Non-Local Description of Pollution Transport in Random Medium[*]

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Abstract

A convection-dispersion equation governing pollution transport in a saturated porous medium of infinite extent is considered. To account for the significant heterogeneity of real media, the liquid convection velocity is assumed to be a random function. The non-local (differential-integral) equation for the mean concentration field is derived and analyzed. Some special cases are specified and discussed along with the numerical illustration of the effect of non-locality and the correlation of the medium properties on the mean concentration field.

Contents

1 Introduction

In order to understand better the evolution of mass (solute, pollutant, etc.) in complicated waterground formations the attention has been focused on the effects of spatial heterogeneity of the medium in question; more specifically - on the spatial variability in the advection velocity which, in turn, is primarily due to the variability in hydraulic conductivity.

It has been commonly accepted that a natural and promising way of quantifying the effects of this complicated heterogeneity consists in stochastic modeling of basic hydraulic properties of porous media. In general, a real water conducting medium is modelled by a stochastic medium, i.e., a medium whose basic properties are described in terms of probability theory, or, more specifically, in terms of random functions of position (cf. [1]). In the analysis of a solute transport in porous media a random heterogeneity of the medium is most often accounted for by assuming that the convection velocity $\mathbf{u}(\mathbf{r})$ in the convection- diffusion equation is a random field, i.e., in general, $\mathbf{u} = \mathbf{u}(\mathbf{r}, t; \gamma)$, where $\mathbf{r} = (x_1, x_2, x_3)$, t is time and $\gamma \in \Gamma$, where Γ is the sample space (or the space of elementary events with family of random events and with the probability defined on the elements of this family).

The governing equation for the random concentration field $C(\mathbf{r}, t; \gamma)$ has the form:

$$\frac{\partial C(\mathbf{r}, t; \gamma)}{\partial t} - \nabla \cdot [\mathbf{D}\nabla C(\mathbf{r}, t; \gamma)] + \mathbf{u}(\mathbf{r}, t; \gamma) \cdot \nabla C(\mathbf{r}, t; \gamma) = 0, \qquad (1.1)$$

where the pore velocity $\mathbf{u}(\mathbf{r}, t; \gamma)$ is assumed to be a given random field and matrix $\mathbf{D}$ has only nonzero diagonal elements which usually are regarded to be constant. In formulation of Eq.(1.1) the porosity of the medium θ has been assumed to be constant. The probabilistic properties of $\mathbf{u}(\mathbf{r}, t; \gamma)$ follow from the properties of random field of hydraulic conductivity $K(\mathbf{r}, t; \gamma)$ and the local relationship between $\mathbf{u}$ and K specified by the Darcy law. The above model for non-reactive solute transport have been investigated by a number of authors (cf. [2]).

One of the important questions is associated with representation of the ensemble mean concentration $\langle C(\mathbf{r}, t; \gamma) \rangle$ by "effective" equation. The existing efforts have mainly been concentrated, however, with derivation from Eq. (1.1) the "effective" convection-diffusion equation of a classical type (Fickian approximation), i.e.,

$$\frac{\partial \langle C(\mathbf{r}, t; \gamma) \rangle}{\partial t} - \nabla \cdot [\mathbf{D}_{eff}\nabla \langle C(\mathbf{r}, t; \gamma) \rangle] + \mathbf{u}_{eff} \bullet \nabla \langle C(\mathbf{r}, t; \gamma) \rangle = 0, \qquad (1.2)$$

where $\mathbf{D}_{eff}$ and $\mathbf{u}_{eff}$ are non-random constant quantities. Although a number of serious attempts have been made to derive equation (1.2) from (1.1) consistently; there are, however serious difficulties in providing the evidence that such Fickian (or Gaussian) representation of $\langle C(\mathbf{r}, t; \gamma) \rangle$ can really be accepted

(cf. [3]). It is also not easy to specify a possible class of variable (deterministic or random) coefficients in (1.1) for which equation (1.2), usually interpreted as describing large-scale concentration, could be derived.

However, there are no sufficient reasons, why one should look for the equation for $\langle C(\mathbf{r}, t; \gamma) \rangle$ just in the form of (1.2). In general, the equation for $\langle C(\mathbf{r}, t; \gamma) \rangle$ should not be a priori restricted as its form is concerned. It can have, to some extent, much more general form, e.g. differential, integral, differential-integral etc.

In this paper we wish to show that under some quite general assumptions the mean concentration field $\langle C(\mathbf{r}, t; \gamma) \rangle$ in stochastic porous medium is governed by a non-local equation (differential- integral one). Only in some special cases it can be "localized" to the differential equation. These special cases will be specified and discussed along with the numerical illustration of the effects of non- locality and the correlation of the medium properties on the mean concentration field.

2 General formulation

Let us write down equation (1.1) in the following (symbolic) form:

$$\mathbf{L}(\gamma)C = g \tag{2.1}$$

where $\mathbf{L} = \mathbf{L}(\gamma)$ is a linear differential operator with random coefficients and $\mathbf{g}$ is non-random element. We are looking for an equation for the mean $\langle C \rangle$ of an unknown random field, i.e., an equation of the form:

$$\mathcal{L} \langle C \rangle = g \tag{2.2}$$

where the operator $\mathcal{L}$ is deterministic and may be called an effective operator for a given operator $\mathbf{L}(\gamma)$; certainly, in general $\mathcal{L} \neq \langle \mathbf{L}(\gamma) \rangle$ (by $\langle \mathbf{L}(\gamma) \rangle$ we mean the modified operator $\mathbf{L}$, where instead of random coefficients we substituted their mean values).

Let us assume that the operator $\mathbf{L}(\gamma)$ is invertible (for almost all $\gamma \in \Gamma$) then from (2.1) we have:

$$C = \mathbf{L}^{-1}(\gamma)g \tag{2.3}$$

Averaging gives

$$\langle C \rangle = < \mathbf{L}^{-1}(\gamma) > g \tag{2.4}$$

or

$$< L^{-1}(\gamma) >^{-1} <C> = g. \tag{2.5}$$

The above formal derivation shows that

$$\mathcal{L} = \ < \mathbf{L}^{-1}(\gamma) >^{-1} \tag{2.6}$$

Unfortunately, it is not easy to evaluate the inverse $\mathbf{L}^{-1}(\gamma)$ of random operator $\mathbf{L}(\gamma)$. To make the equation for the mean field efficient some restrictions concerning the random operator $\mathbf{L}(\gamma)$ have to be introduced. Let us assume that the coefficients in operator $\mathbf{L}(\gamma)$ have a small fluctuating part (characterized by a small parameter ϵ) so, $\mathbf{L} = \mathbf{L}(\gamma, \epsilon)$.

The equation (2.1) can be written as:

$$\left\{\mathbf{L}_0 + \epsilon \left\langle \mathbf{L}_1(\gamma) \right\rangle + \epsilon^2 \left\langle \mathbf{L}_2(\gamma) \right\rangle + ...\right\} C = g \tag{2.7}$$

For $\epsilon = 0$ the operator $\mathbf{L}(\gamma)$ reduces to a deterministic (mean) operator $\mathbf{L}_0$. The terms $\mathbf{L}_1(\gamma), \mathbf{L}_2(\gamma), \ldots$ represent stochastic perturbations of $\mathbf{L}_0$. It can be shown (cf. [1], [4]) that the equation for the mean $\langle C \rangle$ has the form:

$$\left\{\mathbf{L}_0 + \epsilon \left\langle \mathbf{L}_1(\gamma) \right\rangle + \epsilon^2 \left[\left\langle \mathbf{L}_1(\gamma) \right\rangle \mathbf{L}_0^{-1} \left\langle \mathbf{L}_1(\gamma) \right\rangle \ - \right. \right. \tag{2.8}$$
$$\left. \left. - < \mathbf{L}_1(\gamma) \mathbf{L}_0^{-1} \mathbf{L}_1(\gamma) > + \left\langle \mathbf{L}_1(\gamma) \right\rangle \right] \right\} \langle C \rangle = g + O(\epsilon^3)$$

Equation (2.8) is the desired deterministic equation satisfied by the mean field (up to the second order terms).

Let us assume that $\mathbf{L}(\gamma)$ is the sum of non-random operator $\mathbf{L}_0$ and a small operator $\mathbf{L}_1(\gamma)$ with $\langle \mathbf{L}_1(\gamma) \rangle = 0$, i.e.,

$$\mathbf{L}(\gamma) = \mathbf{L}_0 + \epsilon \mathbf{L}_1(\gamma).$$

We obtain from Eq. (2.8) the following equation for $\langle C \rangle$:

$$\left\{\mathbf{L}_0 - \epsilon^2 < \mathbf{L}_1(\gamma) \mathbf{L}_0^{-1} \mathbf{L}_1(\gamma) > \right\} \langle C \rangle = g \tag{2.9}$$

The above equation corresponds to the Bourret approximation in analysis of stochastic wave propagation problems (cf. [1]). In what follows, we shall make use of equation (2.9) to the case of the solute transport equation (1.1). Let us assume that in equation (1.1) matrix $\mathbf{D} = \mathbf{D}_0$ is constant and that only $\mathbf{u}(\mathbf{r}, t; \gamma)$ is random, and that

$$\mathbf{u}(\mathbf{r}, t; \gamma) = \mathbf{u}_0 + \epsilon \mathbf{u}_1(\mathbf{r}, t; \gamma). \tag{2.10}$$

In this case

$$\mathbf{L}_0 = \frac{\partial}{\partial t} - \mathbf{D}_0 \nabla^2 + \mathbf{u}_0 \bullet \nabla, \tag{2.11}$$
$$\mathbf{L}_1(\gamma) = \mathbf{u}_1(\mathbf{r}, t; \gamma) \bullet \nabla,$$

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$$\mathbf{L}^{-1}f = \int \int \mathbf{G}_0\left(\mathbf{r}, \mathbf{r}'; t, t'\right)\ f\left(\mathbf{r}'\right)\ d\mathbf{r}'dt' \tag{2.12}$$

where $\mathbf{G}_0(\mathbf{r},\mathbf{r}'; t, t')$ is the Green function of the operator $\mathbf{L}_0$. Thus, equation (2.9) takes the form (the dot denotes the scalar product of vectors in $\mathbb{R}^3$):

$$\left\{ \frac{\partial}{\partial t} - \mathbf{D}_0 \nabla_r^2 + \mathbf{u}_0 \cdot \nabla_r \right\} \langle C(\mathbf{r},t; \gamma) \rangle - \tag{2.13}$$

$$-\epsilon^2 \int \int \langle \mathbf{u}_1(\mathbf{r},t; \gamma) \cdot \nabla_r \mathbf{G}_0(\mathbf{r} - \mathbf{r}'; t - s)\mathbf{u}_1(\mathbf{r}',s; \gamma) \cdot \nabla_{r'} \rangle$$

$$\langle C(\mathbf{r}',s; \gamma) \rangle\ d\mathbf{r}'ds = 0$$

If the random field $\mathbf{u}(\mathbf{r}, t; \gamma)$ is spatially homogeneous and temporally stationary then

$$< \mathbf{u}_1(\mathbf{r},t; \gamma)\mathbf{u}_1^T(\mathbf{r}',s; \gamma) > = \mathbf{K}(\mathbf{r} - \mathbf{r}',t - s) \tag{2.14}$$

and equation (2.13) yields

$$\left\{ \frac{\partial}{\partial t} - \mathbf{D}_0 \nabla_r^2 + \mathbf{u}_0 \cdot \nabla_r \right\} \langle C(\mathbf{r},t; \gamma) \rangle - \tag{2.15}$$

$$-\epsilon^2 \iint \sum_{i,j=1}^{3} \mathbf{K}_{ij}(\mathbf{r} - \mathbf{r}',t - s)\frac{\partial}{\partial x_i}G_0(\mathbf{r} - \mathbf{r}'; t - s)\times$$

$$\times \frac{\partial}{\partial x_j} \langle C(\mathbf{r}',s; \gamma) \rangle\ d\mathbf{r}'ds = 0.$$

The above differential-integral equation constitutes a non-local, first order approximation of the mean concentration field in randomly heterogeneous medium. In such a description the mean concentration at point $\mathbf{r}$ and time t is affected by its values at other distant points $\mathbf{r}'$ and preceding time instants s. The scale of random heterogeneity is quantified by the correlation radius (in time and space) of random field $\mathbf{u}(\mathbf{r}, t; \gamma)$.

If the random velocity field depends on spatial variable only (what is the most common case in the underground pollution transport problems), that is $\mathbf{u}(\mathbf{r}, t; \gamma) \equiv \mathbf{u}(\mathbf{r}; \gamma)$, then the correlation function in equation (2.15) is $\mathbf{K}(\mathbf{r} - \mathbf{r}')$.

3 Analysis in the one-dimensional case

To make the analysis and results more transparent we restrict ourselves here to one-dimensional transport problem. In this case equation (2.15) takes the form:

$$\left\{ \frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial x^2} + u_0 \frac{\partial}{\partial x} \right\} \langle C(x,t;\gamma) \rangle = \tag{3.1}$$

$$= \epsilon^2 \int\limits_0^t \int\limits_{-\infty}^{\infty} K(x-x',t-s) \frac{\partial G(x-x',t-s)}{\partial x} \frac{\partial \langle C(x',s;\gamma) \rangle}{\partial x'} dx\prime ds,$$

where $G(z,\tau)$, $z = x - x'$, $\tau = t - s$ is the Green function associated with the operator

$$L_0 = \frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial x^2} + u_0 \frac{\partial}{\partial x},$$

that is

$$G(z,\tau) = \frac{1}{2\sqrt{D_0 \pi \tau}} \exp\left\{ \frac{-(z-u_0\tau)^2}{4D_0\tau} \right\}. \tag{3.2}$$

The derivative of $G(z,\tau)$ occurring in equation (3.1) is as follows

$$\frac{\partial G(z,\tau)}{\partial x} = \frac{1}{4D_0\sqrt{D_0\pi\tau}} \left\{ u_0 - \frac{z}{\tau} \right\} \exp\left\{ \frac{-(z-u_0)^2}{4D_0\tau} \right\}. \tag{3.3}$$

Equation (3.1), which is differential-integral with respect to the two variables, can be transformed to the following differential-integral equation:

$$\langle C(x,t;\gamma) \rangle = \int\limits_{-\infty}^{\infty} G(x-x',t)C_0(x')\ dx\prime\ + \epsilon^2 \int\limits_0^t \int\limits_{-\infty}^{\infty} G(x-x',t-s)\ dx\prime\ ds \times$$

$$\tag{3.4}$$

$$\times \int\limits_0^s \int\limits_{-\infty}^{\infty} K(x'-x'',s-p)\frac{\partial G(x'-x'',s-p)}{\partial x'} \frac{\partial \langle C(x'',p) \rangle}{\partial x''} dx'' dp.$$

Equation (3.4) can be useful for determining the concentration for relatively small t. For large t we must apply some other technique to calculate $\langle C(x,t;\gamma) \rangle$. One possibility is to apply the Fourier transform.

Let us remark that equation (3.1) in its integral part is of convolution type. Therefore, making its Fourier transformation with respect to x, we obtain relatively simple equation for $\hat{C}(k,t)$, where

$$\hat{C}(k,t) := \int\limits_{-\infty}^{\infty} \langle C(x,t;\gamma) \rangle\ e^{ikx}dx =: F[C], \tag{3.5}$$

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where i is an imaginary unit number. Taking into account the relationships:

$$F\left[\frac{\partial^n C}{\partial x^n}\right] = (-ik)^n F[C], \tag{3.6}$$
$$F[f * g] = F[f]\ F[g],$$

equation (3.1) yields after transformation

$$\frac{d\hat{C}(k,t)}{dt} + (Dk^2 - iku_0)\ \hat{C}(k,t) = -\ i\epsilon^2 k \int_0^t H(k,t-s)\ \hat{C}(k,t)\ ds \tag{3.7}$$

with the initial condition:

$$\hat{C}(k,0) = \hat{C}_0(k) = \int_{-\infty}^{\infty} C_0(x)\ e^{ikx} dx \tag{3.8}$$

and $H(k,t-s)$ defined as:

$$H(k,t-s) = \int_{-\infty}^{\infty} e^{ikx} K(x,t-s)\frac{\partial G(x,t-s)}{\partial x} dx \tag{3.9}$$

Let us define:

$$S(k,t) = e^{-\left(D_o k^2 - iku_0\right)t}. \tag{3.10}$$

Then the differential-integral equation can be transformed to the following integral one:

$$\hat{C}(k,t) = S(k,t)\ \hat{C}_0(k) - \tag{3.11}$$
$$-\epsilon^2 ik \int_0^t S(k,t-s) \int_0^s H(k,s-p)\ \hat{C}(k,p)\ dp\ ds$$

To obtain the density of concentration for a fixed instant of time t, we must calculate $\hat{C}(k,t)$ from equation (3.11) and then determine its inverse Fourier transform according to the known formula:

$$\langle C(x,t;\gamma)\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{C}(k,t) e^{-ikx} dk. \tag{3.12}$$

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3.1 White-noise temporal variability of the pore velocity

In the idealized case, when temporal random variations are assumed to be characterized by a white-noise, we can derive from the general non-local equation (3.1) a common form of differential dispersion equation with some effective dispersion. To show this let us assume that:

$$K(z,\tau) = \rho(z)\,\delta(\tau) \tag{3.13}$$

where $\delta(\tau)$ is a Dirac delta function. Then the integral part of the equation (3.1) is:

$$\int\limits_0^t \int\limits_{-\infty}^{\infty} K(x-x',t-s)\frac{\partial G(x-x',t-s)}{\partial x}\frac{\partial \langle C(x',s;\gamma)\rangle}{\partial x'}\,dx\prime\,ds = \tag{3.14}$$

$$= \int\limits_0^t \delta(t-s)\int\limits_{-\infty}^{\infty} \rho(x-x')\frac{\partial G(x-x',t-s)}{\partial x}\frac{\partial \langle C(x',s;\gamma)\rangle}{\partial x'}\,dx\prime\,ds =$$

$$= \frac{1}{2}\int\limits_{-\infty}^{\infty} \rho(x-x')\lim_{\tau\to\infty}\frac{\partial G(x-x',\tau,)}{\partial x}\frac{\partial \langle C(x',t;\gamma)\rangle}{\partial x'}\,dx\prime =$$

$$= -\frac{1}{2}\int\limits_{-\infty}^{\infty} \rho(x-x')\delta'_x(x-x')\frac{\partial \langle C(x',t;\gamma)\rangle}{\partial x'}\,dx\prime =$$

$$= \frac{1}{2}\rho(0)\frac{\partial^2 \langle C(x,t;\gamma)\rangle}{\partial x^2} + \frac{1}{2}\rho'(0)\frac{\partial \langle C(x,t;\gamma)\rangle}{\partial x} =$$

$$= \frac{1}{2}\rho(0)\frac{\partial^2 \langle C(x,t;\gamma)\rangle}{\partial x^2},$$

where we used the relation:

$$\lim_{\tau\to 0}\frac{\partial G(x-x',\tau)}{\partial x} = \delta'_x(x-x'), \tag{3.15}$$

which holds since $G(z,\tau)$ is the Green function and for τ tending to zero it approaches $\delta(x-x')$. δ'_x is the derivative of Dirac delta function with respect to its variable x (see [5]) and

$$\rho'(0) = 0, \tag{3.16}$$

due to the symmetry,

$$\rho(x) = \rho(-x).$$

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Therefore, equation (3.1) takes the form:

$$\left\{ \frac{\partial}{\partial t} - \left(D_0 + \frac{1}{2}\rho(0) \right) \frac{\partial^2}{\partial x^2} + u_0 \frac{\partial}{\partial x} \right\} \langle C(x,t;\gamma) \rangle = 0. \qquad (3.17)$$

This result coincides with that obtained in paper [6] for the turbulent diffusion in incompressible fluid. The above equation represents a specific form of the "effective" convection-diffusion, Fickian approximation (1.2)

3.2 Spatial variability of a medium.

Let us assume now that the velocity field u is independent of time t, i.e., $u(x,t;\gamma) \equiv u(x;\gamma)$ and

$$u = u_0 + u_1(x;\gamma), \qquad (3.18)$$

$$\langle u_1(x;\gamma) \rangle = 0, \qquad (3.19)$$
$$< u_1(x;\gamma)\, u_1(x';\gamma) > \quad = K(x - x').$$

Then the equation (3.1) takes the following simplified form:

$$\left\{ \frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial x^2} + u_0 \frac{\partial}{\partial x} \right\} \langle C(x,t;\gamma) \rangle = \qquad (3.20)$$
$$= \epsilon^2 \int_0^t \int_{-\infty}^{\infty} K(x - x') \frac{\partial G(x - x', t - s)}{\partial x} \frac{\partial \langle C(x', s;\gamma) \rangle}{\partial x'} dx'\, ds.$$

An interesting question which arises is concerned with a possible localization of the non-local equation (3.20). A spatial localization in such a case means that an approximate equation arising from the original differential-integral one takes the form in which $\langle C(x,t;\gamma) \rangle$ as a function of x is subjected to differentiation only. The conditions which make such a simplification of the problem (3.20) valid are quite involved. It is required that the spatial radius of correlation is relatively small comparing to other linear dimensions of the system in question. In other words, the numerical values of the parameters in equation (3.20) must be such, that we can replace the integral kernel by a function being the Dirac delta in space. Taking into account the properties of the Green function $G(z, \tau)$ and correlation function $K(z)$ and assuming symmetry in x of the concentration function $\langle C(x,t;\gamma) \rangle$, we deduce that the integral term in equation (3.20) should be approximated of the following way ($f(x)$ is some test function):

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$$\int\limits_{-\infty}^{\infty} K(x - x')\frac{\partial G(x - x', t - s)}{\partial x}f(x')\ dx\prime \cong \qquad (3.21)$$

$$\int\limits_{-\infty}^{\infty} F_{eff}(t - s)\delta(x - x')\frac{\partial}{\partial x'}f(x')\ dx\prime,$$

where, analogously to the results of paper [7]:

$$F_{eff}(t - s) = \int\limits_{-\infty}^{\infty} K(x)\ G(x, t - s)dx. \qquad (3.22)$$

In such a case the approximate localized equation takes the following form:

$$\frac{\partial \langle C(x, t; \gamma)\rangle}{\partial t} - D_0\frac{\partial^2 \langle C(x, t; \gamma)\rangle}{\partial x^2} + u_0\frac{\partial \langle C(x, t; \gamma)\rangle}{\partial x} = \qquad (3.23)$$

$$= \epsilon^2 \int\limits_0^t F_{eff}(t - s)\ \frac{\partial^2 \langle C(x, s; \gamma)\rangle}{\partial x^2}ds.$$

4 Numerical illustration

Consider the particular problem of the propagation of the unit pollution mass which at the initial instant of time $t = 0$ occupies some interval $(-a, a)$, that is

$$\begin{array}{ll} C(x, 0) = 1 & \text{for} \quad x \in (-a, a), \\ C(x, 0) = 0 & \text{otherwise,} \end{array} \qquad (4.1)$$

and the correlation function K has the following exponential Gaussian form:

$$K(x - x') = \sigma^2 e^{-\alpha(x - x')^2}. \qquad (4.2)$$

In this case equation for the mean concentration (3.4) takes the following form:

$$\langle C(x, t; \gamma)\rangle = \Phi\left(\frac{a - x + u_0 t}{\sqrt{2D_0 t}}\right) - \Phi\left(\frac{-a - x + u_0 t}{\sqrt{2D_0 t}}\right) + \qquad (4.3)$$

$$+\epsilon^2 \int\limits_0^t \int\limits_{-\infty}^{\infty} G(x - x', t - s)\ dx\prime\ ds \times$$

$$\times \int\limits_0^s \int\limits_{-\infty}^{\infty} K(x' - x")\frac{\partial G(x' - x", s - r)}{\partial x'}\frac{\partial \langle C(x", r; \gamma)\rangle}{\partial x"}dx"\,dr,$$

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where Φ is the Gaussian probability distribution function.

Also equation (3.11) for the Fourier transform of the density of concentration can be simplified. We can calculate Fourier transform (3.9) of the integral kernel and the Fourier transform (3.8) of the initial condition of the transport equation. The calculated transformations in this particular case are:

$$H(k, t - s) = \int_{-\infty}^{\infty} e^{ikx} K(x) \, G(x, t - s) \, dx =$$

$$= \frac{\sigma^2 (2\alpha u_0 t - ik)}{(1 + 4D_0 \alpha t)^{3/2}} \exp\left\{ \frac{(ik - D_0 k^2)t - \alpha u_0^2 t^2}{1 + 4D_0 \alpha t} \right\}$$

(4.4)

and

$$\hat{C}_0(k) = \int_{-\infty}^{\infty} C_0(x) e^{ikx} dx = \frac{2 \sin ka}{k},$$

(4.5)

and the differential-integral equation (3.11) for the spatial Fourier transform of the averaged concentration density has the following form:

$$\hat{C}(k, t) = \frac{2 \sin ka}{k} \, e^{-\left(D_0 k^2 - iku_0\right)t} -$$

$$-\epsilon^2 \int_0^t e^{-\left(D_0 k^2 - iku_0\right)(t-p)} \int_0^p W(k, p - s) \, \hat{C}(s, k) \, ds \, dp,$$

(4.6)

where

$$W(k, t) = \frac{\sigma^2 (2i\alpha u_0 t k + k^2)}{(1 + 4D_0 \alpha t)^{3/2}} \exp\left\{ \frac{(ik - D_0 k^2)t - \alpha u_0^2 t^2}{1 + 4D_0 \alpha t} \right\}.$$

(4.7)

The above obtained equations (4.3) and (4.6) can be used for numerical discussion of the evolution of the concentration density in a random medium. However, before we do this, let us study the non-random diffusion-convection equation, that is such where both the diffusion coefficient D_0 and velocity u_0 are constant. It is known that these two parameters quantify two qualitatively different aspects of the transport process. Namely, D_0 describes the diffusion due to atomic movement of the particles, guaranteeing mixing phases and tending the process to equilibrium whereas u_0 is responsible for the systematic movement of the phases without mixing effects and graphically it is visualized as the systematic movement of the maximal value of concentration (see Fig.1). In realistic systems (in our case - in real ground water formations) dispersion due to heterogeneity of the medium provides a significant contribution to the process (similarly as turbulent diffusion (see [8])).

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In this paper we take into account the effect of the randomness of the velocity field u. It is seen that the random fluctuations of the velocity field, in average, increase the effect analogous to molecular diffusion and leave the convection diffusivity without change (see Fig. 2, 3). The intensity of the changes strongly depends on the length of the spatial radius of correlation (for the considered correlation function proportional to $1/\alpha$) and it increases along with growth of the radius (see Fig. 2, 3). Let us remark that for long time even in the system with a small radius of correlation, the randomness of the velocity field has the significant effect on the diffusivity.

As we observed, for fixed time t the diffusion due to randomness of the velocity field u has the character of the molecular diffusion with some modified diffusion coefficient D. However, the numerical calculations show that this constant changes in time (see Fig. 4, 5). This fact coincides with the effect of the localization performed in Section 3 of this paper, where equation (3.23) shows that effective medium diffusion depends on time.

In the considered example, for the correlation function of the form (4.2), the required coefficient (3.22) is:

$$F_{eff}(t-s) = \int\limits_{-\infty}^{\infty} K(z)G(z, t-s)\, dz = \tag{4.8}$$

$$= \frac{\sigma^2}{\sqrt{1 + 4D_0\alpha(t-s)}} \exp\left\{\frac{-\alpha u_0^2 (t-s)^2}{1 + 4D_0\alpha(t-s)}\right\}$$

and the localized equation (3.23) for the concentration takes the following form:

$$\frac{\partial \langle C(x,t;\gamma)\rangle}{\partial t} - D_0 \frac{\partial^2 \langle C(x,t;\gamma)\rangle}{\partial x^2} + u_0 \frac{\partial \langle C(x,t;\gamma)\rangle}{\partial x} = \tag{4.9}$$

$$= \epsilon^2 \int\limits_0^t \frac{\sigma^2}{\sqrt{1 + 4D_0\alpha(t-s)}} \exp\left\{\frac{-\alpha u_0^2 (t-s)^2}{1 + 4D_0\alpha(t-s)}\right\} \frac{\partial^2 \langle C(x,s;\gamma)\rangle}{\partial x^2} ds.$$

Figures 6, 7 show the concentration density for the considered example, calculated with the use of the approximate equation (4.9). It is seen, that even for relatively long radius of correlation $1/\alpha$ this local approximation is quite satisfactory and differs from the solution of the non-local equation very slightly..
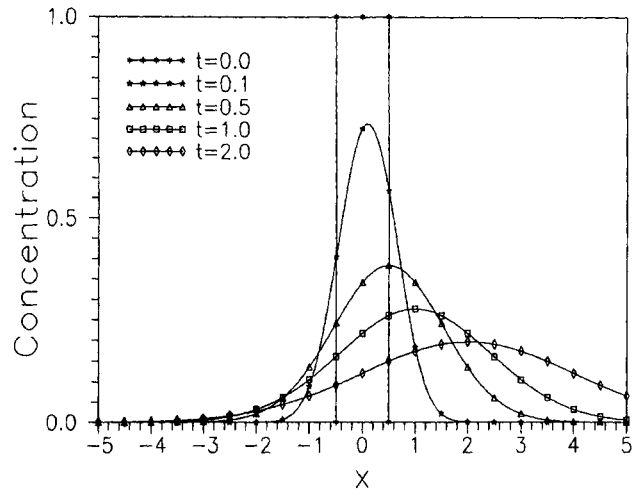
5 Figures

Fig. 1. Evolution in time of the concentration density
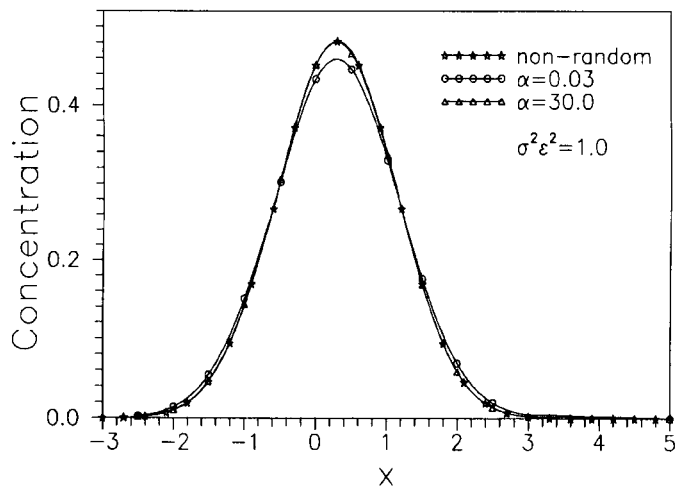for non-random model ($D_0 = 1.0$, $u_0 = 1.0$)

Fig. 2. Distribution of the concentration density at $t = 0.3$ for
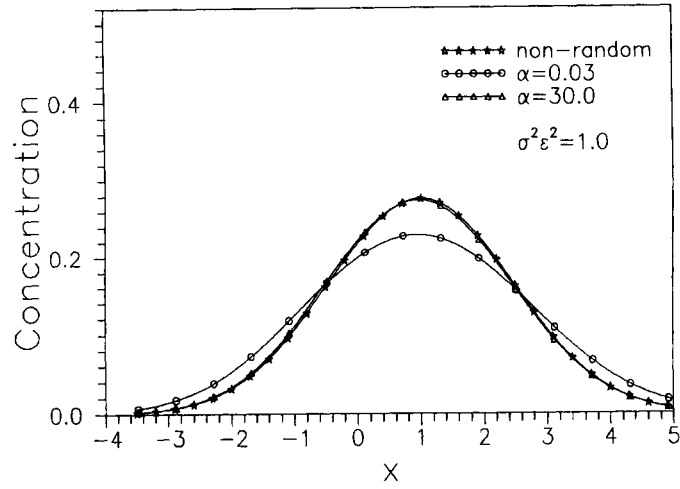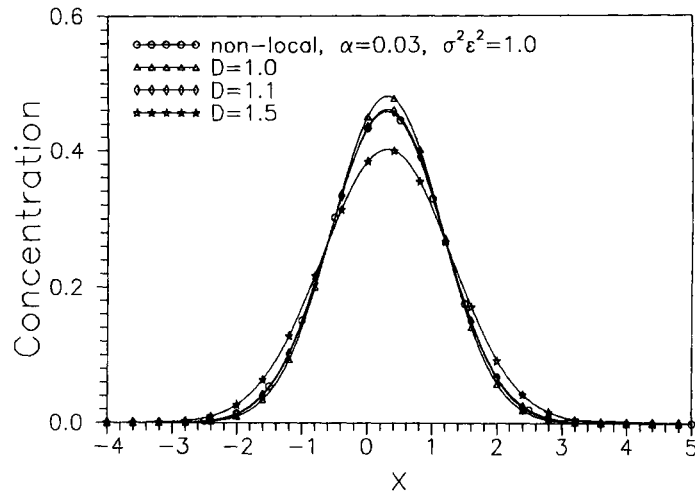the non-local and non-random model ($D_0 = 1.0$, $u_0 = 1.0$)

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Fig. 3. Distribution of the concentration density at $t = 1.0$ for the non-local and non-random model ($D_0 = 1.0$, $u_0 = 1.0$)

Fig. 4. Distribution of the concentration density at $t = 0.3$ for the non-local and non-random model with changing D_0 ($u_0 = 1.0$)
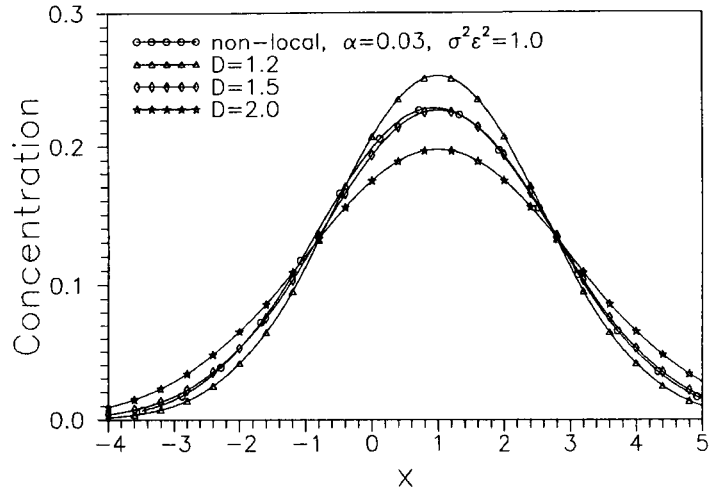
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Fig. 5. Distribution of the concentration density at $t = 1.0$ for the non-local
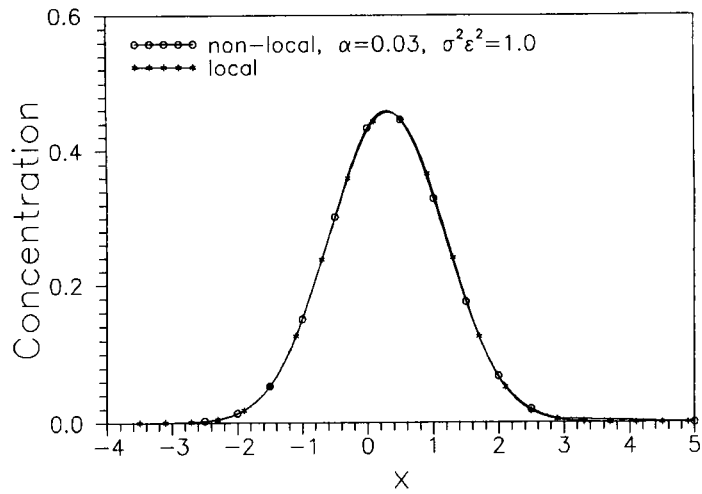and non-random model with changing D_0 ($u_0 = 1.0$)

Fig. 6. Comparison of the concentration density obtained
with the use of the exact and the approximate method
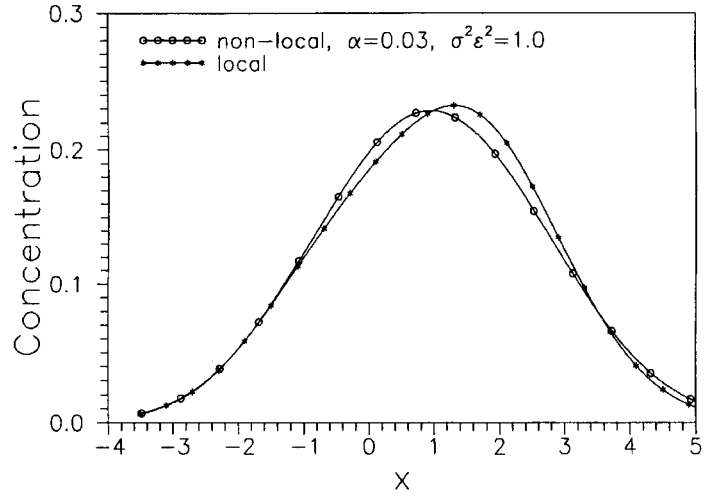for $t = 0.3$ ($D_0 = 1.0$, $u_0 = 1.0$)

Fig. 7. Comparison of the concentration density obtained
with the use of the exact and the approximate method
for $t = 1.0$ ($D_0 = 1.0$, $u_0 = 1.0$)

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