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Diagrammatic Representation of Interval Space in Proving Theorems about Interval Relations

(Extended Abstract)

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Abstract

The paper introduces a two-dimensional graphical representation for the space of intervals (the *IS*-diagram) and arrangement interval relations (the *W*-diagram). The usefulness of the representations is illustrated with the example of proving equivalence of different characterizations of convex interval relations.

1. Introduction

As was stated by Simon in [1]: "*...solving a problem simply means representing it so as to make the solution transparent.*" In fact, much of the progress in science in general, and mathematics in particular, consisted of finding new representations of various phenomena or formal constructs. Devising a new way of representing knowledge about some phenomenon, formal system, or problem class offers new means of effective description of the domain objects and new possibilities of reasoning about them and solving problems involving them.

Since some time, so-called *diagrammatic representations* (and associated *diagrammatic reasoning* methods) gain considerable interest, as they often provide more effective means—for storing, using and presenting complex information and knowledge—than other representations, see the survey paper [2].

In this paper a two-dimensional, diagrammatic representation of the space of intervals, called an *IS-diagram* (introduced in [3]), is elaborated. It constitutes an extension and refinement of the representation proposed by Rit [4] and used to illustrate the concept of *convex* interval relations by Nökel [5]. Another diagrammatic notation based on it, called a *W-diagram*, is useful in depicting interval relations and operations on them [3]. Usefulness of these diagrams is illustrated with the example of proving equivalence of different characterizations of convex interval relations [5, 9-11].

2. Intervals: Notation and Basic Definitions

The basic interval notation and terminology used in the paper is that of Neumaier [6] and Ratschek and Rokne [7]. An *interval* is an ordered pair $u = [e_1, e_2]$, where e_1 and e_2 (*endpoints* of the interval) are elements of some (at least partially) ordered set E (called the *base set*), such that $e_1 \le e_2$. The interval is called *thick* if $e_1 \ne e_2$; *thin* (or *point*) interval if $e_1 = e_2$. The *beginning* and *end* of the interval u are denoted by lb u and ub u, respectively. Thus, $u = [\text{lb } u, \text{ub } u]$. To simplify notation, we will usually identify an interval with a set of elements lying between its endpoints: $u = \{e \mid e_1 \le e \le e_2\}$.

In the paper we will be concerned mostly with *real intervals*, i.e. intervals defined over the set of real numbers. The set of all such intervals will be denoted by *IR* and called a (*real*) *interval space*. For real intervals, the *midpoint*, *radius* and *width* of the interval are defined, respectively, as follows: $\text{mid } u = (\text{lb } u + \text{ub } u) / 2$, $\text{rad } u = (\text{ub } u - \text{lb } u) / 2$, $\text{wid } u = \text{ub } u - \text{lb } u = 2 \text{ rad } u$.

An interval i is called to lie *between* two intervals u and v, if the endpoints of i lie between the corresponding endpoints of u and v, i.e.: $\min(\text{lb } u, \text{lb } v) \le \text{lb } i \le \max(\text{lb } u, \text{lb } v)$ and $\min(\text{ub } u, \text{ub } v) \le \text{ub } i \le \max(\text{ub } u, \text{ub } v)$. The set of all (proper) intervals lying between u and v, denoted by $\Diamond(u, v)$, is called a *lozenge*. Examples of lozenges are shown in the *IS*-diagram figure below.

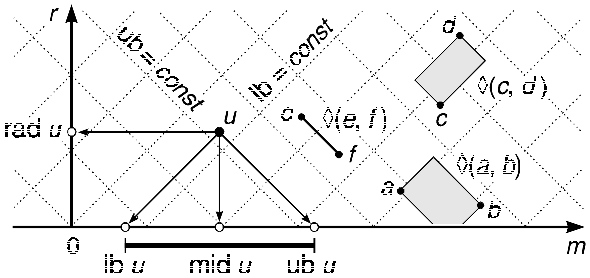
3. Interval Relations

An *arrangement interval relation* is any relation between intervals that can be defined using only the order relation (and equality) defined in their base set between interval endpoints, and logical connectives. Note that some simple relations between intervals, e.g. the "*equal width*" relation, are not arrangement relations. Thirteen of these relations (minimal under the union of relations) constitute the set of *basic* (or *simple*) *interval relations* [8, 9], see the *W*-diagram figure below.

All arrangement relations are unions of basic relations, and the set of arrangement relations is closed under union, intersection and composition of relations. In the sequel, if not otherwise specified, all considered interval relations are assumed to be arrangement relations. They play important role in reasoning about time in AI and Qualitative Physics [4, 5, 8-11].

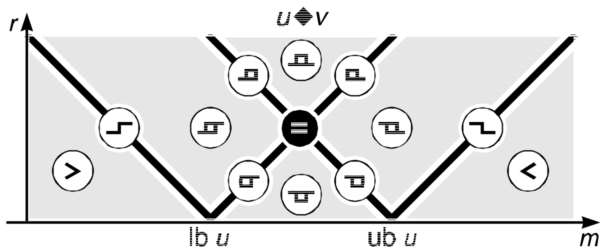
4. Interval Space Diagrams

To uniquely define an interval, two numerical parameters are necessary, hence the space of intervals is two-dimensional. Different pairs of parameters can be used as coordinates of the interval space. The most obvious is the *endpoints space* (e_1, e_2) [4]. It is not very convenient, however, as the set of (proper) intervals occupies the "skewed" half-plane above the diagonal. Other natural choices of coordinates are better in this respect. The *centered* space (mid *u*, rad *u*) has particularly convenient properties for our purposes, and thus constitutes the basis of the *IS-diagram* representation of interval space. In [3], a very similar (mid *u*, wid *u*) space has been considered instead.



As rad $u \geq 0$ always, the space of intervals occupies here the half-plane above the m = mid u axis. Points below this axis may be considered as representations of "inverted" or "improper" intervals (for which $e_1 > e_2$). This interesting possibility will not be further pursued in this paper.

Various interval parameters, relations and transformations have convenient representations on the diagram. In particular, for any given interval *u*, the *images* (or *coimages*) of *u* under basic arrangement relations lead to the *W*-diagram shown below, useful in investigations of the space of arrangement relations and proving theorems about them. The lines and regions of the *IS*-diagram constitute the images of the interval *u* under all 13 basic arrangement relations, and are labelled by graphical symbols proposed in this paper for the appropriate basic relations.



5. Convex Interval Relations

An important subset of all arrangement relations consists of so-called *convex* relations. Their importance comes from at least two facts:

- They constitute a class of relations coinciding more or less exactly with those that are common and natural in everyday reasoning involving intervals (especially time intervals [5, 9-11]).
- Algorithms for solving several important problems involving networks of constraints between intervals are tractable (of polynomial complexity) within the class of convex relations (strictly speaking, this holds also for a wider class of *pointisable* relations), whereas these problems often become *NP*-complete when other relations are also allowed [5, 8-10].

Theorem. For arrangement interval relations, all the definitions (listed below) of *convex* interval relations are equivalent. An interval relation **R** is *convex* iff:

Primary characterization: —for any interval *i* and every pair of intervals *u*, *v* such that *i* **R** *u* and *i* **R** *v*, also *i* **R** *w* for every interval *w* lying between *u* and *v*.

Term characterization: —it can be defined as a conjunction of simple terms involving only relations in the set {≤, ≥, <, >, =} defined in the base set between interval endpoints (i.e., the order relation, its inverse and their negations, plus equality) [10].

W-diagram characterization: —its image (or coimage) in the *W*-diagram includes together with any two intervals also the lozenge defined by these intervals.

Lattice characterization: —it is a union of all relations belonging to some interval (including thin intervals) over a lattice L_{BIR} of basic interval relations [5].

The proof of the above equivalencies constitutes a good example problem to show usefulness of the proposed diagrammatic representation of the interval space.

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