

## SYMMETRIES OF EQUATIONS WITH FUNCTIONAL ARGUMENTS

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The method of determination of the Lie symmetry groups of integro-differential equations is generalized to the case of equations with functional arguments. The method leads to significant applications for instance to the nonlocal NLS equation.

**Key words:** symmetry, PDE's, integro-differential equations, delayed arguments.

### 1. Introduction

The best known examples of such equations are equations with *delayed arguments*. For ordinary differential equations a delayed argument appears usually due to feedback in oscillating systems as in the case of Minorsky's equation

$$\ddot{y}(t) + A \dot{y}(t) + \omega^2 y(t) + B \dot{y}(t - \tau) - C \dot{y}^3(t - \tau) = 0$$

which describes self-excited oscillations in a ship-stabilization device. The wave equation with delayed response

$$\partial_t^2 u(t, x) = a^2 \partial_x^2 u(t, x) + a^2 \partial_x^2 u(t - \tau, x)$$

is a simple example of a partial differential equation of this type. Processes with distributed delay response lead to integro-differential equations. We find a symmetry group for an interesting example of such an equation, namely the nonlocal Schrödinger (nNLS) equation

$$i \partial_t \mathcal{E} + a \partial_x^2 \mathcal{E} + b |\mathcal{E}|^2 \mathcal{E} + \mathcal{E} \frac{1}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \left( \frac{q_{\alpha}}{m_{\alpha}} \right)^2 \int_{-\infty}^{\infty} du \int_{-\infty}^t dt_1 \partial_{t_1} |\mathcal{E}(t_1, x - (t - t_1)u)|^2 \frac{1}{u} \partial_u f_{\alpha 0} = 0, \quad (1)$$

where

$$a = \frac{3}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \int_{-\infty}^{\infty} du u^2 f_{\alpha 0}, \quad b = \frac{-1}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \left( \frac{q_{\alpha}}{m_{\alpha}} \right)^2 \int_{-\infty}^{\infty} du \frac{1}{u} \partial_u f_{\alpha 0},$$

and

$\mathcal{E}(t, x)$  – complex amplitude of the electric vector field  $\mathbf{E} = (E, 0, 0)$ ,  
 $f_{\alpha o}(u)$  – equilibrium distribution function of  $\alpha$ -plasma component,  
 $u$  – component of the vector velocity  $\mathbf{v} = (u, 0, 0)$ ,  
 $q_\alpha, m_\alpha$  – charge and mass of  $\alpha$ -particles, respectively,  
 $\omega_\alpha$  – plasma frequency of  $\alpha$ -component:

The equation (1) describes the envelope  $\mathcal{E}$  of modulated Langmuir waves in plasmas

$$E(x, t) = \mathcal{E}(x, t)e^{-i\omega t} + \text{c.c.},$$

assuming, that the amplitude  $\mathcal{E}(x, t)$  is a slowly varying function of  $t$  with respect to  $1/\omega$ , that is

$$\frac{1}{\omega} |\partial_t \mathcal{E}(x, t)| \ll |\mathcal{E}(x, t)|.$$

The equation (1) was derived in [1] in the framework of the Vlasov description of collision-less, multi-component, one-dimensional plasmas with no magnetic field.

## 2. Extension of a group of point transformations to variables depending on functional arguments

Our aim is to generalize the method, presented in previous papers [2], [3] and [4], of finding symmetry groups of point transformations admitted by integro-differential equations to the case of such equations with *functional arguments*:

$$F(x, y(x), y(x), \dots, y(x), y(\varphi(x)), y(\varphi(x)), \dots, y(\varphi(x))) + \int_X dx^1 \dots dx^l f(x, y(x), y(x), \dots, y(x), y(\varphi(x)), y(\varphi(x)), \dots, y(\varphi(x))) = 0 \quad (2)$$

where  $n, m, k, l, p, q$  are arbitrary natural numbers ( $l \leq n$ ),  $x = (x^1, \dots, x^n)$ , functions  $F$ ,  $f$  and  $\varphi$  are arbitrary but sufficiently regular to secure the existence of solutions to (2), limits of integrations (region  $X$ ) are also arbitrary. The symbol  $y$  denotes the set of all partial derivatives of  $m$ -order:

$$y = \left\{ \frac{\partial^m y}{\partial_{x^{i_1}} \dots \partial_{x^{i_m}}} \equiv \partial_{x^{i_1}} \dots \partial_{x^{i_m}} y \equiv \partial_{i_1} \dots \partial_{i_m} y \right\}.$$

We restrict our considerations to one scalar equation of the type (2) and one functional argument  $\varphi(x)$  for the sake of simplicity of notation. This involves fewer indices in subsequent formulae. Generalization to a system of equations for many dependent variables  $y = \{y^1, \dots, y^r\}$  and to a few functional arguments  $(\varphi, \psi, \dots)$  is obvious.

We look for a Lie symmetry group of *point transformations*

$$\begin{aligned}\tilde{x}^i &= x^i + \epsilon \xi^i(x, y) + \mathcal{O}(\epsilon^2) \\ \tilde{y} &= y + \epsilon \eta(x, y) + \mathcal{O}(\epsilon^2),\end{aligned}\quad (3)$$

which leave the equation (2) invariant. To this end we extend the group of point transformations (3) to a jet space of independent and dependent variables and derivatives of dependent variables in usual way ([5] - [7])

$$\begin{aligned}\tilde{x}^i &= x_i + \epsilon \xi^i(x, y) + \mathcal{O}(\epsilon^2) \\ \tilde{y} &= y + \epsilon \eta(x, y) + \mathcal{O}(\epsilon^2) \\ \tilde{y}_i &= y_i + \epsilon \eta_i(x, y, y_1) + \mathcal{O}(\epsilon^2) \\ &\vdots \\ \tilde{y}_{i_1 \dots i_m} &= y_{i_1 \dots i_m} + \epsilon \eta_{i_1 \dots i_m}(x, y, y_1, \dots, y_m) + \mathcal{O}(\epsilon^2),\end{aligned}\quad (4)$$

where the coefficients  $\eta_i, \dots, \eta_{i_1 \dots i_m}$ , defining the extended group are given by the recursion relations (summation over repeated indices is assumed):

$$\begin{aligned}\eta_i &= D_i \eta - y_j D_i \xi^j \\ &\vdots \\ \eta_{i_1 \dots i_m} &= D_{i_m} \eta_{i_1 \dots i_{m-1}} - y_{i_1 \dots i_{m-1} j} D_{i_m} \xi^j\end{aligned}\quad (5)$$

and the total derivative  $D_i$  is defined as follows

$$D_i = \partial_i + y_i \partial_y + y_{ij} \partial_{(y_j)} + \dots + y_{i i_1 \dots i_n} \partial_{(y_{i_1 \dots i_n})} + \dots$$

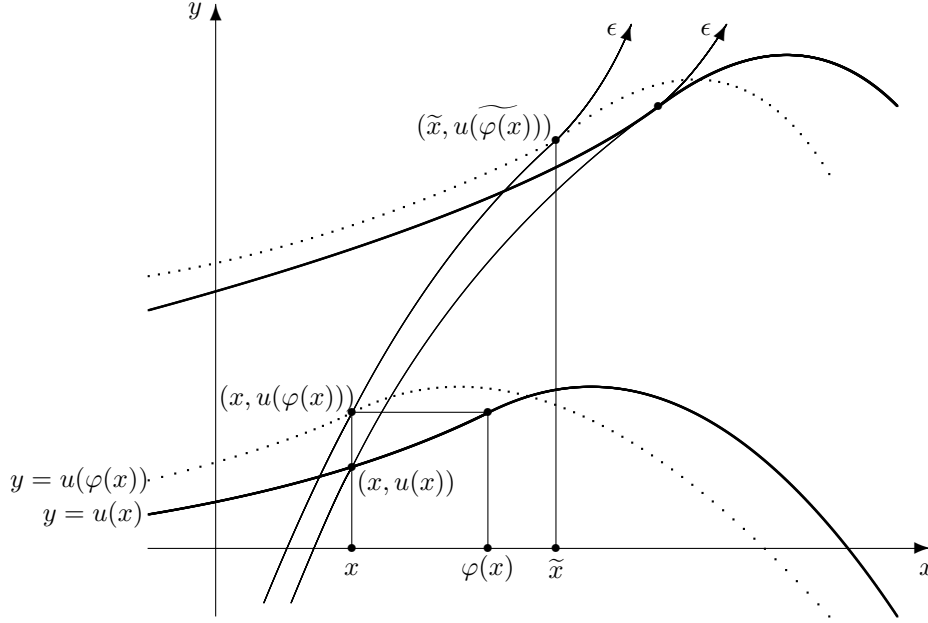
But now further extension of a group (4) to variables depending on functional arguments  $y^\varphi(x) = y(\varphi(x))$ ,  $y_i^\varphi(x) = y_i(\varphi(x))$ , ... is needed. We extend the group action to the new, independent, variables  $y^\varphi$ ,  $y_i^\varphi$ , ... as follows

$$\begin{aligned}\tilde{y}^\varphi &= y^\varphi + \epsilon \eta^\varphi(x, y, y^\varphi) + \mathcal{O}(\epsilon^2) \\ \tilde{y}_i^\varphi &= y_i^\varphi + \epsilon \eta_i^\varphi(x, y, y_1, y^\varphi) + \mathcal{O}(\epsilon^2) \\ &\vdots \\ \tilde{y}_{i_1 \dots i_m}^\varphi &= y_{i_1 \dots i_m}^\varphi + \epsilon \eta_{i_1 \dots i_m}^\varphi(x, y, y_1, \dots, y_m, y^\varphi) + \mathcal{O}(\epsilon^2),\end{aligned}\quad (6)$$

where coefficients  $\eta^\phi, \eta_i^\phi, \dots$  come from transformation of solution of (2) and its derivatives, shifted to the point  $\varphi(x)$ , induced by the point transformations (3). The solution  $y = u(x)$  of (2) is represented by its graph  $\{(x, u(x))\}$  and the shifted solution  $y = u(\varphi(x))$  by the graph  $\{(x, u(\varphi(x)))\}$ . Pairs  $(x, u(\varphi(x)))$  go to  $(\tilde{x}, u(\varphi(x)))$  under the point transformations

$$u(\varphi(x)) = u(\varphi(x)) + \epsilon \eta(x, u(\varphi(x))) + \mathcal{O}(\epsilon^2),\quad (7)$$

as it is illustrated at the following symbolic picture:



We assume that the transformation law for the variable  $y^\varphi$  coincides with (7) in the case when  $(x, y^\varphi) = (x, u(\varphi(x)))$ , so

$$\eta^\varphi(x, y, y^\varphi) = \eta^\varphi(x, y^\varphi) := \eta(x, y(\varphi(x))). \quad (8)$$

In the same way, transformation of  $u_i(\varphi(x))$  under (3) is used to define a transformation of the variable  $y_i^\varphi$

$$\eta_i^\varphi(x, y, y, y^\varphi) = \eta_i(x, y, y_{1, \neq i}, y_i(\varphi(x))) \quad (9)$$

with  $\eta_i$  from (5). In this formula only derivative  $y_i$  is shifted to the point  $\varphi(x)$ . In the same way we define higher order coefficients  $\eta_{i_1 \dots i_m}^\varphi$ . The *infinitesimal generator* of extended transformation (4), (6) is of the form

$$G = \xi^i \partial_{x^i} + \eta \partial_y + \eta_i \partial_{y_i} + \dots + \eta_{i_1 \dots i_m} \partial_{y_{i_1 \dots i_m}} + \eta^\varphi \partial_{y^\varphi} + \dots + \eta_{i_1 \dots i_m}^\varphi \partial_{y_{i_1 \dots i_m}^\varphi}. \quad (10)$$

Now, we are ready to repeat the procedure, presented in [2] and [3], of deriving a criterion of invariance of the equation (2). First, we act on the equation by the extended transformations (4) and (6) writing down explicitly only terms that are linear with respect to the parameter  $\epsilon$ . Next, by expanding functions  $F$  and  $f$  in their Taylor series and changing variables in the integral, we express the change of (2) in terms of extended generator (10). From the definition of symmetry of (2) this change must be equal to zero. Since it has to be valid for all values of  $\epsilon$ , we obtain the following *infinitesimal criterion of invariance* of the equation (2) under the point transformations (3)

$$GF + \int_X dx^1 \dots dx^l \left[ Gf + f \sum_{i=1}^l \partial_i \xi^i \right] = 0 \quad \text{on solutions of (2)}. \quad (11)$$

Other approaches to finding symmetries of IDE's are presented in [8], [9], [10] and in references therein.

### 3. Symmetries of the nonlocal NLS equation

We rewrite the nNLS equation (1) in the form suitable for the criterion (11). The change of the order of integrations is admissible due to the prescription for singularity integration assumed in [1]. But, first of all, it is not necessary and does not affect further considerations.

$$F(\mathcal{E}, \mathcal{E}^*, \mathcal{E}_t, \mathcal{E}_{xx}) + \int_{-\infty}^t dt_1 f(\mathcal{E}, \mathcal{E}(1), \mathcal{E}^*(1), \mathcal{E}_{t_1}(1), \mathcal{E}_{t_1}^*(1)) = 0, \quad (12)$$

where

$$F(\mathcal{E}, \mathcal{E}^*, \mathcal{E}_t, \mathcal{E}_{xx}) = i\mathcal{E}_t + a\mathcal{E}_{xx} + b\mathcal{E}^2\mathcal{E}^*,$$

and

$$\begin{aligned} f(\mathcal{E}, \mathcal{E}(1), \mathcal{E}^*(1), \mathcal{E}_{t_1}(1), \mathcal{E}_{t_1}^*(1)) \\ = \frac{1}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \left( \frac{q_{\alpha}}{m_{\alpha}} \right)^2 \int_{-\infty}^{\infty} du [\mathcal{E}\mathcal{E}_{t_1}(1)\mathcal{E}^*(1) + \mathcal{E}\mathcal{E}(1)\mathcal{E}_{t_1}^*(1)] \frac{1}{u} \partial_u f_{\alpha 0}. \end{aligned}$$

The delayed argument (1) means

$$(1) \equiv \varphi(t_1, x) \equiv (t_1, x - (t - t_1)u),$$

and the usual argument  $(t, x)$  is dropped. The point transformations (3) have the following form ( $x^1 \rightarrow t_1$ ,  $x^2 \rightarrow t$ ,  $x^3 \rightarrow x$ ,  $y \rightarrow \mathcal{E}$ , here  $u$  is a parameter not a variable):

$$\begin{aligned} \tilde{t}_1 &= t_1 + \epsilon\tau(t_1, x, \mathcal{E}) + \mathcal{O}(e^2), \\ \tilde{t} &= t + \epsilon\tau(t, x, \mathcal{E}) + \mathcal{O}(e^2), \\ \tilde{x} &= x + \epsilon\xi(t, x, \mathcal{E}) + \mathcal{O}(e^2), \\ \tilde{\mathcal{E}} &= \mathcal{E} + \epsilon\eta(t, x, \mathcal{E}) + \mathcal{O}(e^2), \end{aligned} \quad (13)$$

with the generator

$$G = \tau(t_1, x, \mathcal{E})\partial_{t_1} + \tau\partial_t + \xi\partial_x + \eta\partial_{\mathcal{E}}. \quad (14)$$

It is essential that there is the same function  $\tau(\cdot)$  in both cases  $t$  and  $t_1$ , which are distinguished only by the value of a variable. The second order extension of the generator (14) has the form

$$\begin{aligned} \overset{1}{G} &= G + \eta^* \partial_{\mathcal{E}^*} + \eta^{(1)} \partial_{\mathcal{E}(1)} + \eta^{(1)*} \partial_{\mathcal{E}^*(1)} + \eta_t \partial_{\mathcal{E}_t} + \eta_{t_1}^{(1)} \partial_{\mathcal{E}_{t_1}(1)} + \eta_{t_1}^{(1)*} \partial_{\mathcal{E}_{t_1}^*(1)}, \\ \overset{2}{G} &= \overset{1}{G} + \eta_{xx} \partial_{\mathcal{E}_{xx}}, \end{aligned} \quad (15)$$

where we omit terms, which do not contribute when they are applied to (12) according to the criterion (11). In (15) we have taken into account additional extension of the group (13) to the complex conjugate dependent variables. Transformation properties of these variable follow from complex conjugation of the formulae (4) and (6). A detailed discussion of the problem can be found in [11].

According to the criterion (11), we apply the generator (15) to the equation (12) to find

$$\begin{aligned}
0 = & \overset{2}{G}F + \int_{-\infty}^t dt_1 \left( \overset{1}{G}f + f\tau_{t_1}(t_1, x, \mathcal{E}) \right) = i\eta_t + a\eta_{xx} + b(2\mathcal{E}\mathcal{E}^*\eta + \mathcal{E}^2\eta^*) \\
& + \frac{1}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \left( \frac{q_{\alpha}}{m_{\alpha}} \right)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{\infty} du \left[ \eta\mathcal{E}_{t_1}(1)\mathcal{E}^*(1) + \eta\mathcal{E}(1)\mathcal{E}_{t_1}^*(1) \right. \\
& \quad + \eta^{(1)}\mathcal{E}\mathcal{E}_{t_1}^*(1) + \eta^{(1)*}\mathcal{E}\mathcal{E}_{t_1}(1) + \eta_{t_1}^{(1)}\mathcal{E}\mathcal{E}^*(1) + \eta_{t_1}^{(1)*}\mathcal{E}\mathcal{E}(1) \\
& \quad \left. + \tau_{t_1}(t_1, x, \mathcal{E})\mathcal{E}\mathcal{E}_{t_1}(1)\mathcal{E}^*(1) + \tau_{t_1}(t_1, x, \mathcal{E})\mathcal{E}\mathcal{E}(1)\mathcal{E}_{t_1}^*(1) \right] \frac{1}{u} \partial_u f_{\alpha 0}. \quad (16)
\end{aligned}$$

From the equation (12), which imposes the constraint onto the variables in the extended space, we calculate the highest order derivative  $\mathcal{E}_{xx}$  so the remaining variables are independent. By equating to zero the independent terms in (16) we derive the system of determining equations. Since the amplitude  $\mathcal{E}(1)$ , which is independent of  $\mathcal{E}(t, x)$ , appears only in the integral terms, thus these terms are independent of others. Hence, we can equate to zero the integral terms and the other terms separately. From non-integral terms we reproduce (see [11]) Tajiri's result [12] for ordinary NLS equation (see also [13])

$$\begin{aligned}
\tau &= 2\alpha t + \gamma_2, \\
\xi &= \alpha x + \beta at + \gamma_1, & \alpha, \beta, \gamma_1, \gamma_2, \delta \in \mathbf{R}^1, \\
\eta &= g(x)\mathcal{E} = \left( \frac{i}{2}\beta x - \alpha + i\delta \right) \mathcal{E}.
\end{aligned} \quad (17)$$

Whereas, the integral terms lead to the following equation:

$$0 = \frac{1}{2\omega^3} \sum_{\alpha} \omega_{\alpha}^2 \left( \frac{q_{\alpha}}{m_{\alpha}} \right)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{\infty} du \left[ -\beta a \mathcal{E}\mathcal{E}_{t_1}(1)\mathcal{E}^*(1) - \beta a \mathcal{E}\mathcal{E}(1)\mathcal{E}_{t_1}^*(1) \right] \frac{1}{u} \partial_u f_{\alpha 0},$$

where (17) was used.

In the integrand, the coefficients  $\beta a$  at monomials of the variables  $\mathcal{E}$ ,  $\mathcal{E}_{t_1}(1)$ ,  $\mathcal{E}^*(1)$ , ... do not depend on these variables. So we may use the Lagrange lemma ([14]) and equate to zero the integrand as the above equation must hold for all values of these variables. Thus, we obtain.

$$\beta = 0.$$

Hence, the final solution of the determining equations for the nNLS equation is following

$$\begin{aligned}
\tau &= 2\alpha t + \gamma_2, \\
\xi &= \alpha x + \gamma_1, \\
\eta &= g\mathcal{E} = (-\alpha + i\delta)\mathcal{E}.
\end{aligned}
\quad \mathbf{R}^1 \ni \alpha, \gamma_1, \gamma_2, \delta - \text{arbitrary constants,}$$

Choosing all parameters  $\alpha, \gamma_1, \gamma_2, \delta$  equal to zero except one, which is assumed to be equal to 1 in each case, we obtain four independent generators

$$G_1 = \partial_t, \quad G_2 = \partial_x, \quad G_3 = i\mathcal{E}\partial_{\mathcal{E}}, \quad G_4 = 2t\partial_t + x\partial_x - \mathcal{E}\partial_{\mathcal{E}}, \quad (18)$$

which span the *Lie algebra* of the symmetry group of nNLS equation (1). The only non-vanishing commutators between the generators (18) are the following:

$$[G_1, G_4] = 2G_1, \quad [G_2, G_4] = G_2, \quad (19)$$

Thus, the symmetry group of the nNLS equation is a 4-parameter subgroup of the 5-parameter symmetry group of the NLS equation. Let us set up finite point transformations corresponding to the four generators (18). The  $G_1$  generates the time translations

$$\tilde{t} = e^{\epsilon G_1} t = t + \epsilon, \quad \tilde{x} = e^{\epsilon G_1} x = x, \quad \tilde{\mathcal{E}} = e^{\epsilon G_1} \mathcal{E} = \mathcal{E},$$

and the  $G_2$  generates the space translations

$$\tilde{t} = e^{\epsilon G_2} t = t, \quad \tilde{x} = e^{\epsilon G_2} x = x + \epsilon, \quad \tilde{\mathcal{E}} = e^{\epsilon G_2} \mathcal{E} = \mathcal{E}.$$

The  $G_3$  generates the following gauge transformations

$$\tilde{t} = e^{\epsilon G_3} t = t, \quad \tilde{x} = e^{\epsilon G_3} x = x, \quad \tilde{\mathcal{E}} = e^{\epsilon G_3} \mathcal{E} = \mathcal{E} e^{i\epsilon}.$$

$G_4$  generates the following scaling transformations

$$\tilde{t} = e^{\epsilon G_4} t = t e^{2\epsilon}, \quad \tilde{x} = e^{\epsilon G_4} x = x e^{\epsilon}, \quad \tilde{\mathcal{E}} = e^{\epsilon G_4} \mathcal{E} = \mathcal{E} e^{-\epsilon}.$$

For ordinary NLS equation there is an additional generator

$$G_5 = at\partial_x + \frac{i}{2}x\mathcal{E}\partial_{\mathcal{E}}, \quad (20)$$

which leads to the following transformations

$$\tilde{t} = e^{\epsilon G_5} t = t, \quad \tilde{x} = e^{\epsilon G_5} x = x + \epsilon at, \quad \tilde{\mathcal{E}} = e^{\epsilon G_5} \mathcal{E} = \mathcal{E} e^{\frac{i}{2}\epsilon(\epsilon \frac{at}{2} + x)},$$

which describe the travelling wave propagation of the amplitude  $\mathcal{E}$  of electric field with the velocity  $\epsilon a$  combined with the change of phase of the amplitude  $\mathcal{E}$  and this change of phase propagates with the velocity  $\epsilon a/2$ . By this transformations we can generate undamped solutions of the NLS equation, which are related to the Zakharov solitons [15].

Mathematically,  $G_5$  is not admitted by the nNLS equation (1) due to the derivative  $\partial_{t_1}$  in the nonlinear term in the integral. This constitutes the difference between the equation (1) and the nonlinear IDE of the Schrödinger type considered in [9]. Moreover, for a local nonlinear term of this type, that is  $\mathcal{E}\partial_t|\mathcal{E}|^2$  added to the NLS equation,  $G_5$  is also excluded.

#### 4. Conclusions

It has been shown that the method of determination of symmetry group of integro-differential equations ([2]-[3]) can be further generalized to equations with complex variables and with functional arguments. The presented method is practically applicable as has been exhibited for nonlocal NLS equation. Comparison of symmetry groups of NLS and nNLS equations shows the symmetry breaking in the case of nNLS equation. It is due to the wave-particle interaction which leads to the Landau damping. Thus, the undamped travelling waves, in particular Zakharov solitons, are excluded from solutions of the nNLS equation.

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