-t/\tau}.$$

We look for invariant solutions in the form (the electric field does not depend on velocity)

$$e^{-2t/\tau} f(x e^{-t/\tau}, v_x e^{-t/\tau}, v_y e^{-t/\tau}), \qquad e^{t/\tau} E(x e^{-t/\tau}).$$

The results of such a classification of solutions invariant with respect to generators of optimal system (3.3) are collected in the following table:

No	Subgroup	Form of the invariant solution
1	$\pm X_1 + a_2X_2 + a_3X_3$	$f(x \mp a_2t, v_x, v_y \mp a_3t), \quad E(x \mp a_2t) \mp a_3\dfrac{m}{e}\omega_c t$
2	$a_2X_2 + a_3X_3$	$f(t, v_x, a_3x - a_2v_y), \quad \dfrac{1}{a_2}E(t) - \dfrac{a_3}{a_2}\dfrac{m}{e}\omega_c x$
3	$\tau X_1 + X_4$	$e^{-2t/\tau}f(xe^{-t/\tau}, v_xe^{-t/\tau}, v_ye^{-t/\tau}),$ $e^{t/\tau}E(xe^{-t/\tau})$
4	$a_2X_2 + a_3X_3 + X_6$	$f(t, x\omega\cos(\omega t) - v_x(a_2 + \sin(\omega t)),$ $x(a_3 + \omega_c\sin(\omega t)) - v_y(a_2 + \sin(\omega t)),$ $[a_2 + \sin(\omega t)]^{-1}\left[E(t) + \dfrac{m}{e}(\omega^2\sin(\omega t) - a_3\omega_c)x\right]$

where $\tau, a_2, a_3 \in \mathbf{R}^1$.

4. Cold Electron Plasma Limit

Considerable extension of symmetry takes place in the cold electron plasma limit, i.e. for the distribution function of the form

$$f(t, x, v_x, v_y) = n(t, x)\,\delta(v_x - u(t, x))\,\delta(v_y - v(t, x)). \tag{4.1}$$

The equations (2.1) are then reduced to partial differential equations for the functions u, v and E of the variables t and x:

$$\partial_\tau u = -\frac{e}{m}\,E - \omega_c v, \qquad \partial_\tau v = \omega_c u, \qquad \partial_\tau E = 4\pi e n_0 u, \tag{4.2}$$

where $\partial_\tau = \partial_t + u\partial_x$. A solution of (4.2) by the use of Lagrangean variables is made possible by adding the equation for the orbit $x(t)$:

$$\partial_\tau x = u \tag{4.3}$$

and treating x, u, v and E as the functions of τ and the initial value of $x(\tau)$, i.e. $x_0 = x(0)$. It is readily seen that

$$\partial_\tau^2 u + \omega^2 u = 0$$

and the general solution has the form

$$\begin{aligned} x &= I_1\cos(\omega t) + I_2\sin(\omega t) + I_3, \\ u &= \omega\,\left[-I_1\sin(\omega t) + I_2\cos(\omega t)\right], \\ E &= 4\pi e n_0\,\left[I_1\cos(\omega t) + I_2\sin(\omega t) + I_4\right], \\ v &= \omega_c\left[I_1\cos(\omega t) + I_2\sin(\omega t)\right] - \frac{\omega_p^2}{\omega_c}\,I_4, \end{aligned}$$

where $\mathbf{I} = \{I_1, \ldots, I_4\}$ are functions of x_0 and are determined by the initial conditions, e.g. $I_2(x_0) = u(0, x_0)/\omega$. In this way complicated expressions for x, u, v and E, presented in [10], can be obtained. From the point of view of the symmetry approach, this possibility of finding the general solution in Lagrangean variables is related to the fact that the system of Eqs. (4.2) and (4.3) can be written in the form

$$\partial_\tau \mathbf{I} = 0,$$

which is invariant, as it is readily seen, under the wide class of transformations depending on arbitrary functions $\mathbf{F}$ and G:

$$\mathbf{I}' = \mathbf{F}(\mathbf{I}), \qquad \tau' = G(\tau, \mathbf{I}). \tag{4.4}$$

The transformations (4.4) allow us to obtain the general solution of the system of Eqs. (4.2) and (4.3) starting from the trivial zero solution. In fact, the trivial solution

$$I_2 = 0, \ldots, I_4 = 0$$

is generalized by the transformation (4.4) to

$$F_2(I_1, \ldots, I_4) = 0, \ldots, F_4(I_1, \ldots, I_4) = 0,$$

with arbitrary functions $F_2, \ldots, F_4$. Then we can express the solution in the form

$$I_2 = g_2(I_1), \ldots, I_4 = g_4(I_1).$$

Finally, we determine the functions $g_2, \ldots, g_4$ from the initial conditions for u, v and E thus reproducing the solution obtained in Lagrangean variables. So, the possibility of solving the cold electron plasma equations is due to the sufficiently large extension of symmetry of the model.

5. Langmuir Electron Plasma Waves

For comparison with the above results we review shortly the previous results, obtained in [5], for longitudinal electron plasma waves. In this case, i.e. for the system of equations (see, e.g. [10])

$$\begin{aligned} &\partial_t f + v\,\partial_x f - \frac{e}{m}\,E\,\partial_v f = 0, \\ &\partial_t E = 4\pi e \int dv\, v f, \\ &\partial_x E = 4\pi e \left(n_0 - \int dv\, f \right), \end{aligned} \tag{5.1}$$

where $v = v_x$ and the distribution function is already integrated over v_y and v_z, the symmetry group is generated by the following operators (see [5]):

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = \partial_x, \\ &X_3 = x\,\partial_x + v\,\partial_v - f\,\partial_f + E\,\partial_E, \\ &X_4 = \cos(\omega_p t)\,\partial_x - \omega_p \sin(\omega_p t)\,\partial_v + \frac{m}{e}\,\omega_p^2 \cos(\omega_p t)\,\partial_E, \\ &X_5 = \sin(\omega_p t)\,\partial_x + \omega_p \cos(\omega_p t)\,\partial_v + \frac{m}{e}\,\omega_p^2 \sin(\omega_p t)\,\partial_E. \end{aligned}$$

In the cold electron plasma limit

$$f(t,x,v) = n(t,x)\,\delta(v - w(t,x))$$

the system (5.1) reduces to the partial differential equations

$$\partial_\tau w = -\frac{e}{m}\,E, \qquad \partial_\tau E = 4\pi e n_0 w, \tag{5.2}$$

where $\partial_\tau = \partial_t + w\partial_x$. In this case invariants $\mathbf{I} = \{I_1, I_2, I_3\}$ have the form

$$\begin{aligned} I_1 &= x - \frac{E}{4\pi e n_0}\,, \\ I_2 &= \frac{w}{\omega_p}\,\cos(\omega_p\,\tau) + \frac{E}{4\pi e n_0}\,\sin(\omega_p\,\tau)\,, \\ I_3 &= \frac{w}{\omega_p}\,\sin(\omega_p\,\tau) - \frac{E}{4\pi e n_0}\,\cos(\omega_p\,\tau)\,. \end{aligned}$$

The invariance under the transformations (4.4) made it possible to obtain the general solution of (5.2) from the trivial one or to solve this system in Lagrangean variables as it was done in [10].

6. Conclusions

In this paper we have studied the invariance of the integro-differential equations for the nonlinear upper hybrid waves (2.2). Lie symmetry group of point transformations (3.2) has been found. Among the symmetries appear the time and space homogeneity (X_1 and X_2), the non-relativistic remnant of the Lorentz transform in the y direction (X_3), similarity transformation (X_4) related to the fact that in the equations (2.1) no assumptions are made *a priori* about the plasma temperature and thus no characteristic thermal velocity is contained in the system. Transformations X_5 and X_6 are the specific ones for upper hybrid waves. They mean that the spatially homogeneous upper hybrid oscillations can be included in arbitrary solution of the model as e.g. the nonlinear reaction of the system to the rapid homogeneous external current flash.

The optimal system and corresponding invariant solutions for the system (2.2) were found.

It was shown also that in the cold electron plasma limit (4.1) the symmetry extension made it possible to obtain the general solution, which is equivalent to the better known procedure of solving equations in Lagrangean variables.

Comparison of the above mentioned results with symmetries and solutions for the simpler model of electron plasma high frequency waves in absence of the external magnetic field shows very close qualitative analogy.

Acknowledgments

One of the authors (V. Taranov) is very grateful to the Mianowski Fund for financial support, to the Space Research Center and the Institute of Fundamental Technological Research of Polish Academy of Sciences for their hospitality. The authors are indebted to Professor A. J. Turski for his inspiration and helpful discussions.

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Received July 4, 2002.