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Generalized Shapiro–Loginov formula and moment stability of a string equation with random telegraphic parameter

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IN THIS PAPER the results concerning the moments of stochastic linear differential equations with the multiplicative parameter in the form of a stochastic telegraph process (Shapiro–Loginov formula) are generalized to the case of Hilbert-space-valued evolution equations. The obtained results are then applied to the investigation of the moment stability of some string equation with stochastic parametric excitation. The results obtained for exact and modal approaches are compared showing the possibility of the simplified analysis as well as differences. Additionally, the system with the appropriate white-noise excitation is considered, and, with the aid of an "equivalent" white-noise process the conditions of an approximation of the telegraphic stochastic process are studied.

W pracy uogólniono wyniki dotyczące momentów rozwiązania liniowego stochastycznego równania różniczkowego z multiplikatywnym parametrem w postaci stochastycznego procesu telegraficznego (wzór Szapiro–Loginowa), na przypadek równania ewolucyjnego o wartościach w przestrzeni Hilberta. Uzyskane rezultaty zastosowano następnie do badania momentowej stabilności równania struny ze szczególnym stochastycznym wymuszeniem parametrycznym. Porównując wyniki otrzymane przy zastosowaniu metody nieskończenie-wymiarowej i w przybliżeniu modalnym, pokazano możliwość uproszczonej analizy zagadnienia i różnice w wynikach. Ponadto rozważono układ z odpowiednim („równoważnym") wymuszeniem białoszumowym i zbadano możliwość aproksymacji procesu telegraficznego białym szumem.

В работе обобщены результаты, касающиеся моментов решения линейного стохастического дифференциального уравнения с мультипликативным параметром в виде стохастического телеграфного процесса (формула Шапиро–Логинова), на случай эволюционного уравнения со значениями в гильбертовом пространстве. Полученные результаты применены затем для исследования моментной стабильности уравнения струны с особенным стохастическим параметрическим вынуждением. Сравнивая результаты, полученные при применении бесконечноразмерного метода и в модальном приближении, показана возможность упрощенного анализа проблемы и указаны разницы в результатах. Кроме этого рассмотрена система с соответствующим („эквивалентным") вынуждением типа белого шума и исследована возможность аппроксимации телеграфного процесса белым шумом.

1. Introduction

AMONG SEVERAL definitions of stochastic stability (cf. [5, 13]), the concept of moment stability is very intuitive. We say that a solution of a stochastic equation is stable in the sense of k-th mean if its k-th moment is stable. There are some cases where the moment stability of an equation is relatively easy for investigation. We have to do with such a situation when exact equations for the moments of the solution can be obtained. In the class of linear stochastic differential equations with a multiplicative stochastic coefficient (parametric excitation), the example could be the stochastic Langevin equations (see [14]). For this kind of equations the moment equations have been derived in the literature. In papers [1] the ordinary differential equations with white-noise coefficients have been treated. CHOW in papers [3, 4] has dealt with a partial differential equations of the parabolic type with a function-valued white-noise. Finally, in papers [9, 10] the moment equations for general evolution equations with a Hilbert-space valued white-noise have been derived. Such equations include, as particular cases, both ordinary and partial differential equations.

Another example of the stochastic linear differential equation, for which the exact moment equations can be written in closed form, is the one where the multiplicative parameter has the form of a stochastic telegraph process (cf. [7]). For this example the moment equations have been derived in [12] and [11] for the ordinary and the partial differential equations, respectively.

In this paper we generalize the results concerning the moments of the stochastic equations with the multiplicative telegraphic parameter to the case of Hilbert-space-valued evolution equations. Then we consider the example of a string equation and investigate the mean and mean-square stability of its solution using both exact partial differential equations for the moments of its modes. The obtained results are then compared. Finally, we introduce the moment equations for the string with the "equivalent" white-noise process instead of the telegraph one in order to find corresponding stability conditions. (The literature concerning the stability of stochastic equations with discrete parametric excitations is cited in Ref. [11]).

2. Generalized Shapiro–Loginov formula

Consider the stochastic evolution equation of the form

$$\frac{dU(t,\gamma)}{dt} = \mathscr{A}U(t,\gamma) + P(t,\gamma)\mathscr{B}U(t,\gamma), \qquad t\in(0,T), \qquad \gamma\in\Gamma, \tag{2.1}$$
$$U(0,\gamma) = U_0\in\mathbf{X},$$

where $(\Gamma,\mathscr{F},\mathscr{P})$ is a complete probabilistic space, Γ is the set of elementary events, $\mathscr{F}$ is the σ-algebra of its measurable subsets and $\mathscr{P}$ is the probabilistic measure, $U(t)$ is an $\mathbf{X}$-valued stochastic process, $\mathbf{X}$ is a separable, real Hilbert space, $\mathscr{A}$ and $\mathscr{B}$ are linear, possibly unbounded operators (generators of strongly continuous semigroups of linear operators) acting from $\mathscr{D}(\mathscr{A})(\mathscr{D}(\mathscr{B}))\subset\mathbf{X}$ into $\mathbf{X}$, and $P(t,\gamma)$ is a stochastic telegraph process defined as (cf. [7]):

(2.2) $$P(t,\gamma) = a(-1)^{N(t,\gamma)}, \qquad P(0,\gamma) = a, \qquad P^2(t,\gamma) = a^2,$$

where a is a constant and $N(t,\gamma)$ is a homogeneous Poisson process with intensity ν. This means that the process $P(t,\gamma)$ takes the values a or $-a$, jumping between the states at random instants of time constituting a Poissonian point process.

The mean value and the covariance of $P(t,\gamma)$ are

(2.3) $$\langle P(t,\gamma)\rangle = a\, e^{-2\nu t}, \qquad \langle P(t_1,\gamma)P(t_2,\gamma)\rangle = a^2 e^{-2\nu|t_1-t_2|},$$

and the higher order moments satisfy the following recurrent relations:

(2.4) $$m(t_1,\dots,t_n) = \langle P(t_1,\gamma)\dots P(t_n,\gamma)\rangle$$
$$= \langle P(t_1,\gamma)P(t_2,\gamma)\rangle\, m_{n-2}(t_3,\dots,t_n), \qquad t_1 \ge t_2 \ge \dots \ge t_n > 0.$$

($\langle\cdot\rangle$ denotes the mathematical expectation of a random variable).

Let $R_t[P(\tau)]$ be some H-valued functional depending on the values of $P(t)$ for $\tau < t$. (H is a real separable Hilbert space). The functional $R_t[P]$ can be represented in the form of the following functional Taylor series:

(2.5) $$R_t[P] = R_t[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t dt_1 \dots \int_0^t dt_n\; K_t^{(n)}(t_1,\dots,t_n)\; P(t_1)\dots P(t_n),$$

where $K_t^{(n)}(t_1,\dots,t_n)$ are H-valued deterministic functions of n arguments $t_1,\dots,t_n$ from the interval $(0,t)$ and one real positive parameter t, defined as

(2.6) $$K_t^{(n)}(t_1,\dots,t_n) = \left.\frac{\delta^n R_t[P]}{\delta P(t_1)\dots\delta P(t_n)}\right|_{P=0},$$

where $\delta/[\delta P(t)]$ is the Volterra variational derivative.

Since $K_t^{(n)}$ is a symmetric function of all its arguments, the formula (2.5) can be written as

(2.7) $$R_t[P] = R_t[0] + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n\; K_t^{(n)}(t_1,\dots,t_n)\; P(t_1)\dots P(t_n).$$

Consider the product of $P(t)$ and $R_t[P]$. We have

(2.8) $$\langle P(t)R_t[P]\rangle = \langle P(t)R_t[0]\rangle$$
$$+ \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n\; K_t^{(n)}(t_1,\dots,t_n)\; \langle P(t)P(t_1)\dots P(t_n)\rangle.$$

Differentiating Eq. (2.7) with respect to t, multiplying by $P(t)$ and taking expectation, we have

$$(2.9) \qquad <P(t) \frac{d}{dt} R_t[P]> = <P(t) \frac{d}{dt} R[0]>$$
$$+ \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n \frac{d}{dt} K_t^{(n)}(t_1,...,t_n) <P(t)P(t_1)...P(t_n)>$$
$$+ \sum_{n=2}^{\infty} \int_0^t dt_2 \int_0^{t_2} dt_3 ... \int_0^{t_{n-1}} dt_n K_t^{(n)}(t,t_2,...,t_n) <P^2(t)P(t_2)...P(t_n)>.$$

Differentiating Eq. (2.8), we have (since $R_t[0]$ is deterministic)

$$(2.10) \qquad \frac{d}{dt} <P(t) R_t[P]> = \frac{d}{dt} <P(t)> R_t[0] + <P(t)> \frac{d}{dt} R_t[0]$$
$$+ \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n \frac{d}{dt} K_t^{(n)}(t_1,...,t_n) <P(t)P(t_1)...P(t_n)>$$
$$+ \sum_{n=2}^{\infty} \int_0^t dt_2 \int_0^{t_2} dt_3 ... \int_0^{t_{n-1}} dt_n K_t^{(n)}(t,t_2,...,t_n) <P^2(t)P(t_2)...P(t_n)>$$
$$+ \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n K_t^{(n)}(t_1,...,t_n) \frac{d}{dt} <P(t)P(t_1)...P(t_n)>$$
$$= <P(t) \frac{d}{dt} R_t[P]> + \frac{d}{dt} <P(t)> R_t[0]$$
$$+ \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n K_t^{(n)}(t_1,...,t_n) \frac{d}{dt} <P(t)P(t_1)...P(t_n)>,$$

where in the second part of Eq. (2.10) the formula (2.9) has been used.

Using now the property (2.4) of the moments of the telegraphic process $P(t)$ and the following two equations satisfied by the mean and the covariance (deduced from Eq. (2.3))

$$(2.11) \qquad \begin{aligned} \frac{d}{dt} <P(t,\gamma)> &= -2\nu < P(t,\gamma)>, \\ \frac{d}{dt} < P(t,\gamma) P(t_1,\gamma)> &= -2\nu < P(t,\gamma) P(t_1,\gamma)>, \end{aligned}$$

we finally arrive at the following shape [illegible] formula [illegible] original real-valued [illegible]

$$(2.12)\qquad \frac{d}{dt} < P(t,\gamma)\, R_t[P] > \; = \; < P(t,\gamma) \frac{d}{dt} R_t[P] > -2\nu < P(t,\gamma)\, R_t[P] > .$$

The derivation of the equation for the mean value of the solution to Eq. (2.1) runs in two steps. First, taking the expectation of Eq. (2.1), we obtain the equation for two unknown functions $<U(t,\gamma)>$ and $<P(t,\gamma)U(t,\gamma)>$. Next, using the Shapiro–Loginov formula (2.12) for $R_t[P] \equiv U(t)$ and the governing equation (2.1), we obtain the equation for $<P(t,\gamma)U(t,\gamma)>$ (the property (2.2): $P^2(t,\gamma) = a^2$ closes the hierarchy of the equations). The result is the following:

$$(2.13)\qquad \begin{aligned} \frac{d}{dt} <U(t,\gamma)> &= \mathscr{A} <U(t,\gamma)> + \mathscr{B} < P(t,\gamma)U(t,\gamma)>, \\ \frac{d}{dt} < P(t,\gamma)U(t,\gamma)> &= \mathscr{A} <P(t,\gamma)U(t,\gamma)> -2\nu <P(t,\gamma)U(t,\gamma)> \\ &\quad + a^2 \mathscr{B} <U(t,\gamma)>, \end{aligned}$$

with the initial conditions

$$<U>(0) = U_0, \qquad <PU>(0) = aU_0.$$

Analogously, differentiating the tensor product of the functions $U(t,\gamma)$ and substituting $R_t[P] \equiv \underbrace{U(t,\gamma) \otimes \ldots \otimes U(t,\gamma)}_{l \text{ times}}$, $l = 2,3,\ldots$, we arrive at the following equations for the moments of any order of the solution to Eq. (2.1):

$$(2.14)\qquad \begin{aligned} \frac{d}{dt} \Gamma_l &= \sum_{i=1}^{l} \mathscr{A}^i \Gamma_l + \sum_{i=1}^{l} \mathscr{B}^i \Gamma_l^p, \\ \frac{d}{dt} \Gamma_l^p &= \sum_{i=1}^{l} \mathscr{A}^i \Gamma_l^p + a^2 \sum_{i=1}^{l} \mathscr{B}^i \Gamma_l - 2\nu \Gamma_l^p, \end{aligned}$$

and corresponding initial conditions

$$\Gamma_l(0) = \underbrace{U_0 \otimes \ldots \otimes U_0}_{l \text{ times}},$$

$$\Gamma_l^p(0) = a\underbrace{U_0 \otimes \ldots \otimes U_0}_{l \text{ times}},$$

where the moments are defined as

$$\Gamma_l(t) = <\underbrace{U(t,\gamma) \otimes \ldots \otimes U(t,\gamma)}_{l \text{ times}}>,$$

$$\Gamma_l^p(t) = <P(t,\gamma)U(t,\gamma) \underbrace{\otimes \dots \otimes}_{l \text{ times}} U(t,\gamma)>,$$

and the used operators $\mathscr{A}^i$ and $\mathscr{B}^i$ are acting on the simple tensors of the form

$$\Gamma_l = \gamma_1 \otimes \dots \otimes \gamma_l$$

in the following way:

$$A^i \Gamma_l = \gamma_1 \otimes \dots \otimes \mathscr{A}\gamma_i \otimes \dots \otimes \gamma_l,$$

$$B^i \Gamma_l = \gamma_1 \otimes \dots \otimes \mathscr{B}\gamma_i \otimes \dots \otimes \gamma_l.$$

3. The string equation

As an illustrative application of the obtained moment equations we will deal with the problem of the moment stability of a dynamical system. Consider the equation of vibrations of a string in a medium with the viscosity fluctuating according to the telegraph process

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} - p(t,\gamma)\frac{\partial u}{\partial t}, \qquad t \in (0,T], \qquad x \in [0,L],$$

with the following initial and boundary conditions:

$$u(0,x) = u_0(x),$$

(3.2)
$$\frac{\partial u}{\partial t}(0,x) = v_0(x)$$

$$u(t,0) = u(t,L) = 0,$$

where L is the length of the string and $p(t,\gamma)$ is the stochastic process of the following form:

(3.3)
$$p(t,\gamma) = b[1 - a_0(-1)^{N(t,\gamma)}], \qquad 0 < a_0 < 1,$$

and $N(t,\gamma)$ is the Poisson point process with intensity ν.

Substituting the velocity $v = \partial_u/\partial_t$ into Eqs. (3.1), we arrive at the system of two evolutionary differential equations of the form

(3.4)
$$\frac{\partial u}{\partial t} = v,$$

$$\frac{\partial v}{\partial t} = c\frac{\partial^2 u}{\partial x^2} - bv + P(t,\gamma)v.$$

where

$$P(t,\gamma) = a(-1)^{N(t,\gamma)} \tag{3.5}$$

is the telegraphic stochastic process defined in Eq. (2.2), $a = ba_0$, c and b are arbitrary positive constants.

The system of evolution equations (3.4) written in the abstract form is as follows:

$$\begin{aligned} &\frac{dU(t,\gamma)}{dt} = \mathscr{A}U(t,\gamma) + P(t,\gamma)\mathscr{B}U(t,\gamma), \\ &U(0,\gamma) = U_0, \end{aligned} \tag{3.6}$$

where

$$\mathscr{A} = \begin{bmatrix} 0 & , & 1 \\ c\dfrac{\partial^2}{\partial x^2} & , & -b \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 0 & , & 0 \\ 0 & , & 1 \end{bmatrix}.$$

In the above problem the Hilbert space where the operators $\mathscr{A}$ and $\mathscr{B}$ are acting is $X = H_0^1(0,L) \times L^2(0,L)$. Here $H_0^1(0,L)$ is the Sobolev space of square integrable functions possessing square-integrable derivatives, with the support contained within the interval $(0,L)$, and $L^2(0,L)$ is the space of square-integrable functions on the same interval.

In order to apply our generalized moment equations (2.14) to the derivation of equations for the mean value and the covariance of the solution of Eq. (3.1), we introduce the following denotations for the required moments:

$$\Gamma^1 = \langle u \rangle, \quad \Gamma^2 = \langle v \rangle, \quad \Gamma^3 = \langle pu \rangle, \quad \Gamma^4 = \langle pv \rangle,$$

and

$$\Gamma^{11} = \langle uu \rangle, \quad \Gamma^{12} = \langle uv \rangle, \quad \Gamma^{21} = \langle vu \rangle, \quad \Gamma^{22} = \langle vv \rangle,$$
$$\Gamma^{011} = \langle puu \rangle, \quad \Gamma^{012} = \langle puv \rangle, \quad \Gamma^{021} = \langle pvu \rangle, \quad \Gamma^{022} = \langle pvv \rangle.$$

The obtained equations for the mean value are

$$\begin{aligned} &\partial_t \Gamma^1 = \Gamma^2, \\ &\partial_t \Gamma^2 = c\partial^2 \Gamma^1 - b\Gamma^2 + \Gamma^4, \\ &\partial_t \Gamma^3 = -2\nu\Gamma^3 + \Gamma^4, \\ &\partial_t \Gamma^4 = c\partial^2 \Gamma^3 - (2\nu + b)\Gamma^4 + a^2\Gamma^2, \end{aligned} \tag{3.7}$$

with the following initial conditions:

$$\Gamma^1(0) = u_0, \qquad \Gamma^2(0) = v_0, \qquad \Gamma^3(0) = au_0, \qquad \Gamma^4(0) = av_0,$$

where

$$\partial_t \overset{\text{def}}{=} \frac{\partial}{\partial t}, \qquad \partial \overset{\text{def}}{=} \frac{\partial}{\partial x}.$$

The equations for the second order moments are the following:

$$\begin{aligned}
\partial_t \Gamma^{11} &= \Gamma^{12} + \Gamma^{21}, \\
\partial_t \Gamma^{12} &= c\partial_2^2 \Gamma^{11} - b\Gamma^{12} + \Gamma^{22} + \Gamma^{02}, \\
\partial_t \Gamma^{21} &= c\partial_1^2 \Gamma^{11} - b\Gamma^{21} + \Gamma^{22} + \Gamma^{021}, \\
\partial_t \Gamma^{22} &= c\partial_1^2 \Gamma^{12} + c\partial_2^2 \Gamma^{21} - 2b\Gamma^{22} + 2\Gamma^{022}, \\
\partial_t \Gamma^{011} &= \Gamma^{012} + \Gamma^{021} - 2\nu\Gamma^{011}, \\
\partial_t \Gamma^{012} &= c\partial_2^2 \Gamma^{011} - (2\nu + b)\Gamma^{012} + \Gamma^{022} + a^2\Gamma^{12}, \\
\partial_t \Gamma^{021} &= c\partial_1^2 \Gamma^{011} - (2\nu + b)\Gamma^{021} + \Gamma^{022} + a^2\Gamma^{21}, \\
\partial_t \Gamma^{022} &= c\partial_1^2 \Gamma^{012} + c\partial_2^2 \Gamma^{021} - 2(\nu + b)\Gamma^{022} + 2a^2\Gamma^{22},
\end{aligned} \tag{3.8}$$

$$\partial_1 \overset{\text{def}}{=} \frac{\partial}{\partial x_1}, \qquad \partial_2 \overset{\text{def}}{=} \frac{\partial}{\partial x_2},$$

along with the deterministic initial conditions

$$\begin{aligned}
\Gamma^{11}(0) &= u(x_1)u(x_2), & \Gamma^{12}(0) &= u(x_1)v(x_2), \\
\Gamma^{21}(0) &= v(x_1)u(x_2), & \Gamma^{22}(0) &= v(x_1)v(x_2), \\
\Gamma^{011}(0) &= au(x_1)u(x_2), & \Gamma^{012}(0) &= au(x_1)v(x_2), \\
\Gamma^{021}(0) &= av(x_1)u(x_2), & \Gamma^{022}(0) &= av(x_1)v(x_2).
\end{aligned}$$

Equations (3.7) and (3.8) can be easily used for the investigation of the mean and the mean-square stability of the solution of the string equation (3.1). In the considered example the moment stability is the (Lyapunov) stability of the deterministic systems of partial differential equations (3.7) and (3.8). As appropriate Lyapunov functionals we use the squared norm in, respectively, the spaces $\mathbf{X} \times \mathbf{X}$ and $(\mathbf{X} \times \mathbf{X}) \otimes (\mathbf{X} \times \mathbf{X})$ for the mean and the mean-stability.

Such functionals are very natural; they guarantee that energy carried by our dynamical system (the string) remains bounded.

At the beginning consider Eq. (3.7) for the mean value. In this case the Lyapunov functional is defined as

$$V = \|\Gamma_2\|^2 = \int [c(\partial\Gamma^1)^2 + (\Gamma^2)^2 + c(\partial\Gamma^3)^2 + (\Gamma^4)^2]\,dx. \tag{3.9}$$

Differentiating the functional V along the trajectories of Eq. (3.7), we obtain

$$\begin{aligned} \dot V &= \int [2c\partial\Gamma^1\partial\dot\Gamma^1 + 2\Gamma^2\dot\Gamma^2 + 2c\partial\Gamma^3\partial\dot\Gamma^3 + 2\Gamma^4\dot\Gamma^4]\,dx \\ &= \int [2c\partial\Gamma^1\partial\Gamma^2 + 2\Gamma^2(c\partial^2\Gamma^1 - b\Gamma^2 + \Gamma^4) + 2c\partial\Gamma^3\partial(\Gamma^4 - 2\nu\Gamma^3) \\ &\qquad + 2\Gamma^4(c\partial^2\Gamma^3 - (2\nu + b)\Gamma^4 + a^2\Gamma^2]\,dx \\ &= \int [-4\nu c(\partial\Gamma^3)^2 + 2(1+a^2)\Gamma^2\Gamma^4 - 2b(\Gamma^2)^2 - 2(2\nu+b)(\Gamma^4)^2]\,dx. \end{aligned} \tag{3.10}$$

From Eq. (3.10) we have that the solution of Eq. (3.7) is stable if the matrix

$$A = \begin{bmatrix} 2b, & -(1+a^2) \\ -(1+a^2), & 2(2\nu+\mathrm{b}) \end{bmatrix}$$

is positive definite. Therefore the condition of the mean stability of the solution of Eq. (3.1) is

$$4b(2\nu+b) > (1+a^2)^2, \tag{3.11}$$

or, after substituting the definition of a from Eq. (3.5),

$$4b(2\nu+b) > (1+b^2a_o^2)^2. \tag{3.12}$$

To investigate the stability of Eq. (3.8), we choose as the Lyapunov functional the following expression:

$$\begin{aligned} V = \iint [&c^2(\partial_1\partial_2\Gamma^{11})^2 + c(\partial_1\Gamma^{12})^2 + c(\partial_1\Gamma^{21})^2 + (\Gamma^{22})^2 \\ &+ c^2(\partial_1\partial_2\Gamma^{011})^2 + c(\partial_1\Gamma^{012})^2 + c(\partial_1\Gamma^{021})^2 + (\Gamma^{022})^2]\,dx_1dx_2. \end{aligned} \tag{3.13}$$

Differentiating Eq. (3.13) along trajectories of the equation, we have

$$\begin{aligned} V' = 2\iint [&c^2\partial_1\partial_2\Gamma^{11}\partial_1\partial_2\dot\Gamma^{11} + c\partial_1\Gamma^{12}\partial_1\dot\Gamma^{12} + c\partial_1\Gamma^{21}\partial_1\dot\Gamma^{21} \\ &+ \Gamma^{22}\dot\Gamma^{22} + c^2\partial_1\partial_2\Gamma^{011}\partial_1\partial_2\dot\Gamma^{011} + c\partial_1\Gamma^{012}\partial_1\dot\Gamma^{012} + c\partial_1\Gamma^{021}\partial_1\dot\Gamma^{021} \end{aligned} \tag{3.14}$$

$$+\Gamma^{022}\dot{\Gamma}^{022}]\,dx_1dx_2 = 2\iint[-bc(\partial_1\Gamma^{12})^2 + c\partial_1\Gamma^{12}\partial_1\Gamma^{012} - bc(\partial_2\Gamma^{21})^2 + c\partial_2\Gamma^{21}\partial_2\Gamma^{021} - 2b(\Gamma^{22})^2 + 2\Gamma^{22}\Gamma^{022} - 2vc^2(\partial_1\partial_2\Gamma^{011})^2 - (2v+b)c(\partial_1\Gamma^{012})^2 + a^2c\partial_1\Gamma^{12}\partial_1\Gamma^{012} + a^2c\partial_2\Gamma^{21}\partial_2\Gamma^{021} - 2(v+b)(\Gamma^{022})^2 + 2a^2\Gamma^{022}\Gamma^{22} - (2v+b)c(\partial_2\Gamma^{021})^2]\,dx_1dx_2. \tag{3.14 [cont.]}$$

The mean-square stability of Eq. (3.1) is guaranteed if the matrices constituting in Eq. (3.14) the quadratic forms with respect to the moments and the spatial derivatives of the moments, respectively, are negative definite.

The appropriate matrices of the quadratic forms are

$$\mathbf{A} = \begin{bmatrix} -2b, & (1+a^2) \\ (1+a^2), & -2(v+b) \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -2bc & 0 & c(1+a^2) & 0 \\ 0 & -2bc & 0 & c(1+a^2) \\ c(1+a^2) & 0 & -2(2v+b)c & 0 \\ 0 & c(1+a^2) & 0 & -2(2v+b)c \end{bmatrix}.$$

The nontrivial inequalities obtained as the conditions of the stability are: from matrix **A**:

$$4b(v+b) > (1+a^2)^2,$$

and, from matrix **B**:

$$4b(2v+b) > (1+a^2)^2.$$

Finally, since we consider only positive values of the parameters, the condition of mean-square stability is

$$4b(v+b) > (1+a^2)^2, \tag{3.16}$$

or, with the use of Eq. (3.5)

$$4b(v+b) > (1+a_o^2b^2)^2. \tag{3.17}$$

In this section we have used moment equations for studying the stability of the solution to the partial differential equation. In practical problems of mechanics the evolution of systems is very often investigated using the modal approach. Such a treatment leads from partial to ordinary differential equations and, usually in stability problems, yields conditions easier to derive. The considered case, which is relatively simple in analysis, gives us the opportunity to compare the results obtained using both methods. The following section is devoted to the modal treatment of our string equation.

4. String equation. Modal approach

Assume that the solution of the system (3.4) can be expanded into the series of the form

$$(4.1)\qquad \begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} y_n(t) \sin \frac{n\pi x}{L}, \\ v(t,x) &= \sum_{n=1}^{\infty} z_n(t) \sin \frac{n\pi x}{L}. \end{aligned}$$

Substituting Eq. (4.1) into Eq. (3.4) results in the sequence of equations of motion for all the modes

$$(4.2)\qquad \begin{aligned} \frac{dy_n(t)}{dt} &= z_n(t), \\ \frac{dz_n(t)}{dt} &= -\frac{n^2\pi^2 c}{L^2} y_n(t) - bz_n(t) + p(t)z_n(t), \end{aligned}$$

$n = 1,2,...$, and the initial conditions

$$\begin{aligned} y_n(0) &= \int u_0(x) \sin \frac{n\pi x}{L}\, dx = y_{n0}, \\ z_n(0) &= \int v_0(x) \sin \frac{n\pi x}{L}\, dx = z_{n0}. \end{aligned}$$

The systems of ordinary differential equations (4.2) written down in abstract form (2.1) are the following:

$$(4.3)\qquad \begin{aligned} &\frac{d}{dt} Y_n = \mathscr{A}_n Y_n + p(t)\mathscr{B} Y_n, \\ &Y_n(0) = Y_{n0}, \qquad n = 1, 2, 3, ..., \end{aligned}$$

where $\mathscr{A}_n$, $n = 1, 2, ...$ and $\mathscr{B}$ are the following matrices:

$$\mathscr{A}_n = \begin{bmatrix} 0, & 1 \\ \frac{n^2\pi^2 c}{L^2}, & -b \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 0, 0 \\ 0, 1 \end{bmatrix}, \tag{4.4}$$

and

$$Y_n = \begin{bmatrix} y_n \\ z_n \end{bmatrix}, \quad Y_{n0} = \begin{bmatrix} y_{n0} \\ z_{n0} \end{bmatrix}.$$

To study the approximate conditions of the moment stability of the solution of Eq. (3.1), we consider the moments of the modes. Equations for the mean values of the solutions to Eq. (4.3) (also possible to obtain with the use of Eq. (2.7)) are

$$\begin{aligned} <\dot{y}_n> &= <z_n>, \\ <\dot{z}_n> &= -b<z_n> - \frac{n^2\pi^2 c}{L^2}<y_n> + <pz_n>, \\ <p\dot{y}_n> &= -2v<py_n> + <pz_n>, \\ <p\dot{z}_n> &= -b<pz_n> - \frac{n^2\pi^2 c}{L^2}<py_n> - 2v<pz_n> + a^2<z_n>, \end{aligned} \tag{4.5}$$

with the initial conditions

$$<y_n>(0) = y_{n0}, \qquad <z_n>(0) = z_{n0},$$
$$<py_n>(0) = ay_{n0}, \qquad <pz_n>(0) = az_{n0},$$

where the angle bracket denotes the mathematical expectation of a random variable. The characteristic polynomial of the matrix $\mathfrak{U}$ of the system (4.5) is

$$\mathrm{Det}(\mathfrak{U} - \lambda I) = \lambda^4 + a_3^n\lambda^3 + a_2^n\lambda^2 + a_1^n\lambda + a_0^n, \tag{4.6}$$

where

$$\begin{aligned} a_0^n &= \frac{n^2\pi^2 c}{L^2}\left(2v(2v+b) + \frac{n^2\pi^2 c}{L^2}\right), \\ a_1^n &= (2v+b)\left(2vb + 2\frac{n^2\pi^2 c}{L^2}\right) - 2va^2, \end{aligned} \tag{4.7}$$

(4.17) [cont.]
$$a_2^n = 2v(2v+b) + (4v+b)b + 2\frac{n^2\pi^2 c}{L^2} - a^2,$$
$$a_3^n = 2(2v+b).$$

Applying the Routh-Hurvitz criterion (cf. [2]) to Eqs. (4.5), we have that for $n = 1, 2, \ldots$ their solutions are stable if simultaneously three conditions are satisfied:

$$a_1^n > 0, \quad W_2^n = a_1^n a_2^n - a_0^n a_3^n > 0, \quad W_3^n = W_2^n a_3^n - (a_1^n)^2 > 0. \tag{4.8}$$

Hence we say that the solution of our string equation is stable in the mean (in k-th approximation) if the equations for the mean values of modes (4.5) are stable for $n = 1, 2, \ldots, k$. This means in fact that the approximated solution of Eq. (3.1) (the truncated series (4.1)) is stable in the mean.

To investigate mean-square stability, we should consider the mutual moments of the solutions of Eq. (4.2) for arbitrary fixed n and m (the pairs (n,m) $n = 1, 2, \ldots, k$, $m = 1, 2, \ldots, k$), because the second moment of the truncated series (4.1) contains the terms $\langle y_n y_m\rangle$, $\langle y_n z_m\rangle$, $\langle z_n z_m\rangle$, $n,m = 1, 2, \ldots, k$. The adequate moment equations are

$$\langle y_n y_m\rangle = \langle y_n z_m\rangle + \langle z_n y_m\rangle,$$
$$\langle y_n{}^{\cdot} z_m\rangle = -\frac{m^2\pi^2 c}{L^2}\langle y_n y_m\rangle - b\langle y_n z_m\rangle + \langle z_n z_m\rangle + \langle p y_n z_m\rangle,$$
$$\langle z_n{}^{\cdot} y_m\rangle = -\frac{n^2\pi^2 c}{L^2}\langle y_n y_m\rangle - b\langle z_n y_m\rangle + \langle z_n z_m\rangle + \langle p z_n y_m\rangle,$$
$$\langle z_n{}^{\cdot} z_m\rangle = -\frac{n^2\pi^2 c}{L^2}\langle y_n z_m\rangle - \frac{m^2\pi^2 c}{L^2}\langle z_n y_m\rangle - 2b\langle z_n z_m\rangle + 2\langle p z_n z_m\rangle,$$
(4.9)
$$\langle p y_n{}^{\cdot} y_m\rangle = -2v\langle p y_n y_m\rangle + \langle p y_n z_m\rangle + \langle p z_n y_m\rangle,$$
$$\langle p y_n{}^{\cdot} z_m\rangle = a^2\langle y_n z_m\rangle - \frac{m^2\pi^2 c}{L^2}\langle p y_n y_m\rangle - (2v+b)\langle p y_n z_m\rangle + \langle p z_n z_m\rangle,$$
$$\langle p z_n{}^{\cdot} y_m\rangle = a^2\langle z_n y_m\rangle - \frac{n^2\pi^2 c}{L^2}\langle p y_n y_m\rangle - (2v+b)\langle p z_n y_m\rangle + \langle p z_n z_m\rangle,$$
$$\langle p z_n{}^{\cdot} z_m\rangle = 2a^2\langle z_n z_m\rangle - \frac{n^2\pi^2 c}{L^2}\langle p y_n z_m\rangle - \frac{m^2\pi^2 c}{L^2}\langle p z_n y_m\rangle - 2(v+b)\langle p z_n z_m\rangle.$$

Applying the Routh–Hurvitz criterion of stability of the system of equations (4.9) for all pairs (n,m) such that $n,m \leq k$, we obtain the analytical conditions on the parameters of Eq. (3.1) which guarantee the stability of its solution. Of course, they are too involved to express them in an explicit form. The results will be presented graphically in Sect. 5 for some fixed parameters. The characteristic polynomial of the matrix of the system (4.9), the coefficients of which are required in the Routh–Hurvitz criterion, is given in the Appendix.

5. Numerical example

Consider the string equation (3.1) with the initial condition (3.2) and telegraph parametric excitation (3.3) and the corresponding moment equations with the following parameters fixed: $c = 1.0$, $L = 1.0$, $a_0 = 0.1$. For such constants the areas of mean and mean-square stability for the exact criterion introduced in Sect. 3 are shown in the b-ν-system of coordinates (Fig.1).

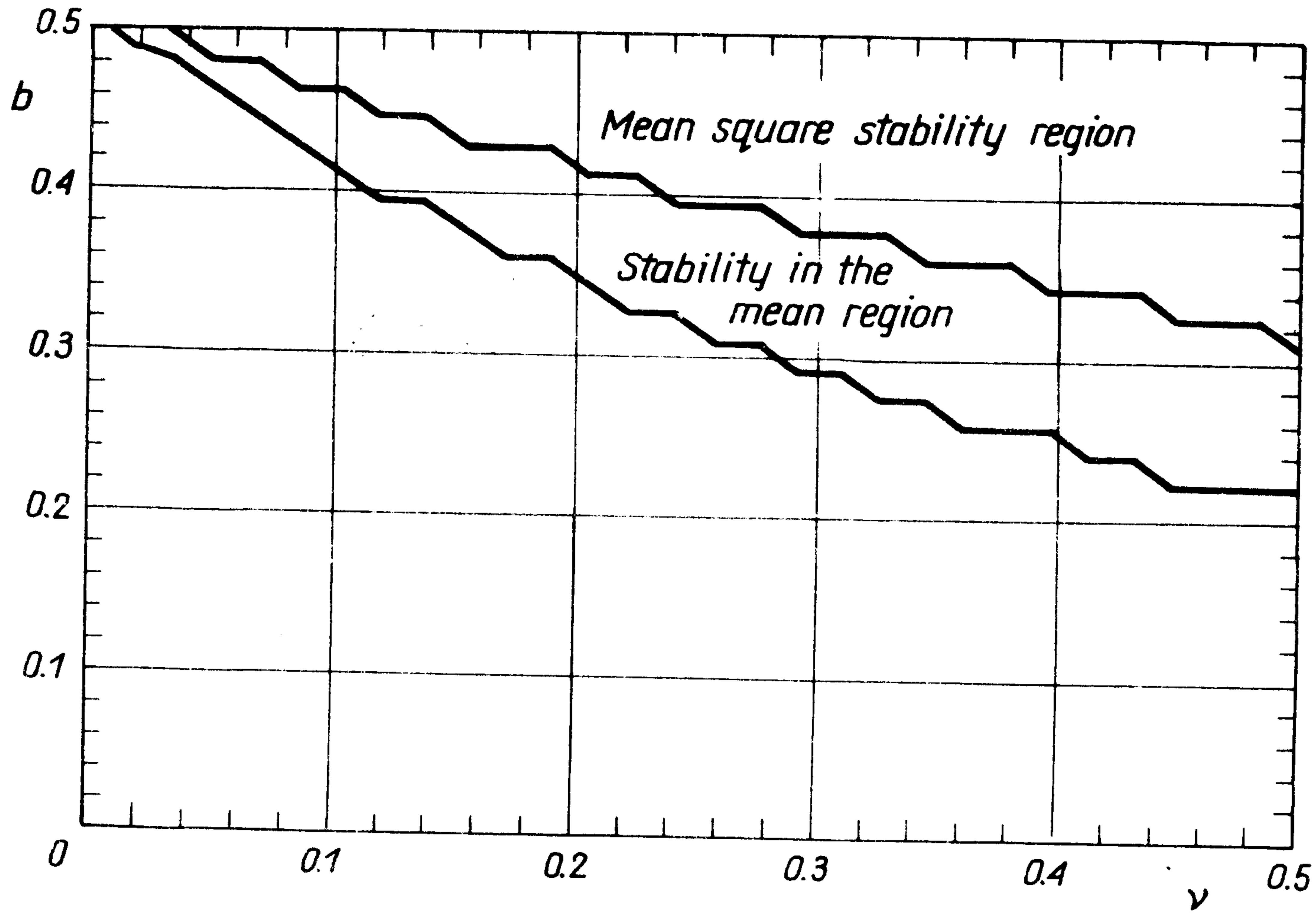

FIG. 1 Infinite-dimensional criterion of the mean and mean-square stability; $L = 1$, $c = 1$, $a_0 = 0.1$.

It is seen that the region of mean-square stability is contained within the region of the mean one. The dependence of the stability conditions on a_0 is relatively simple (see the inequalities (3.12) and (3.17)). For $b = 0$ the string is always unstable. The conditions of stability are independent of L and c.

In the case of the modal approach introduced in Sect. 4 the situation is much more complicated. The conditions of stability depend on the parameters of the string equation c, L, a_0 and on the rang of approximation in a very involved way.

Consider again our numerical example. Substituting the given values $L = 1$ and $c = 1$, we have the mean stability of equations for each mode for all a_0, b

and v (except of $b=v=0$). The modal equations loose the stability if b is sufficiently small and c/L^2 tends to zero (c is small and L is big). We conclude that in the case of stability in the mean the modal approach and the exact criterion give quite different results.

When mean-square stability is considered, we have a more interesting situation. In Fig. 2 the regions of the mean-square stability of the vibrating

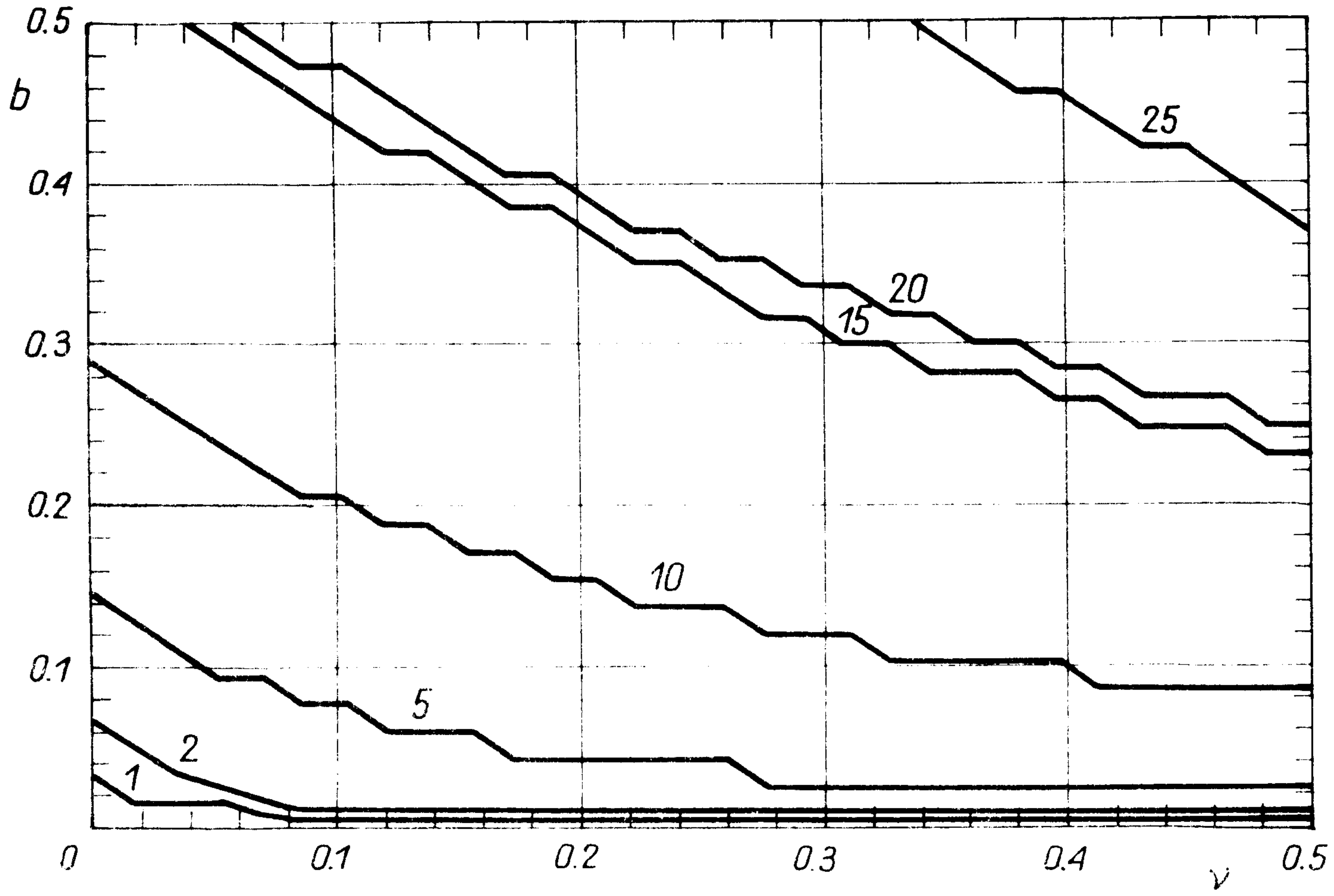

FIG. 2. Regions of mean-square stability for 1, 2, 5, 10, 15, 20, 25 modes; $L=1$, $c=1$, $a_0=0.1$.

string for $n=1,2,5,10,15,20,25$ modes are shown. Taking into account only the first mode gives a result far from the one obtained in the infinite-dimensional method (see Fig. 1). It is seen that for n tending to 20 the bounds of the stability regions concentrate near the curve obtained with the use of the infinite-dimensional criterion; next they diverge and give the stability regions in the b-v-system of coordinates more restrictive than in the exact case. As a conclusion we can say that the first mode approximation gives in our example some information about the instability of the string, but it may happen that taking too many modes we loose information about instability and do not gain certainty about stability.

Let us remark that our conclusion differs from the classical results of Lyapunov stability of the string with deterministic parameters, where the positivity of the damping coefficient b is the condition of the asymptotic stability. In the considered approach the solution not only tends to zero (with its temporal derivative), but also makes the fluctuating parameter $P(t,y)$ (the mixed moments) tend to zero. The obtained conditions guarantee that a numerical procedure for solving the exact set of the moment equations converge for the long period of time.

6. Relation to white-noise coefficient

As it is seen from Eq. (2.3), the correlation time of the random telegraph process is $1/2\nu$ and it tends to zero as the intensity ν in the Poisson stream of pulses tends to infinity. Therefore we can say that the process $P(t,\gamma)$ tends to a white-noise when $\nu \to \infty$ but a^2/ν remains bounded, and, what follows, for very large ν the telegraph process could be approximated by a white-noise one with the "equivalent" intensity $I = a^2/\nu$ (cf. [6]).

Let us assume that the process $P(t,\gamma)$ in Eqs. (3.4) is replaced by a white-noise with intensity $I = a^2/\nu$. The equations for the two lowest order moments of the solution of such a modified equation are (cf. [8]) the following:

$$\begin{aligned} &\partial_t \Gamma^1 = \Gamma^2, \\ &\partial_t \Gamma^2 = c\partial^2 \Gamma^1 - b\Gamma^2 + \frac{1}{2} I\, \Gamma^2, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &\partial_t \Gamma^{11} = \Gamma^{12} + \Gamma^{21}, \\ &\partial_t \Gamma^{12} = c\partial_2^2 \Gamma^{11} - b\Gamma^{12} + \Gamma^{22} + \frac{1}{2} I\, \Gamma^{12}, \\ &\partial_t \Gamma^{21} = c\partial_1^2 \Gamma^{11} - b\Gamma^{21} + \Gamma^{22} + \frac{1}{2} I\, \Gamma^{21}, \\ &\partial_t \Gamma^{22} = c\partial_1^2 \Gamma^{12} + c\partial_2^2 \Gamma^{21} - 2b\Gamma^{22} + 2I\Gamma^{22}. \end{aligned} \tag{6.2}$$

(The denotations in Eqs. (6.1) and (6.2) are the same as in Eqs. (3.7) and (3.8)).

The stability conditions obtained from Eqs. (6.1) and (6.2) (in a way analogous to that in Sect 3) for the mean and the mean-square are, respectively,

$$\frac{a^2}{\nu} < 2b \tag{6.3}$$

and

$$\frac{a^2}{\nu} < b, \tag{6.4}$$

what, with the use of Eq. (3.5), gives

$$a_0^2 < 2\nu \tag{6.5}$$

and

$$a_0^2 < \nu. \tag{6.6}$$

Using the modal approach we obtain the following equations for the moments (see the series (4.1)) (cf. [1]):

$$\begin{aligned} <\dot{y}> &= <z_n>, \\ <\dot{z}_n> &= -b<z_n> - \frac{n^2\pi^2 c}{L^2}<y_n> + \frac{1}{2}I<z_n>, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} <y_n{}^\cdot y_m> &= <y_n z_m> + <z_n y_m>, \\ <y_n{}^\cdot z_m> &= -\frac{m^2\pi^2 c}{L^2}<y_n y_m> - b<y_n z_m> + <z_n z_m> \\ &\qquad + \frac{1}{2}I<y_n z_m>, \\ <z_n{}^\cdot y_m> &= -\frac{n^2\pi^2 c}{L^2}<y_n y_m> - b<z_n y_m> + <z_n z_m> \\ &\qquad + \frac{1}{2}I<z_n y_m>, \\ <z_n{}^\cdot z_m> &= -\frac{n^2\pi^2 c}{L^2}<y_n z_m> - \frac{m^2\pi^2 c}{L^2}<z_n y_m> - 2b<z_n z_m> \\ &\qquad + 2I<pz_n z_m>. \end{aligned} \tag{6.8}$$

As it is seen, the condition of stability of Eqs. (6.7) for $n = 1,2,\ldots$ is

$$b - \frac{1}{2}I > 0, \tag{6.9}$$

or

$$\frac{a^2}{\nu} < 2b, \tag{6.10}$$

and it is independent of n.

From the inequalities (6.3) and (6.10) we have that the exact and approximated conditions of mean stability in the case of the white-noise process in the parameter are equivalent.

The condition of mean-square stability in the case of the modal approach is much more involved. From the Routh–Hurvitz criterion we have that the system of equations (6.8) is stable if simultaneously W_0, W_1, W_2 and W_3 are positive, where these quantities in terms of the parameters of the "equivalent" white-noise are $\left(c_n^2 = \frac{n^2\pi^2 c}{L^2},\ c_m^2 = \frac{m^2\pi^2 c}{L^2}\right)$:

$$W_0 = (c_n^2 - c_m^2)^2 + (c_n^2 + c_m^2)(a_0^4 b^2 - 3a_0^2 \nu b + 2\nu^2)\frac{b^2}{\nu^2},$$

$$W_1 = (c_n^2 + c_m^2)(4\nu - 3a_0^2 b)\frac{b}{\nu} + (-a_0^6 b^3 + 5a_0^4 \nu b^2 - 8a_0^2 \nu^2 b + 4\nu^3)\frac{b^3}{2\nu^3},$$

$$W_2 = (4\nu - 3a_0^2 b)(c_n^4 + 6c_m^2 c_n^2 + c_m^4)\frac{b}{\nu}$$
$$+ (c_n^2 + c_m^2)(-19a_0^6 b^3 + 88a_0^4 \nu b^2 - 132a_0^2 \nu^2 b + 64\nu^3)\frac{b^3}{4\nu^3}$$
$$+ (-9a_0^{10} b^5 + 73a_0^8 \nu b^4 - 232a_0^6 \nu^2 b^3 + 360a_0^4 \nu^3 b^2 - 272a_0^2 \nu^4 b + 80\nu^5)\frac{b^5}{8\nu^5},$$

$$W_3 = 4c_n^2 c_m^2 (9a_0^4 b^2 - 24a_0^2 \nu b + 16\nu^2)\frac{b^2}{\nu^2}$$
$$+ (c_n^2 + c_m^2)(45a_0^8 b^4 - 264a_0^6 \nu b^3 + 572a_0^4 \nu^2 b^2 - 544a_0^2 \nu^3 b + 192\nu^4)\frac{b^4}{4\nu^4}$$
$$+ (25a_0^{12} b^6 - 235a_0^{10} \nu b^5 + 906a_0^8 \nu^2 b^4 - 1832a_0^6 \nu^3 b^3 + 2048a_0^4 \nu^4 b^2$$
$$- 1200a_0^2 \nu^5 b + 288\nu^6)\frac{b^6}{8\nu^6}.$$

For the parameters taken in the numerical example ($a_0 = 0.1$, $L = 1$, $c = 1$), we always have the mean-square stability of each mode except $b = 0$ and $\nu = 0$. This fact shows that approximation of the telegraph process with white noise requires great caution. It may happen that the result of such a substitution gives some unexpected result — like here the stability of the originally unstable system. This means in fact that even if the telegraph process in the limit case gives white noise, its nonlinear transformation (generated by the string equation) might not converge to the analogous nonlinear transformation of white-noise. Therefore such an approximation could be performed only within the general problem where it is used.

Appendix

The characteristic polynomial of the matrix of the system of Eqs. (4.9) is

$$\mathrm{Det}(\mathfrak{U} - \lambda Id) = \lambda^8 + a_7\lambda^7 + a_6\lambda^6 + a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where its coefficients are $\left(c_n^2 = \dfrac{n^2\pi^2 c}{L^2},\ c_m^2 = \dfrac{m^2\pi^2 c}{L^2}\right)$:

$$a_7 = 8(v+b),$$

$$a_6 = 24v^2 + 26b^2 + 56vb + 4(c_n^2 + c_m^2) - 6a^2,$$

$$a_5 = 32v^3 + 144v^2b + 156vb^2 + 44b^3 + 24(v+b)(c_n^2 + c_m^2) - 4a^2(9v + 7b),$$

$$\begin{aligned} a_4 = {} & 9a^4 - 2a^2(36v^2 + 70vb + 25b^2 + 4(c_n^2 + c_m^2)) + 2(c_n^2 - c_m^2)^2 \\ & + 4(c_n^2 + c_m^2)^2 + (c_n^2 + c_m^2)(56v^2 + 120vb + 56b^2) \\ & + 16v^4 + 160v^3b + 332v^2b^2 + 220vb^3 + 41b^4, \end{aligned}$$

$$\begin{aligned} a_3 = {} & a^4(36v + 20b) - a^2(32(v+b)(c_n^2 + c_m^2) + (48v^3 + 224v^2b + 200vb^2 \\ & + 40b^3)) + (24(c_n^4 + c_m^4) + 16c_n^2c_m^2)(v+b) + (c_n^2 + c_m^2)(64(v^3 + b^3) \\ & + 224vb(v+b)) + 64v^4b + 288v^3b^3 + 368v^2b^3 + 164vb^4 + 20b^5, \end{aligned}$$

$$\begin{aligned} a_2 = {} & -4a^6 + a^4(4(c_n^2 + c_m^2) + 36v^2 + 60vb + 12b^2) - a^2(-2(c_n^4 + c_m^4) \\ & + 36c_n^2c_m^2 + (c_n^2 + c_m^2)(56v^2 + 96vb + 40b^2) + 112v^3b + 232v^2b^2 \\ & + 120vb^3 + 12b^4) + 4(c_n^6 + c_m^6 - c_n^4c_m^2 - c_n^2c_m^4) + (c_n^4 + c_m^4)(40v^2 \\ & + 72vb + 34b^2) + c_n^2c_m^2(-16v^2 + 48vb + 28b^2) + (c_n^2 + c_m^2)(32v^4 \\ & + 192v^3b + 320v^2b^2 + 192vb^3 + 36b^4) + 80v^4b^2 + 224v^3b^3 \\ & + 196v^2b^4 + 60vb^5 + 4b^6, \end{aligned}$$

$$\begin{aligned} a_1 = {} & -8a^6v + a^4(8(c_n^2 + c_m^2)(v+b) + 40v^2b + 24vb^2) - a^2(4(c_n^4 + c_m^4)(b - v) \\ & + c_n^2c_m^2(72v + 56b) + (c_n^2 + c_m^2)(48v^3 + 112v^2b + 80vb^2 + 16b^3) \\ & + 64v^3b + 80v^2b^3 + 24vb^4) + 8(c_n^6 + c_m^6 - c_n^4c_m^2 - c_n^2c_m^4)(v+b) \\ & + (c_n^4 + c_m^4)(32v^3 + 80v^2b + 68vb^2 + 20b^3) + c_n^2c_m^2(-64v^3 - 32v^2b \\ & + 56vb^2 + 24b^3) + (c_n^2 + c_m^2)(64v^4b + 192v^3b^2 + 192v^2b^3 + 72vb^4 \\ & + 8b^5) + 32v^4b^3 + 64v^3b^4 + 40v^2b^5 + 8vb^6, \end{aligned}$$

$$\begin{aligned} a_0 = {} & a^4(4(c_n^2 + c_m^2)^2 + (c_n^2 + c_m^2)(16v^2 + 8vb)) - a^2(4(-c_n^6 - c_m^6 + c_n^4c_m^2 \\ & + c_n^2c_m^4) + (c_n^4 + c_m^4)(-16v^2 + 4vb + 8b^2) + c_n^2c_m^2(32v^2 + 56vb + 16b^2) \\ & + (c_n^2 + c_m^2)(48v^3b + 56v^2b^2 + 16vb^2)) + (c_n^8 + c_m^8 + 6c_n^4c_m^4 - 4c_n^6c_m^2 - 4c_n^2c_m^6) \\ & + (c_n^6 + c_m^6 - c_n^4c_m^2 - c_n^2c_m^4)(8v^2 + 8vb + 4b^2) + (c_n^4 + c_m^4)(16v^4 + 32v^3b \\ & + 36v^2b^2 + 20vb^3 + 4b^4) - c_n^2c_m^2(32v^4 + 64v^3b + 8v^2b^2 - 24vb^3 - 8b^4) \\ & + (c_n^2 + c_m^2)(32v^4b^2 + 64v^3b^3 + 40v^2b^4 + 8vb^5). \end{aligned}$$

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