The work unfortunately did not survive to our times; the only (short) relation about its contents was given by Proclus, the author of an extensive commentary on Euclid's work. Unfortunately, due in part to long neglect of these problems, especially when the diagrams were seemingly safely expelled from mathematics at the end of the XIXth century, little has been done in this matter since Euclid. The necessary knowledge is still very inadequate and even that limited information that we do have is not well known to a wider audience, hence the (unwarranted) feeling that diagrams are somehow intrinsically unreliable as tools of rigorous and precise reasoning.

Various possible error situations that can occur in diagrams are discussed in Sections II.3 and II.4, see also Section II.2. A general, although probably not yet full and precise summary of the most important error causes is provided below.

Readability errors. Here the error comes from unreliable reading of diagrams due to their sloppy design or execution, see Section II.2.3, unaccounted for visual illusions (Fig. II.16), or too intricate or graphically inadequate visual language. (Fig. II.7c).

Imprecision errors. These are due to the argument relying on imprecise metric features, like in the "64=65" puzzle, see Example II.11a in Section II.3.2.1. This kind of error becomes often an important cause of several other kinds of errors like *accidental alignments* (Section II.3.2.4), *unreliable emergence* (Section II.4.2.2), or *false divergence* (Section II.4.3.2).

Emergence errors. The emergence effect (Section II.4.2), although generally beneficial, may nevertheless lead to errors as well. Here we have *false emergence* which can be due to improper visual language (Section II.2.2) or diagram design (Section II.4.2.1), and *unreliable emergence* (Section II.4.2.2), usually caused by diagram imprecision.

Divergence errors. The common technique of reasoning by cases (called here the *divergence effect*, Section II.4.3) can be also conducted erroneously. The errors can be, first, of the *overlooked divergence* type (Section II.4.3.1), due to an inadequate case analysis or *particularity* (Section II.3.2.3), or some kind of *accidental features* effect (Example II.14 in Section II.3.2.4). Second, there is also *false divergence* (Section II.4.3.2), due to imprecision or limited analogicity of the representation (Section II.3.1.3).

Particularity errors. These errors come from the necessity of using a particular diagram to stay for a whole class of situations (Section II.3.2.3), and more generally, problems with precise representation of sets of objects or configurations (Section II.3.2.2). The errors of this kind are due to imprecise (possibly due to diagram imprecision, Section II.3.2.1) or erroneous delineation of the required set, and manifest themselves often as *overlooked divergence* errors (Section II.4.3.1).

Some kinds of errors listed above occur also in other representations, not only the diagrammatic ones (like errors due to sloppy execution of the representation or language misunderstanding errors), while some are rather specific to diagrammatic representations (like emergence errors). All these errors can be avoided by careful design and use of the appropriate visual language well adapted to the given task (see Section II.2), augmented by knowledge of possible error situations and methods of avoiding them. Therefore, Hilbert's warning:

> The proof can indeed be given by calling on a suitable figure ... [It merely] makes the interpretation easier, and it is a fruitful means of discovering new propositions. *Nevertheless, care, since it can easily be misleading.*
>
> [David Hilbert, *Lecture on Geometry* (1894)]

should clearly be observed, though *not* to the extreme of banning diagrams altogether from formal mathematical reasoning, as can be ascertained from the following quote from [Hadamard 1945] about the work of Hilbert himself:

> Logically, of course—and this is all that is essential—the result announced is fully attained and every intervention of geometrical sense eliminated: that is, theoretically unnecessary to follow the reasoning from the beginning to the end. Is it the same from the psychological point of view? Certainly not. There is no doubt that Hilbert, in working out his *Principles of Geometry*, has been constantly guided by his geometrical sense. If anybody could doubt that (which no mathematician will), he ought simply to cast one glance at Hilbert's book. Diagrams appear at practically every page. They do not hamper mathematical readers in ascertaining that, logically speaking, no concrete picture is needed.
>
> [Jacques Hadamard, *The Psychology of Invention in the Mathematical Field* (1945)]

The last sentence of the above quote needs additional comment. Possibly, "logically speaking" no concrete picture is needed—but then equally logically speaking no concrete formulae are needed in mathematical texts, as all their content can be presented with sentences in plain natural language, say English. Actually, so it was done through centuries of history of mathematics. Intricate formulae and mathematical notations are a quite recent invention, and as yet nobody proposed to get rid of them as "logically speaking" not needed. So why trying to ban diagrams? See also the next Section II.5.3, where the remaining part of the above Hilbert's quote is discussed.

II.5.2.1 Are formulae reliable?

The reasoning based on sentential representations—e.g. mathematical formulae—is also prone to error and may be equally unreliable, as it is signified by the long history of wrong proofs published by professional mathematicians, and often amusing blunders constantly produced by schoolchildren and college students.

If the mere existence of fallacious inferences would be sufficient to dismiss the reasoning method, the propositional reasoning, including mathematical formulae, would qualify for such a dismissal as well. A wide variety of mistaken proofs and fallacious inferences has been produced without any help of diagrams. These range from the traditional informal fallacies, to the misapplication of formal rules, to mistakes far harder to classify or categorize. Moreover, the sources, and indeed the very existence of such fallacies, was in no way self-evident, but come from careful attention and analysis of this form of reasoning.

Table II.7: A sentential "proof" of the equality $1 = 2$.

$$\begin{aligned} -2 &= -2 \\ 1-3 &= 4-6 \\ 1-3+9/4 &= 4-6+9/4 \\ 1^2 - 2\cdot 1\cdot (3/2) + (3/2)^2 &= 2^2 - 2\cdot 2\cdot (3/2) + (3/2)^2 \\ (1-3/2)^2 &= (2-3/2)^2 \\ 1-3/2 &= 2-3/2 \\ 1 &= 2 \end{aligned}$$

As an amusing example, see the "proof" that $1 = 2$, shown in Table II.7.[25] To avoid such blunders one must also observe certain rules and precautions, just like those sketched above for diagrammatic reasoning. Which of these rules (and where) were violated in Table II.7 is left as an exercise to the reader. Finally, consider that the diagrammatic proof of the same equality looks more elegant and entertaining, see Fig. II.39 on the right (and also Section I.5.2).

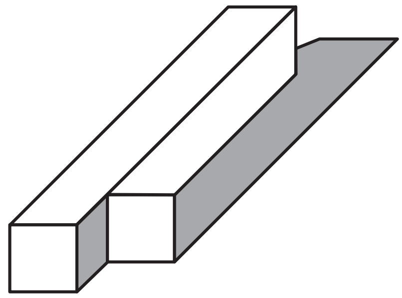

Figure II.39: A diagrammatic "proof" of the equality $1 = 2$.

Unreliability argument: conclusion. To sum up this section, the unreliability objection also cannot be credibly held against diagrams. Diagrams are no more unreliable than formulae, provided the proper caution is observed, and reasoning rules suitable for the specific features of diagrammatic reasoning are used. The only remaining difficulty so far is due to the fact that the proper error-avoidance rules for diagrams are not yet fully investigated, codified and taught. This problem constitutes then another challenge to be confronted by the researchers in the field of diagrammatics.

II.5.3 Are diagrams intrinsically informal?

The more formal we made the visit
the less information we might obtain.
[Arthur Conan Doyle, *The Hound of the Baskervilles* (1902)]

The most serious argument against the use of diagrams as a standard tool in rigorous mathematical work says that by their nature diagrams cannot be made formal enough to be acceptable as valid components of rigorous mathematical proofs. To that argument one can answer shortly that it is already disproved by many fully formalized systems of diagrammatic reasoning that have been developed, see e.g. [Shin 1994, Hammer 1996,

[25] One of many similar examples circulating around. This particular version is adapted, with modifications, from a book on entertaining mathematics [Jeleński 1968].