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Wave Transmission Properties of the Randomly Segmented Elastic Bar

by

Zbigniew KOTULSKI

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Summary. In the paper the propagation of wave pulses through bars with random properties is investigated. It is assumed that the bar is built of several homogeneous segments whose lengths as well as the material parameters are random variables. The overall properties of such bars are studied. The convergence of the segmented bar to the homogenized one is studied numerically both in case of the periodic and randomly periodic bar.

1. Introduction. Mechanical phenomena taking place in media with complicated (random) structure make a lot of difficulties in the mathematical modeling. Therefore authors propose methods of an approximate (in a certain sense) description of such problems by some simplified models where the media of complicated structure is replaced by some homogeneous one. The procedure of such an approximation is called homogenization and the resulting homogeneous medium — the effective one. Although in the literature there is many papers on homogenization, the most of them is devoted to static problems and in dynamic case — to stationary ones.

In this paper we consider the dynamic problem which is strongly nonstationary — the wave pulse propagation in a segmented bar. We formulate the equations for the amplitudes of the reflected and transmitted pulses generated by some incident pulse. They are valid for any combination of the dimensions of segments and their material parameters. Generally the values of the parameters can be random variables. We also consider in detail the particular case of a randomly periodic bar and its limiting case with the number of segments tending to infinity. In such a case we obtain the effective parameters of the bar, sufficient to characterize the generated and

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transmitted pulses. We also consider the numerical example comparing the transmitted pulses in case of the periodic and randomly periodic bar.

2. Formulation of the problem. We consider the wave propagating along an elastic bar of a constant cross-section, described by the system of two differential equations (cf. [1])

$$\tag{1} \begin{aligned} \frac{\partial f}{\partial x} &= A\varrho\frac{\partial v}{\partial t}, \\ \frac{\partial v}{\partial x} &= \frac{1}{AE}\frac{\partial f}{\partial t}, \end{aligned}$$

where f denotes the force, v is the particle velocity in the medium, A is the area of the perpendicular cross-section of the bar and ϱ is the material density, E is the Young modulus. As usually, x is the spatial variable along the length of the bar and t is time. If we introduce as a new parameter the impedance $Z = A\sqrt{\varrho E}$ and as a new variable the travel time from 0 to x defined as $\xi = x/c$, where $c = \sqrt{E/\varrho}$, and then perform the Fourier transform of Eq. (1), the wave equation takes the form

$$\tag{2} \frac{\partial}{\partial \xi}\begin{bmatrix} \widehat{f} \\ \widehat{v} \end{bmatrix} = \mathrm{i}\omega \begin{bmatrix} 0 & Z \\ \frac{1}{Z} & 0 \end{bmatrix} \begin{bmatrix} \widehat{f} \\ \widehat{v} \end{bmatrix},$$

where $\widehat{f}$ and $\widehat{v}$ are the Fourier transform of the force and velocity.

The solution of Eq. (2) at a point ξ inside the bar can be represented as

$$\tag{3} \widehat{\mathbf{s}}(\xi,\omega) = \boldsymbol{P}(\xi,\omega)\widehat{\mathbf{s}}(0,\omega),$$

where $\boldsymbol{P}(\xi,\omega)$ is the fundamental matrix of Eq. (2), $\widehat{\mathbf{s}}(\xi,\omega)$ is the solution vector, and $\widehat{\mathbf{s}}(0,\omega)$ is the initial pulse at the front of the bar at time 0

$$\tag{4} \boldsymbol{P}(\xi,\omega) = \frac{1}{2}\begin{bmatrix} \cos\omega\xi & -\mathrm{i}Z\sin\omega\xi \\ -\frac{\mathrm{i}}{Z}\sin\omega\xi & \cos\omega\xi \end{bmatrix}, \qquad \widehat{\mathbf{s}}(\xi,\omega) = \begin{bmatrix} \widehat{f}(\xi,\omega) \\ \widehat{v}(\xi,\omega) \end{bmatrix}.$$

If the bar is built of several homogeneous segments, then the wave partially reflects from the interfaces of the segments and partially transmits through them in such a way that the vector field $\widehat{\mathbf{s}}(\xi,\omega)$ remains continuous.

The solution of Eq. (4) can be represented in the following form

$$\tag{5} \begin{bmatrix} \widehat{f}(\xi,\omega) \\ \widehat{v}(\xi,\omega) \end{bmatrix} = \begin{bmatrix} \widehat{f}_R(\omega)\mathrm{e}^{-\mathrm{i}\omega\xi} + \widehat{f}_L(\omega)\mathrm{e}^{\mathrm{i}\omega\xi} \\ \frac{1}{Z}(-\widehat{f}_R(\omega)\mathrm{e}^{-\mathrm{i}\omega\xi} + \widehat{f}_L(\omega)\mathrm{e}^{\mathrm{i}\omega\xi}) \end{bmatrix},$$

where $\widehat{f}_R(\omega)$ and $\widehat{f}_L(\omega)$ are the amplitudes of the incident and reflected force waves, respectively.

Assume now that the bar is built of N homogeneous panels; in a j-th panel the impedance is Z_j, the wave travel time through the segment is h_j; generally, both Z_j and h_j can be random variables. The beginning of the bar is located at the point 0. Let the wave pulse $\widehat{f}_R^0(\omega)\mathrm{e}^{-\mathrm{i}\omega\xi}$ comes from the surrounding media (with the impedance indexed by 0) to the front end of the bar. Then, using the transition matrix method analogous to the one applied for the harmonic waves in [2], we obtain the following matrix equation for $\widehat{f_L^0}(\omega)$ and $\widehat{f_R^0}(\omega)$ — the amplitudes of the pulse reflected from the bar and transmitted through it

$$(6)\qquad \begin{bmatrix} \widehat{f_R^0} \\ \widehat{f_L^0} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\mathrm{i}Z_0 \\ 1 & \mathrm{i}Z_0 \end{bmatrix} \prod_{j=1}^{N} M_j \begin{bmatrix} 1 \\ \frac{\mathrm{i}}{Z_0} \end{bmatrix} \widehat{f_R^{fin}} \exp\left\{ -\mathrm{i}\omega \sum_{j=1}^{N} h_j \right\},$$

where M_j, for $j = 1, 2, \ldots, N$, is the transition matrix through j-th segment

$$(7)\qquad M_j = \begin{bmatrix} \cos\omega h_j & Z_j \sin\omega h_j \\ -\frac{1}{Z_j}\sin\omega h_j & \cos\omega h_j \end{bmatrix}.$$

3. The model of the bar with random properties. Consider the bar built of $2K$ segments with the lengths $l_1(\gamma), l_2(\gamma), \ldots, l_{2K}(\gamma)$, where $l_i(\gamma)$, $i = 1, 2, \ldots, K$, are random variables. We assume that $N = 2K$, γ is the random element. Assume additionally that the material parameters of the segments and the areas of their cross-sections form the vector random variables $(\varrho_{2j-1}(\gamma), E_{2j-1}(\gamma), A_{2j-1}(\gamma), \varrho_{2j}(\gamma), E_{2j}(\gamma), A_{2j}(\gamma))$, so are independent and identically distributed (i.i.d.) for $j = 1, 2, \ldots, 2K$. Moreover, we assume that the lengths of the segments can be represented as: $(l_{2j-1}(\gamma), l_{2j}(\gamma)) = (\frac{L_{2j-1}(\gamma)}{2K}, \frac{L_{2j}(\gamma)}{2K})$, and for $j = 1, 2, \ldots, K$, they are i.i.d. two-dimensional random variables with $E\{L_{2j-1}(\gamma)\} = L_1$, $E\{L_{2j}(\gamma)\} = L_2$. In this particular case, we can define the transition matrices through the couples of segments $(2j-1, 2j)$, $j = 1, 2, \ldots, K$, in the following form ($h_j(\gamma)$ is the travel time corresponding to the segment of the length $l_j(\gamma)$)

$$(8)\quad M_j(\omega,\gamma) = \begin{bmatrix} \cos\omega h_{2j-1}(\gamma)\cos\omega h_{2j}(\gamma) - \frac{Z_{2j-1}(\gamma)}{Z_{2j}(\gamma)}\sin\omega h_{2j-1}(\gamma)\sin\omega h_{2j}(\gamma), & Z_{2j-1}(\gamma)\sin\omega h_{2j-1}(\gamma)\cos\omega h_{2j}(\gamma) + Z_{2j}(\gamma)\cos\omega h_{2j-1}(\gamma)\sin\omega h_{2j}(\gamma) \\ -\frac{\sin\omega h_{2j-1}(\gamma)\cos\omega h_{2j}(\gamma)}{Z_{2j-1}(\gamma)} - \frac{\cos\omega h_{2j-1}(\gamma)\sin\omega h_{2j}(\gamma)}{Z_{2j}(\gamma)}, & \cos\omega h_{2j-1}(\gamma)\cos\omega h_{2j}(\gamma) - \frac{Z_{2j}(\gamma)}{Z_{2j-1}(\gamma)}\sin\omega h_{2j-1}(\gamma)\sin\omega h_{2j}(\omega) \end{bmatrix}.$$

To study the asymptotic behavior of the equation for the wave's amplitudes, we apply the law of large numbers for the products of random matrices obtained in [3]. It says that under certain assumptions, the product of K random matrices $M_{j,K}(\gamma)$, $j = 1, 2, \ldots, K$, possessing the asymptotic representation

$$(9)\qquad M_{j,K}(\gamma) = Id + \frac{1}{K}B_{j,K}(\gamma) + R_j(K,\gamma),$$

where $\boldsymbol{B}_{j,K}(\gamma)$, for $j = 1, 2, \ldots, K$, are i.i.d. random matrices and $|\boldsymbol{R}_j(K,\gamma)| = o(K^{-1})$, for large K satisfying the following law of large numbers

$$\lim_{K\to\infty} \prod_{j=1}^{K} \boldsymbol{M}_{j,K}(\gamma) = \exp\left(E\{\boldsymbol{B}_{j,K}(\gamma)\}\right) \quad \text{in distribution.} \tag{10}$$

In the considered case, the matrices $\boldsymbol{B}_j$ obtained from expansion (9) are

$$\boldsymbol{B}_j = \begin{bmatrix} 0 & Z_{2j-1}(\gamma)\omega H_{2j-1}(\gamma) + Z_{2j}(\gamma)\omega H_{2j}(\gamma) \\ -\frac{\omega H_{2j-1}(\gamma)}{Z_{2j-1}(\gamma)} - \frac{\omega H_{2j}(\gamma)}{Z_{2j}(\gamma)} & 0 \end{bmatrix} \tag{11}$$

and the exponent of the matrix $E(\boldsymbol{B}_j)$ obtained in the limit, being the transition matrix through a single homogenized bar, is of the following form

$$\mathrm{e}^{E\{\boldsymbol{B}_j\}} = \begin{bmatrix} \cos\omega a & b\sin\omega a \\ -\frac{1}{b}\sin\omega a & \cos\omega a \end{bmatrix}, \tag{12}$$

where a is the effective travel time and b is the effective impedance of the bar

$$a = \sqrt{\left(E\{Z_1(\gamma)H_1(\gamma)\} + E\{Z_2(\gamma)H_2(\gamma)\}\right)\left(E\left\{\frac{H_1(\gamma)}{Z_1(\gamma)}\right\} + E\left\{\frac{H_2(\gamma)}{Z_2(\gamma)}\right\}\right)}, \tag{13}$$

$$b = \sqrt{\frac{E\{Z_1(\gamma)H_1(\gamma)\} + E\{Z_2(\gamma)H_2(\gamma)\}}{E\left\{\frac{H_1(\gamma)}{Z_1(\gamma)}\right\} + E\left\{\frac{H_2(\gamma)}{Z_2(\gamma)}\right\}}}. \tag{14}$$

In the above $H_i = \frac{A\varrho_i L_i}{Z_i}$ is the travel time corresponding to the length $L_i(\gamma)$, $i = 1, 2$.

Analogously to the above considerations we can obtain the effective parameters in case of the deterministic periodic bar. They have then the following form

$$a = \sqrt{\frac{(H_1Z_1 + H_2Z_2)(H_1Z_2 + H_2Z_1)}{Z_1Z_2}}, \tag{15}$$

$$b = \sqrt{Z_1Z_2\frac{H_1Z_1 + H_2Z_2}{H_1Z_2 + H_2Z_1}}, \tag{16}$$

where H_1, H_2, Z_1, Z_2 are deterministic counterparts of the parameters in stochastic model equal, for comparison of the deterministic and stochastic model, to the expected values of $H_1(\gamma)$, $H_2(\gamma)$, $Z_1(\gamma)$, $Z_2(\gamma)$.

The effective parameters can also be obtained in the more complicated model, where some panels built of k segments are the repeated periodically parts of the bar. Then the parameters a and b are the following

$$(17)\qquad a = \sqrt{\sum_{i=1}^{k} E\{\varrho_i(\gamma)A_i(\gamma)L_i(\gamma)\}}\sqrt{\sum_{i=1}^{k} E\left\{\frac{L_i(\gamma)}{E_i(\gamma)A_i(\gamma)}\right\}},$$

$$(18)\qquad b = \sqrt{\frac{\sum_{i=1}^{k} E\{\varrho_i(\gamma)A_i(\gamma)L_i(\gamma)\}}{\sum_{i=1}^{k} E\left\{\frac{L_i(\gamma)}{E_i(\gamma)A_i(\gamma)}\right\}}}.$$

In formulae (17) and (18) the parameters are characterized by the original random parameters: densities ϱ, lengths L, cross-sections A and Young moduli E.

The amplitude of the transmitted wave can be easily obtained from Eq. (6). In case of a finite number of bar's segments, one should solve the equation with respect to $\widehat{f}_R^{fin}(\omega)$ and then, calculating the inverse Fourier transform, obtain the shape of the transmitted pulse.

In the particular case of the bar built of K periodically repeated panels, the formula for the Fourier transformed amplitude is

$$(19)\qquad \widehat{f}_R^{fin}(\omega) = \frac{2\widehat{f}_R^{0}(\omega)e^{i\omega(H_1+H_2)}}{\left[\left(M_{11}^K + M_{22}^K \frac{Z_0}{Z_{fin}}\right) + i\left(M_{12}^K \frac{1}{Z_{fin}} - M_{21}^K Z_0\right)\right]},$$

where M_{ij}^K is the i,j-th element of the K-th power of M — the transition matrix through the panel.

Analogously, if the homogenized bar is considered, we replace the product of matrices in Eq. (6) by effective transition matrix (12) with parameters (13)–(14). Then the formula for the amplitude of the transmitted wave is

$$(20)\qquad \widehat{f}_R^{fin}(\omega) = \frac{2\widehat{f}_R^{0}(\omega)e^{i\omega(H_1+H_2)}}{\left[\left(1 + \frac{Z_0}{Z_{fin}}\right)\cos\omega a + i\left(\frac{b}{Z_{fin}} + \frac{Z_0}{b}\right)\sin\omega a\right]}$$

4. Illustrative example. As a numerical example (for details see [4]) consider the bar built of two kinds of material with the impedances $Z_1 = 2.0$ and $Z_2 = 4.0$ and average travel time $H_1 = H_2 = 1.0$, surrounded by the material of impedance $Z_0 = 1.0$. We study the bar in two cases: the periodic one, where the travel time of each segment is constant, equal to H_i/K, $i = 1, 2$, K is the number of couples of segments within the bar, and periodic stochastic one, where the travel time is a random variable (in the calculated example — the random variable uniformly distributes on the interval $(0, \frac{2H_i}{K})$) with the mean value H_i/K, $i = 1, 2$. In the both cases, the effective parameters calculated according to formulae (13) and (14) are the same, equal to $a = 2.12132$ and $b = 2.828427$. It is seen that the effective travel time is greater than the sum of the travel times through the components. This means that the wave pulse localizes in the bar travelling through it. From (13) we can also deduce that the bar with random structure in a general case localizes the wave stronger than the periodic one. Figure 1 shows the pulse transmitted through the homogenized bar, generated by the travelling force pulse of the unique amplitude and duration. The amplitude is calcualted according to (20), modified in such a way that the result is presented in real time. It is seen the effect of localization of the wave pulse in the segmented bar — the first transmitted pulse as well as next pulses, generated by reflected the ones, come with some delay (the arrival times of the pulses in an adequate non-segmented bar would be: 2, 6, 10, etc).

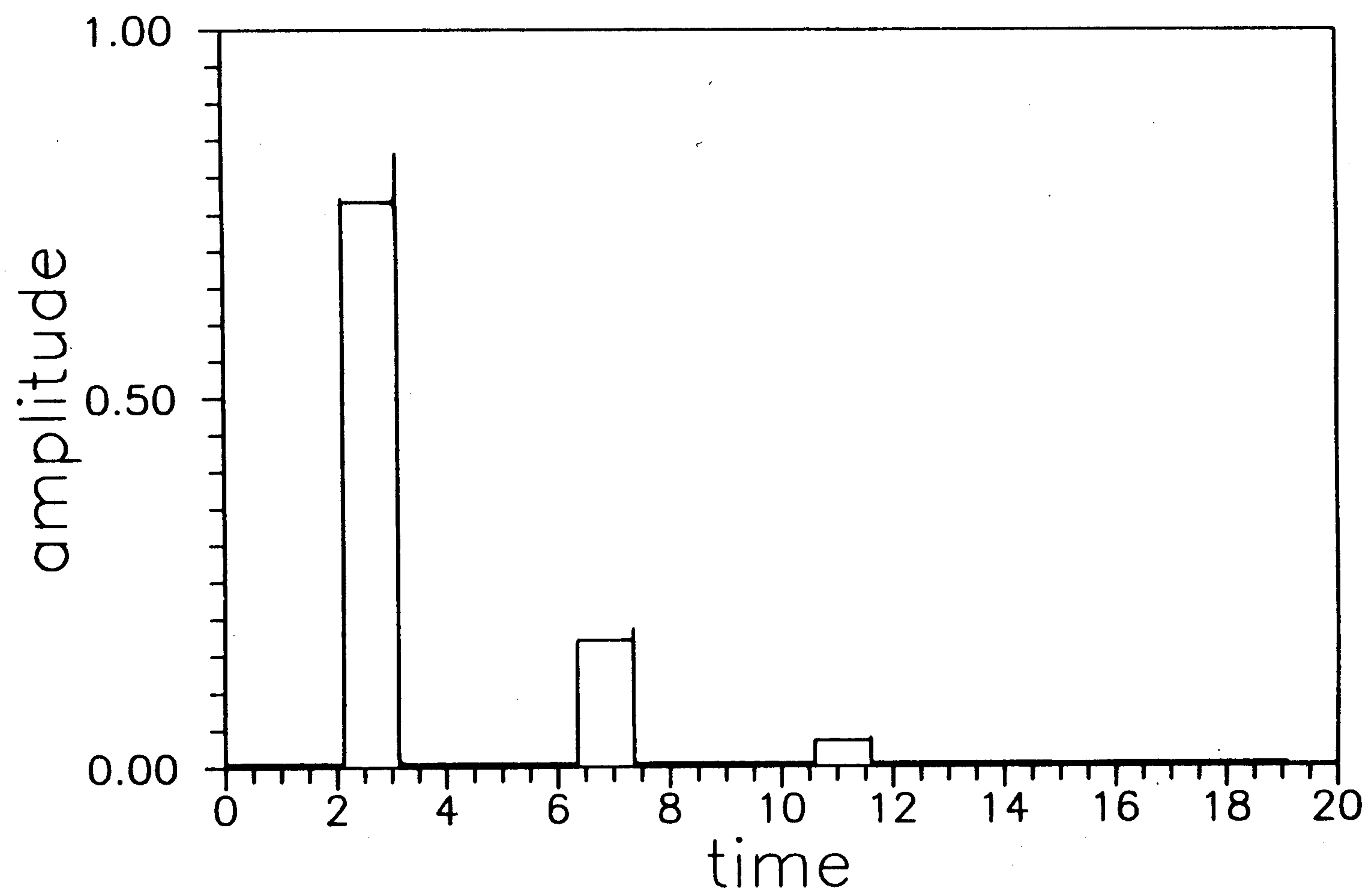

Fig. 1. The amplitude of the pulse transmitted through the homogenized bar

Formula (19) for the amplitude of the transmitted wave make it possible to study the changes of the transmitted pulse if the number of segments in the bar grows, both in the deterministic periodic and stochastic periodic cases (in the stochastic case we take the product of matrices instead of the power of the matrix). Figures 2 and 3 show the pulses transmitted through the segmented bars built of respectively 30 and 100 couples of segments (figures (a) show the amplitudes in the periodic case, figures (b) — the mean amplitudes obtained by simulation in case of the random lengths of segments). It is seen that the greater number of bar's segments, the wave pulse more similar to that in the homogenized bar. Therefore it is seen how the bar homogenizes in the dynamic non-stationary problem of the wave pulse propagation. The analogy with the stationary case of harmonic wave propagation is observed e.g. in [5]. One can also compare the shapes of the wave

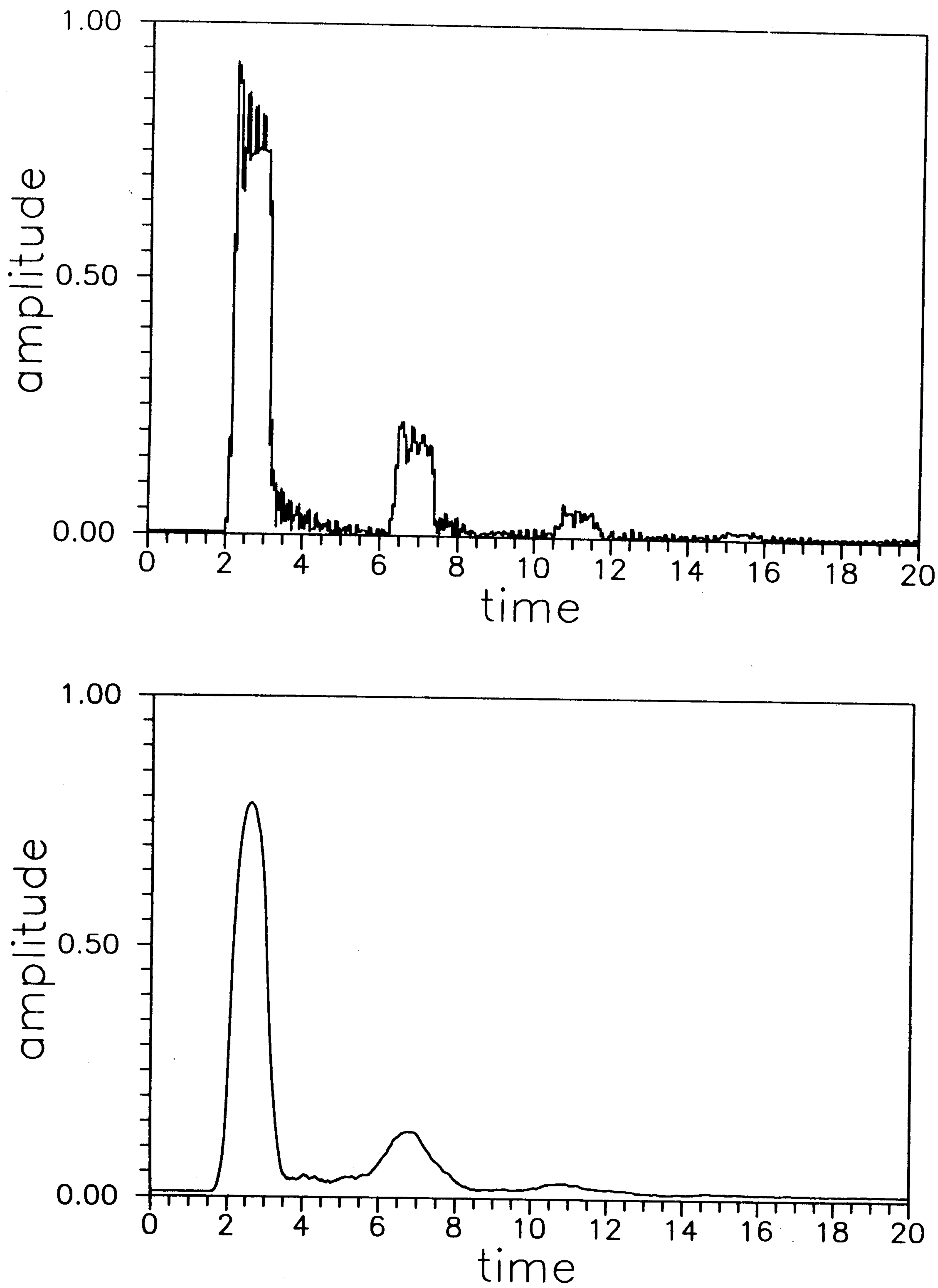

Fig. 2. The pulse transmitted through the bar built of 30 couples of segments; a — the amplitude in the deterministic periodic case, b — the mean value of the amplitude in the case of the bar where the lengths of the segments are random variables

pulses in the deterministic and stochastic cases. The shape of the averaged stochastic wave pulses obtained in computer simulation (Figs 2, 3 b) is the result of a very non-regular structure of the sample-path in the stochastic case where some additional small transmitted wave peaks in between the dominating ones can be observed. It is seen that the averaged stochastic pulse is more regular than deterministic one — due to the elimination rapid changes of the amplitude by averaging. This averaging makes that, in between the dominating pulse, it appears a 'permanent transmitted pulse' of the level tending to zero, when the number of segments in the bar tends to infinity. The averaging makes also that the sharp edges of the transmitted pulses are fuzzy (see Figs 2, 3b).

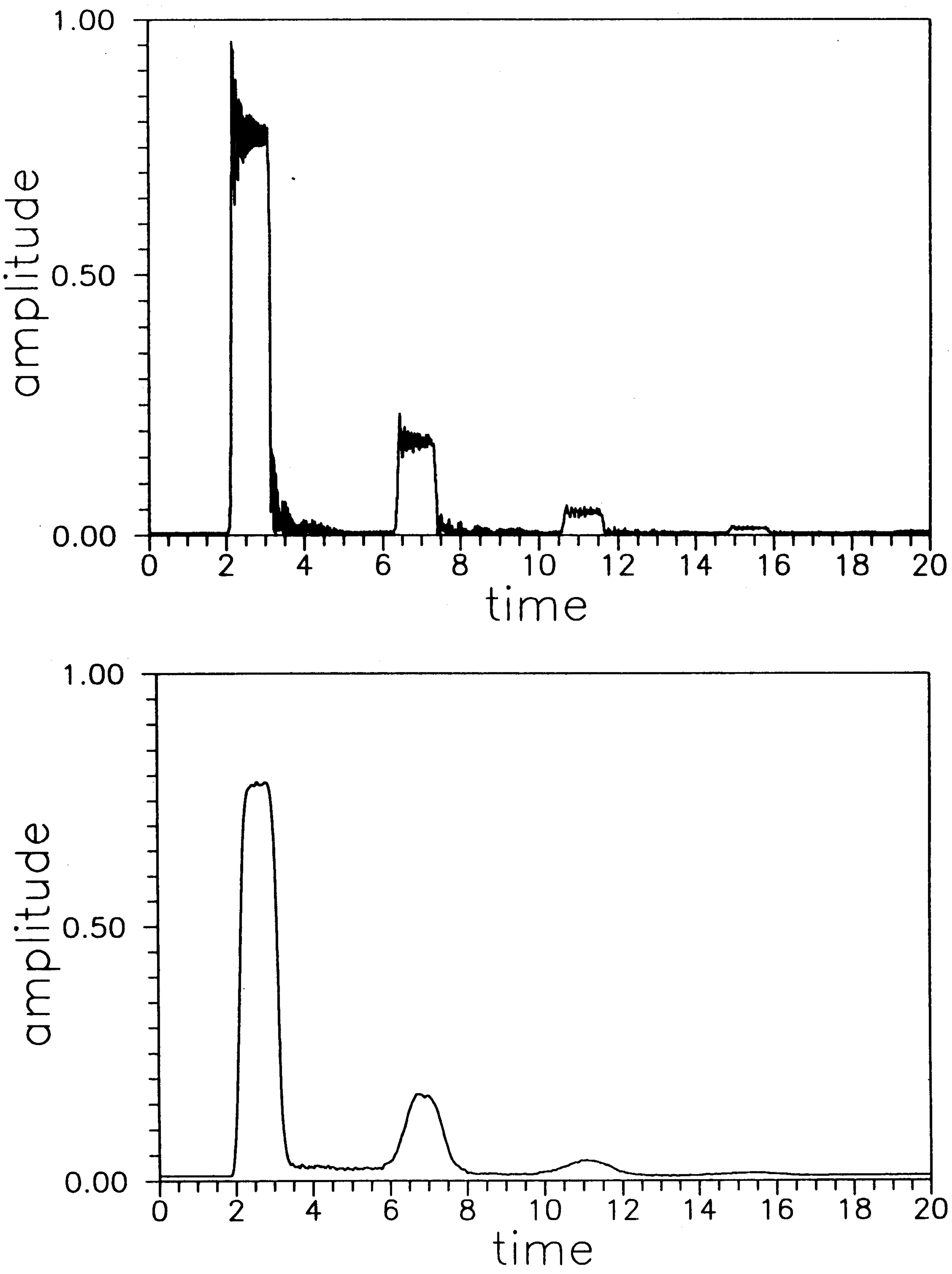

Fig. 3. As Fig. 2 but for 100 couples of segments in the bar

In the above considerations we have studied the averaged properties of the transmitted wave pulses. The obtained equations for the amplitude make it possible to analyse also the individual paths of the wave process. For instance, from Eq. (6) we can observe some particular properties of the pulses paths and under certain assumptions calculate their probabilities. Generation of a strong pulse inside the bar, analogous to the phenomenon studied in case of continuously inhomogeneous stochastic stratified media (see [6]), is an example of such a phenomenon.

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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH, POLISH ACADEMY OF SCIENCES, ŚWIĘTOKRZYSKA 21, PL-00-049 WARSZAWA
(INSTYTUT PODSTAWOWYCH PROBLEMÓW TECHNIKI PAN)

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