Finite Characterizations of Some Linear Problems with Inexact Data*

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Notations

- $A \le B$ means $A_{ij} \le B_{ij}$ for each i, j,
- $e = (1, 1, \ldots, 1)^T$,
- $Y = \{y \in R^n;\ |y| = e\}$
  $=$ the set of all ± 1-vectors.

Basic tool: matrices A_{yz}

For a given square interval matrix

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \le A \le \overline{A}\}$$

and for $y, z \in Y$ we define a matrix A_{yz} by

$$(A_{yz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases}$$

$(i, j = 1, \dots, n)$.

Hence $A_{yz} \in \mathbf{A}$ for each $y, z \in Y$.

Main point

All the subsequent properties will be formulated in terms of matrices A_{yz}.

Each A_{yz} satisfies

$$(A_{yz})_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\},$$

i.e., it is a "vertex matrix".

But there are "only" 2^{2n} matrices A_{yz} compared to 2^{n^2} vertex matrices.

Part One:
Matrix Properties

Regularity

Definition. An interval matrix $\mathbf{A}$ is called regular if each $A \in \mathbf{A}$ is nonsingular.

Theorem 1. $\mathbf{A}$ *is regular if and only if*

$$(\det A_{yz})(\det A_{y'z'}) > 0$$

for each $y, z, y', z' \in Y$.

Theorem 2. *The problem is NP-hard.*

Positive (semi)definiteness

Definition. An interval matrix **A** is called positive (semi)definite if each $A \in \mathbf{A}$ is positive (semi)definite.

Theorem 1. **A** *is positive (semi)definite if and only if*

$$\frac{1}{2}(A_{yy} + A_{yy}^T)$$

is positive (semi)definite for each $y \in Y$.

Theorem 2. *The problem is NP-hard.*

Stability

Definition. An interval matrix $\mathbf{A}$ is called stable if each $A \in \mathbf{A}$ is stable, i.e., $\mathrm{Re}\,\lambda < 0$ for each eigenvalue λ of A.

Theorem 1. *A symmetric* $\mathbf{A} = [\underline{A}, \overline{A}]$ *(i.e., both* $\underline{A}$*,* $\overline{A}$ *symmetric) is stable if and only if* $A_{-y,y}$ *is stable for each* $y \in Y$*.*

Theorem 2. *The problem is NP-hard.*

Part Two:
Linear Equations and Matrix Inversion

Reminder and add'l notation

For $\mathbf{A} = [\underline{A}, \overline{A}]$ we have defined

$$(A_{yz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases}$$

$(i, j = 1, \ldots, n)$.

Similarly, for $\mathbf{b} = [\underline{b}, \overline{b}] = \{b;\ \underline{b} \le b \le \overline{b}\}$ we define

$$(b_y)_i = \begin{cases} \overline{b}_i & \text{if } y_i = 1, \\ \underline{b}_i & \text{if } y_i = -1 \end{cases}$$

$(i = 1, \ldots, n)$ for each $y \in Y$.

Again, $b_y \in \mathbf{b}$ for each $y \in Y$.

Linear interval equations

Definition. For $\mathbf{A}x = \mathbf{b}$, we define $[\underline{x}, \overline{x}]$ as the interval hull of the solution set
$X = \{x;\ Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}$.

Theorem 1. *If* $\mathbf{A}$ *is regular, then we have*

$$\underline{x} = \min_{y,z \in Y} A_{yz}^{-1} b_y$$

$$\overline{x} = \max_{y,z \in Y} A_{yz}^{-1} b_y$$

*(*min/max *componentwise).*

Theorem 2. *The problem is NP-hard.*

Inverse interval matrix

Definition. For a regular **A**, we define $[\underline{B}, \overline{B}]$ as the interval hull of the set $\{A^{-1};\ A \in \mathbf{A}\}$.

Theorem 1. *For a regular* **A** *we have*

$$\underline{B} = \min_{y,z \in Y} A_{yz}^{-1}$$

$$\overline{B} = \max_{y,z \in Y} A_{yz}^{-1}$$

*(*min/max *componentwise).*

Theorem 2. *The problem is NP-hard.*

Nonnegative invertibility

Definition. $\mathbf{A}$ is called nonnegative invertible if $A^{-1} \geq 0$ for each $A \in \mathbf{A}$.

Theorem 1. $\mathbf{A} = [\underline{A}, \overline{A}]$ *is nonnegative invertible if and only if* $\underline{A}^{-1} \geq 0$ *and* $\overline{A}^{-1} \geq 0$.

Theorem 2. *The problem is solvable in polynomial time* **(!)**.

Part Three:
Interval Linear Systems
(Rectangular Case)

Reminder and clarification

From now on we assume that **A** is rectangular $m \times n$ and that **b** is m-dimensional.

Again,

$$(A_{yz})_{ij} = \begin{cases} \overline{A}_{ij} & \text{if } y_i z_j = -1, \\ \underline{A}_{ij} & \text{if } y_i z_j = 1 \end{cases}$$

where y and z are m-dimensional and n-dimensional ± 1-vectors, respectively.

Solvability of linear equations

Definition. $\mathbf{A}x = \mathbf{b}$ is called solvable if $Ax = b$ has a solution for each $A \in \mathbf{A}$, $b \in \mathbf{b}$.

Theorem 1. *$\mathbf{A}x = \mathbf{b}$ is solvable if and only if*

$$A_{ye}x' - A_{-y,e}x'' = b_y$$

$$x' \geq 0,\; x'' \geq 0$$

has a solution for each $y \in Y$.

Theorem 2. *The problem is NP-hard.*

Solvability of linear inequalities

Definition. $\mathbf{A}x \le \mathbf{b}$ is called solvable if $Ax \le b$ has a solution for each $A \in \mathbf{A}$, $b \in \mathbf{b}$.

Theorem 1. $\mathbf{A}x \le \mathbf{b}$ *is solvable if and only if*

$$\overline{A}x' - \underline{A}x'' \le \underline{b}$$

$$x' \ge 0,\ x'' \ge 0$$

has a solution. (In this case $x = x' - x''$ solves each $Ax \le b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$.)

Theorem 2. *The problem is solvable in polynomial time* **(!)**.

Tolerance solutions

Definition. x^* is called a tolerance solution to $\mathbf{A}x = \mathbf{b}$ if $Ax^* \in \mathbf{b}$ for each $A \in \mathbf{A}$.

Theorem 1. $\mathbf{A}x = \mathbf{b}$ *has a tolerance solution if and only if*

$$\overline{A}x' - \underline{A}x'' \leq \overline{b}$$

$$-\underline{A}x' + \overline{A}x'' \leq -\underline{b}$$

$$x' \geq 0,\ x'' \geq 0$$

has a solution ($x^* = x' - x''$ is then a tolerance solution).

Theorem 2. *The problem is solvable in polynomial time* **(!)**.

Control solutions

Definition. x^* is called a control solution to $\mathbf{A}x = \mathbf{b}$ if for each $b \in \mathbf{b}$ there is an $A \in \mathbf{A}$ with $Ax^* = b$.

Theorem 1. $\mathbf{A}x = \mathbf{b}$ *has a control solution if and only if*

$$A_{ez}x \le \underline{b}$$

$$A_{-e,z}x \ge \overline{b}$$

$$(\mathrm{diag}(z))x \ge 0$$

has a solution for some $z \in Y$ *(*x *is then a control solution).*

Theorem 2. *The problem is NP-hard.*

Linear programming I

Definition. For a linear programming problem

$$\text{minimize}\, c^T x$$

s.t.

$$Ax = b,\ x \geq 0$$

put

$$f(A,b,c) = \inf\{c^T x;\ Ax = b,\ x \geq 0\}.$$

Given **A**, **b**, **c**, define

$$\underline{f} = \inf\{f(A,b,c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}$$

$$\overline{f} = \sup\{f(A,b,c);\ A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}\}$$

("range of optimal value").

Linear programming II

Theorem 1. *Let the values*

$$\begin{aligned} \underline{\varphi} &= \inf\{\underline{c}^T x;\ \underline{A}x \le \overline{b},\ \overline{A}x \ge \underline{b},\ x \ge 0\} \\ \overline{\varphi} &= \sup_{y \in Y} f(A_{ye}, b_y, \overline{c}) \end{aligned}$$

be finite. Then

$$\begin{aligned} \underline{f} &= \underline{\varphi} \\ \overline{f} &= \overline{\varphi}. \end{aligned}$$

Theorem 2. *Computing $\underline{f}$ is polynomial-time and computing $\overline{f}$ is NP-hard* **(!)**.

Summary

Features observed:

- all the problems considered can be characterized in terms of matrices A_{yz} and vectors b_y $(y, z \in Y)$,
- most of these problems are NP-hard,
- only a few are solvable in polynomial time.

Conclusion

Hence, the main task of validated computations consists in providing "reasonable" sufficient conditions or enclosures for NP-hard problems.

Your turn now!