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Wave pulses in one-dimensional randomly defected thermoelastic media

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IN THE PAPER the propagation of wave pulses in a one-dimensional thermoelastic medium is considered. The problem is described by the transition matrix method. The transition matrix for such a wave problem is obtained and then the equation for the reflected and transmitted wave field is derived. Finally, the effective thermoelastic constants are derived with the use of the law of large numbers for the product of random matrices.

1. Introduction

THERMOMECHANICAL phenomena in elastic media conducting heat are usually modelled by the equations of thermoelasticity (see [31]). Using such a theory, many authors have attempted to solve both the stationary (harmonic) wave problems and non-stationary wave problems (wave pulses). In this short review let us confine the analysis to the problems of one-dimensional (or planar) thermoelastic waves.

In the paper [37] the problem of harmonic wave propagation in a semi-infinite and infinite bar has been considered for various boundary conditions and, consequently, various excitations generating the waves. Analogous problems for the thermoelastic medium have been considered in [31], where the effect of planar mass forces in an infinite space and the effect of planar heat sources acting on the layer was studied.

The non-stationary problems of thermoelasticity are more complicated from the mathematical point of view; since here the additional, time variable appears, they need more involved computations. The most effective method used in such problems is the application of the Fourier transformation with respect to spatial variables and (or) the Laplace transform with respect to time. As an example of the nonstationary thermoelastic problem, we can present the propagation of plane wave generated by a sudden heating of the plane boundary of the half-space (the Danilovskaya problem). Such a problem has been considered in [5, 11, 12, 29]. Another example can be the propagation of the longitudinal wave in infinite and semi-infinite bar, considered in [13, 37]. In the paper by IGNACZAK [13] the method of spatial Fourier transformation of the thermoelastic wave equation was applied to the analysis of the problem.

In the literature, thermomechanical phenomena have been mostly analyzed in the homogeneous spaces. The description of the problem in spatially non-homogeneous case is not simple, because then one must analyze the equations with spatially variable coefficients. However, considering the stratified media we can apply the methods known from the homogeneous media theory, using equations with constant coefficients and applying the suitable continuity conditions. The stratified models can be approximations of the continuously variable media, as well as they can describe physical situations where the stratified scheme is natural, like wave processes in layered soils, defected or compound elements of structures, etc.

The mathematical description of the thermoelastic wave processes in stratified media and the methods of solution of the problems are analogous to the methods of analysis of the purely thermal or elastic problems. In the literature there is a number of papers where such problems are considered. In this short introduction we only mention the most important of them, and give the references to the literature where more information about the state of art can be found.

The problems of heat transmission through multi-layered structures and their interactions with environment are presented, among others, in the papers [2, 6, 7, 32]. The most effective methods proved to be the transition matrix method and the thermal factors method.

The first method of the analysis of heat transfer in layered bodies with periodic boundary conditions is shown in [6, 32].

The thermal factors method, originally presented in [27], is discussed in [7, 30, 15, 16]. Applying this method, the processes of heat transfer through multi-layered walls is analyzed in [17]. Analogously, using the factors method, the computer simulation of heat transmission in the solar wall is performed in [18, 19].

The literature concerning elastic waves propagation in stratified media is very rich (see e.g. [14]). In this presentation we restrict ourselves to the presentation of the transition matrix method, widely applied as the method of analysis of this problem. The earliest papers where the method was used for the surface harmonic waves are [34, 36]. Later the method was applied to SH [9] and P + SV [10] waves. The method proved to be useful for the stochastic models: one-dimensional [39, 3, 20, 21] and two-dimensional [22, 23]. It was also effective in the case of wave pulses propagation in stratified media: deterministic [1, 28] and stochastic [24, 25, 26].

In this paper we apply the transition matrix method to the analysis of the dynamic problem of thermoelastic wave pulse propagation in a randomly segmented one-dimensional medium. Starting from the one-dimensional equations of thermoelasticity ([31]) and the appropriate continuity conditions, we obtain the transfer matrices for the randomly stratified medium. We apply two methods: we use either the Legendre interpolation polynomials, or alternatively, solve the appropriate system of continuity equations. Then we are able to write the solution of the wave problem applying the derived matrices. Finally, using the limit theorem (see [4]), we obtain the equation for the homogenized problem.

2. Governing equations

2.1. The equations of motion

Consider the linear thermoelastic wave propagating in a one-dimensional medium. The equations describing the changes of the displacement of the medium $u(t,x)$ and the temperature fluctuations of the medium $\vartheta(t,x)$ are the following (see [31]):

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - (3\lambda + 2\mu)\alpha\frac{\partial \vartheta}{\partial x}, \tag{2.1}$$

$$\rho c_\varepsilon\frac{\partial \vartheta}{\partial t} = \beta\frac{\partial^2 \vartheta}{\partial x^2} - T_0(3\lambda + 2\mu)\alpha\frac{\partial}{\partial t}\frac{\partial u}{\partial x}, \tag{2.2}$$

where λ, μ — Lamé elastic constants, T_0 — the reference temperature, c_ε — the specific heat at constant strain for the unit of mass, ρ — the density of the material, β — the heat conductivity coefficient, α — the coefficient of linear expansion of the medium.

The system of equations (2.1)–(2.2) can be alternatively written in the form of two following continuity equations:

the principle of the conservation of linear momentum:

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial\sigma}{\partial x}, \tag{2.3}$$

and the heat equation:

$$\rho c_\varepsilon\frac{\partial\vartheta}{\partial t} = \frac{\partial\varphi}{\partial x}, \tag{2.4}$$

where we have introduced, as new variables, the stress σ and the heat flux φ, defined as

$$\sigma = (\lambda + 2\mu)\frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha\vartheta \tag{2.5}$$

and

$$\varphi = \beta\frac{\partial\vartheta}{\partial x} - T_0(3\lambda + 2\mu)\alpha\frac{\partial}{\partial t}u. \tag{2.6}$$

Using Eqs. (2.3)–(2.6) we can write the following system of thermoelastic equations:

$$\frac{\partial u}{\partial x} = \frac{1}{(\lambda + 2\mu)}\sigma + \frac{(3\lambda + 2\mu)\alpha}{(\lambda + 2\mu)}\vartheta, \tag{2.7}$$

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{2.8}$$

$$\frac{\partial\vartheta}{\partial x} = \frac{1}{\beta}\varphi + \frac{T_0(3\lambda + 2\mu)\alpha}{\beta}\frac{\partial u}{\partial t}, \tag{2.9}$$

$$\frac{\partial\varphi}{\partial x} = \rho c_\varepsilon\frac{\partial\vartheta}{\partial t}. \tag{2.10}$$

Introducing new definitions of constants,

$$A_1 = \frac{1}{(\lambda + 2\mu)}, \tag{2.11}$$

$$A_2 = \frac{(3\lambda + 2\mu)\alpha}{(\lambda + 2\mu)}, \tag{2.12}$$

$$A_3 = \rho, \tag{2.13}$$

$$B_1 = \frac{T_0(3\lambda + 2\mu)\alpha}{\beta}, \tag{2.14}$$

$$B_2 = \frac{1}{\beta}, \tag{2.15}$$

$$B_3 = \rho c_\varepsilon, \tag{2.16}$$

we can write the system of equations (2.7)–(2.10) in the following vector-matrix form:

$$\frac{\partial}{\partial x}\begin{bmatrix} u \\ \sigma \\ \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 & A_1 & A_2 & 0 \\ A_3\frac{\partial^2}{\partial t^2} & 0 & 0 & 0 \\ B_1\frac{\partial}{\partial t} & 0 & 0 & B_2 \\ 0 & 0 & B_3\frac{\partial}{\partial t} & 0 \end{bmatrix}\begin{bmatrix} u \\ \sigma \\ \vartheta \\ \varphi \end{bmatrix}. \tag{2.17}$$

Applying the Fourier transforms to the system of equations with respect to the time variable t, according to the following definition:

$$\widehat{s}(\omega) = \int \exp\{-i\omega t\} s(t)\, dt, \tag{2.18}$$

$$\frac{\partial \widehat{s}}{\partial t} = i\omega\widehat{s}, \tag{2.19}$$

we obtain the following system of linear ordinary differential equations:

$$\frac{\partial}{\partial x}\begin{bmatrix} \widehat{u} \\ \widehat{\sigma} \\ \widehat{\vartheta} \\ \widehat{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & A_1 & A_2 & 0 \\ -\omega^2 A_3 & 0 & 0 & 0 \\ i\omega B_1 & 0 & 0 & B_2 \\ 0 & 0 & i\omega B_3 & 0 \end{bmatrix}\begin{bmatrix} \widehat{u} \\ \widehat{\sigma} \\ \widehat{\vartheta} \\ \widehat{\varphi} \end{bmatrix}, \tag{2.20}$$

where the variable of the Fourier transformation ω is now the parameter.

Let us define the matrix of the system of equations (2.20) as

$$\mathcal{A} = \begin{bmatrix} 0 & A_1 & A_2 & 0 \\ -\omega^2 A_3 & 0 & 0 & 0 \\ i\omega B_1 & 0 & 0 & B_2 \\ 0 & 0 & i\omega_3 B_3 & 0 \end{bmatrix}. \tag{2.21}$$

Then the eigenvalues of the matrix $\mathcal{A}$ of the system of equations are the solutions of the characteristic equation

$$\det(A - r\mathbf{Id}) = 0, \tag{2.22}$$

($\mathbf{Id}$ is a 4×4 identity matrix), or explicitly

$$r^4 - r^2\omega[-A_1A_3\omega + (A_2B_1 + B_2B_3)i] - B_2B_3A_1A_3 i\omega^3 = 0. \tag{2.23}$$

The solutions of the characteristic equation are

$$r_{1,2} = \pm\frac{\sqrt{\omega}}{\sqrt{2}}\sqrt{bi - a\omega + \sqrt{a^2\omega^2 - b^2 + 2id\omega}} \tag{2.24}$$

and

$$r_{3,4} = \pm\frac{\sqrt{\omega}}{\sqrt{2}}\sqrt{bi - a\omega - \sqrt{a^2\omega^2 - b^2 + 2id\omega}}, \tag{2.25}$$

where the following notations have been introduced:

$$a = A_1A_3, \tag{2.26}$$

$$b = A_2B_1 + B_2B_3, \tag{2.27}$$

$$d = A_1A_3(A_2B_1 - B_2B_3). \tag{2.28}$$

2.2. Discontinuity surfaces and continuity conditions

The governing equations (2.1)–(2.2) are valid for the homogeneous media, that is for such media where the coefficients in the equations are constants. If the medium is built of several regions where the coefficients are constant, in every region the suitable governing equation of the form (2.1)–(2.2) is valid. If the solution of the wave problem exists in the entire medium, then on the interfaces of the homogeneous subregions (being the surfaces of discontinuity) some continuity conditions must be satisfied. The conditions are: the mechanical variables — displacements (u) and normal stresses (σ), and two thermal variables — temperature (ϑ) and normal heat flux (φ), are continuous across the surface of discontinuity of the medium.

2.3. Excitation, initial and boundary conditions

The complete description of the thermoelastic problem necessitates the governing equation and, additionally, appropriately described excitations acting on the system, and initial and boundary conditions. Usually one defines the mechanical excitations acting on the structure (displacements or stresses) and heat sources distributed in the medium. Analogously, one defines the initial displacements, stresses and the temperature fluctuations over the body.

In this paper we assume there are no external excitations acting on the body in the x-direction. Moreover, we assume that at $t = 0$ the medium is in equilibrium (homogeneous initial conditions).

The wave processes analyzed in the paper are one-dimensional; we consider the wave pulse in the (stratified) slab or bar. The pulse is generated by suitable changes of the boundary conditions. The boundary conditions of thermoelastic problems are of two types: mechanical and thermal. In the literature authors usually assume known displacements or normal stresses (on non-overlapping surfaces of the boundary), and known temperature and normal projections of the heat flux (also on non-overlapping surfaces). Some combinations of the above conditions are also possible.

In our wave problem, since we consider a one-dimensional model, we should assume only one mechanical and one thermal boundary condition on the whole surface of the slab. The wave problem considered requires precise specification of the boundary conditions (being in our model the excitations generating the thermoelastic wave pulse in the slab or bar).

3. Continuity equations and transition matrix

To construct the transition matrix for the solution of Eq. (2.20), we postulate the form of two components of the solution as

$$\widehat{u}(x) = \sum_{i=1}^{4} C_i^u \exp(r_i x), \tag{3.1}$$

$$\widehat{\vartheta}(x) = \sum_{i=1}^{4} C_i^\vartheta \exp(r_i x). \tag{3.2}$$

Then the spatial derivatives of the functions are

$$\frac{d\widehat{u}(x)}{dx} = \sum_{i=1}^{4} C_i^u r_i \exp(r_i x), \tag{3.3}$$

$$\frac{d\widehat{\vartheta}(x)}{dx} = \sum_{i=1}^{4} C_i^\vartheta r_i \exp(r_i x), \tag{3.4}$$

where $r_i, i = 1,2,3,4$ are the eigenvalues of the matrix $\mathcal{A}$ of the system of equations (2.20), given by formulae (2.24), (2.25), and the constants C_i^u and C_i^ϑ, $i = 1,2,3,4$, must be determined from the boundary conditions. Then the value of the solution of Eq. (2.20) at the plane $x = 0$ is

$$\widehat{u}(0) \equiv \widehat{u}_0 = \sum_{i=1}^{4} C_i^u, \tag{3.5}$$

$$\widehat{\vartheta}(0) \equiv \widehat{\vartheta}_0 = \sum_{i=1}^{r} C_i^\vartheta, \tag{3.6}$$

$$\widehat{\sigma}(0) \equiv \widehat{\sigma}_0 = \frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i - \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^\vartheta, \tag{3.7}$$

$$\widehat{\varphi}(0) \equiv \widehat{\varphi}_0 = \frac{1}{B_1}\sum_{i=1}^{4} C_i^\vartheta r_i - \frac{B_2}{B_1} i\omega \sum_{i=1}^{4} C_i^u \tag{3.8}$$

and the value for $x = L$ is

$$\widehat{u}(L) \equiv \widehat{u}_L = \sum_{i=1}^{4} C_i^u \exp(r_i L), \tag{3.9}$$

$$\widehat{\vartheta}(L) \equiv \widehat{\vartheta}_L = \sum_{i=1}^{4} C_i^\vartheta \exp(r_i L), \tag{3.10}$$

$$\widehat{\sigma}(L) \equiv \widehat{\sigma}_L = \frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i \exp(r_i L) - \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^\vartheta \exp(r_i L), \tag{3.11}$$

$$\widehat{\varphi}(L) \equiv \widehat{\varphi}_L = \frac{1}{B_1}\sum_{i=1}^{4} C_i^\vartheta r_i \exp(r_i L) - \frac{B_2}{B_1} i\omega \sum_{i=1}^{4} \exp(r_1 L). \tag{3.12}$$

Our purpose is to express the values of the solution at $x = 0$ in terms of its value at $x = L$. To do this we must eliminate from the equations (3.5)–(3.12) constants C_i^u and C_i^ϑ, $i = 1,2,3,4$. From the above equations we obtain the following matrix equation for the constants C_i^u, $i = 1,2,3,4$:

$$\begin{bmatrix} \widehat{u}_0 \\ \widehat{u}_L \\ \widehat{\sigma}_0 + \dfrac{A_2}{A_1}\widehat{\vartheta}_0 \\ \widehat{\sigma}_L + \dfrac{A_2}{A_1}\widehat{\vartheta}_L \end{bmatrix} = \mathbf{A}^u \begin{bmatrix} C_1^u \\ C_2^u \\ C_3^u \\ C_4^u \end{bmatrix}, \tag{3.13}$$

where matrix $\mathbf{A}^u$ has the following form:

$$\mathbf{A}^u = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \exp(r_1 L) & \exp(r_2 L) & \exp(r_3 L) & \exp(r_4 L) \\ \dfrac{1}{A_1} r_1 & \dfrac{1}{A_1} r_2 & \dfrac{1}{A_1} r_3 & \dfrac{1}{A_1} r_4 \\ \dfrac{1}{A_1} r_1 \exp(r_1 L) & \dfrac{1}{A_1} r_2 \exp(r_2 L) & \dfrac{1}{A_1} r_3 \exp(r_3 L) & \dfrac{1}{A_1} r_4 \exp(r_4 L) \end{bmatrix}. \tag{3.14}$$

Now we can obtain the following formula for the coefficients C_i^u:

$$\begin{bmatrix} C_1^u \\ C_2^u \\ C_3^u \\ C_4^u \end{bmatrix} = [\mathbf{A}^u]^{-1} \begin{bmatrix} \widehat{u}_0 \\ \widehat{u}_L \\ \widehat{\sigma}_0 + \dfrac{A_2}{A_1}\widehat{\vartheta}_0 \\ \widehat{\sigma}_L + \dfrac{A_2}{A_1}\widehat{\vartheta}_L \end{bmatrix}. \tag{3.15}$$

Analogously we can obtain the following matrix equation for the constants C_i^ϑ, $i = 1,2,3,4$:

$$\begin{bmatrix} \widehat{\vartheta}_0 \\ \widehat{\vartheta}_L \\ \widehat{\varphi}_0 + \dfrac{B_2}{B_1} i\omega\widehat{u}_0 \\ \widehat{\varphi}_L + \dfrac{B_2}{B_1} i\omega\widehat{u}_L \end{bmatrix} = \mathbf{A}^\vartheta \begin{bmatrix} C_1^\vartheta \\ C_2^\vartheta \\ C_3^\vartheta \\ C_4^\vartheta \end{bmatrix}, \tag{3.16}$$

where matrix $\mathbf{A}^\vartheta$ has the form

$$\mathbf{A}^\vartheta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \exp(r_1 L) & \exp(r_2 L) & \exp(r_3 L) & \exp(r_4 L) \\ \dfrac{1}{B_1} r_1 & \dfrac{1}{B_1} r_2 & \dfrac{1}{B_1} r_3 & \dfrac{1}{B_1} r_4 \\ \dfrac{1}{B_1} r_1 \exp(r_1 L) & \dfrac{1}{B_1} r_2 \exp(r_2 L) & \dfrac{1}{B_1} r_3 \exp(r_3 L) & \dfrac{1}{B_1} r_4 \exp(r_4 L) \end{bmatrix}. \tag{3.17}$$

Now we can obtain the following formula for the coefficients C_i^{ϑ}:

$$
\begin{bmatrix} C_1^{\vartheta} \\ C_2^{\vartheta} \\ C_3^{\vartheta} \\ C_4^{\vartheta} \end{bmatrix} = [\mathbf{A}^{\vartheta}]^{-1} \begin{bmatrix} \widehat{\vartheta}_0 \\ \widehat{\vartheta}_L \\ \widehat{\varphi}_0 + \dfrac{B_2}{B_1} i\omega \widehat{u}_0 \\ \widehat{\varphi}_L + \dfrac{B_2}{B_1} i\omega \widehat{u}_L \end{bmatrix}. \tag{3.18}
$$

The formulae for the derivatives are

$$
\left.\frac{d\widehat{\sigma}}{dx}\right|_{x=0} = -A_3\omega^2\widehat{u}_0 = \frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i^2 - \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i, \tag{3.19}
$$

$$
\left.\frac{d\widehat{\sigma}}{dx}\right|_{x=L} = -A_3\omega^2\widehat{u}_L = \frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i^2 \exp(r_i L) - \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i \exp(r_i L), \tag{3.20}
$$

$$
\left.\frac{d\widehat{\varphi}}{dx}\right|_{x=0} = B_3 i\omega\widehat{\vartheta}_0 = \frac{1}{B_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i^2 - \frac{B_2}{B_1}\sum_{i=1}^{4} C_i^u r_i, \tag{3.21}
$$

$$
\frac{d\widehat{\varphi}}{dx}|_{x=L} = B_3 i\omega\widehat{\vartheta}_L = \frac{1}{B_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i^2 \exp(r_i L) - \frac{B_2}{B_1}\sum_{i=1}^{4} C_i^u r_i \exp(r_i L). \tag{3.22}
$$

Now we can obtain the expressions for the functions sought for at the front plane $x = 0$,

$$
\widehat{u}_0,\ \widehat{\sigma}_0,\ \widehat{\vartheta}_0,\ \widehat{\varphi}_0, \tag{3.23}
$$

under the given values of the functions at the rear plane $x = L$:

$$
\widehat{u}_L,\ \widehat{\sigma}_L,\ \widehat{\vartheta}_L,\ \widehat{\varphi}_L. \tag{3.24}
$$

The formulae constitute the following system of linear algebraic equations:

$$
A_3\omega^2\widehat{u}_0 = -\frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i^2 + \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i, \tag{3.25}
$$

$$
A_3\omega^2\widehat{u}_L = -\frac{1}{A_1}\sum_{i=1}^{4} C_i^u r_i^2 \exp(r_1 L) + \frac{A_2}{A_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i \exp(r_i L), \tag{3.26}
$$

$$
B_3 i\omega\widehat{\vartheta}_0 = \frac{1}{B_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i^2 - \frac{B_2}{B_1}\sum_{i=1}^{4} C_i^u r_i, \tag{3.27}
$$

$$
B_3 i\omega\widehat{\vartheta}_L = \frac{1}{B_1}\sum_{i=1}^{4} C_i^{\vartheta} r_i^2 \exp(r_i L) - \frac{B_2}{B_1}\sum_{i=1}^{4} r_i \exp(r_i L). \tag{3.28}
$$

where the constants C_i^u and C_i^{ϑ} are the solutions of Eq. (3.13) and (3.16), respectively.

The above system of equations can be written in the following matrix form:

$$
\begin{bmatrix} \widehat{u}_0 \\ \widehat{\sigma}_0 \\ \widehat{\vartheta}_0 \\ \widehat{\varphi}_0 \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} \widehat{u}_L \\ \widehat{\sigma}_L \\ \widehat{\vartheta}_L \\ \widehat{\varphi}_L \end{bmatrix}, \tag{3.29}
$$

where matrix $\mathbf{T}^{-1}$ is the inverse of the transition matrix for a single layer. It can be represented in the following form:

$$\mathbf{T}^{-1} = [\mathbf{P}]^{-1}\mathbf{Q}, \tag{3.30}$$

where matrices **P** and **Q** are defined by the formulae

$$\mathbf{P} = [\mathbf{P}_{ij}], \quad i,j = 1,2,3,4, \tag{3.31}$$

$$\mathbf{P}_{11} = -A_3\omega^2 - \frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^u_{k1}r_k^2 + \frac{A_2}{A_1}\frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k3}r_k, \tag{3.32}$$

$$\mathbf{P}_{21} = -\frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^u_{k1}r_k^2\exp(r_kL) + \frac{A_2}{A_1}\frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k3}r_k\exp(r_kL), \tag{3.33}$$

$$\mathbf{P}_{31} = -\frac{B_2}{b_1^2}i\omega\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k3}r_k^2 + \frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^u_{k1}r_k, \tag{3.34}$$

$$\mathbf{P}_{41} = -\frac{B_2}{B_1^2}i\omega\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k3}r_k^2\exp(t_kL) + \frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^u_{k1}r_k\exp(r_kL), \tag{3.35}$$

$$\mathbf{P}_{12} = -\frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2, \tag{3.36}$$

$$\mathbf{P}_{22} = -\frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2\exp(r_kL), \tag{3.37}$$

$$\mathbf{P}_{32} = \frac{B_2}{B_1}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2, \tag{3.38}$$

$$\mathbf{P}_{42} = \frac{B_2}{B_1}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2\exp(r_kL), \tag{3.39}$$

$$\mathbf{P}_{13} = -\frac{A_2}{A_1^2}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2 + \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k1}r_k, \tag{3.40}$$

$$\mathbf{P}_{23} = -\frac{A_2}{A_1^2}\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k^2\exp(r_kL) + \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k1}r_k\exp(r_kL), \tag{3.41}$$

$$\mathbf{P}_{33} = B_3i\omega - \frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k1}r_k^2 + \frac{A_2}{A_1}\frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k, \tag{3.42}$$

$$\mathbf{P}_{43} = -\frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k1}r_k^2\exp(r_kL) + \frac{A_2}{A_1}\frac{B_2}{B_1}i\omega\sum_{k=1}^{4}\mathbf{A}^u_{k3}r_k\exp(r_kL), \tag{3.43}$$

$$\mathbf{P}_{14} = \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^\vartheta_{k3}r_k^2, \tag{3.44}$$

$$\mathbf{P}_{24} = \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k3} r_k^2 \exp(r_k L), \tag{3.45}$$

$$\mathbf{P}_{34} = -\frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k3} r_k^2, \tag{3.46}$$

$$\mathbf{P}_{44} = -\frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k3} r_k^2 \exp(r_k L) \tag{3.47}$$

and

$$\mathbf{Q} = [\mathbf{Q}_{ij}], \qquad i,j = 1,2,3,4, \tag{3.48}$$

$$\mathbf{Q}_{11} = \frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k2} r_k^2 - \frac{A_2}{A_1}\frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k4} r_k, \tag{3.49}$$

$$\mathbf{Q}_{21} = A_3\omega^2 + \frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k2} r_k^2 \exp(r_k L) - \frac{A_2}{A_1}\frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k4} r_k \exp(r_k L), \tag{3.50}$$

$$\mathbf{Q}_{31} = \frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k4} r_k^2 - \frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{u}_{k2} r_k, \tag{3.51}$$

$$\mathbf{Q}_{41} = \frac{B_2}{B_1^2} i\omega \sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k4} r_k^2 \exp(r_k L) - \frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{u}_{k2} r_k \exp(r_k L), \tag{3.52}$$

$$\mathbf{Q}_{12} = \frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2, \tag{3.53}$$

$$\mathbf{Q}_{22} = \frac{1}{A_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2 \exp(r_k L), \tag{3.54}$$

$$\mathbf{Q}_{32} = -\frac{B_2}{B_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2, \tag{3.55}$$

$$\mathbf{Q}_{42} = -\frac{B_2}{B_1}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2 \exp(r_k L), \tag{3.56}$$

$$\mathbf{Q}_{13} = \frac{A_2}{A_1^2}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2 - \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k2} r_k, \tag{3.57}$$

$$\mathbf{Q}_{23} = \frac{A_2}{A_1^2}\sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k^2 \exp(r_k L) - \frac{A_2}{A_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k2} r_k \exp(r_k L), \tag{3.58}$$

$$\mathbf{Q}_{33} = \frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k2} r_k^2 - \frac{A_2}{A_1}\frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k, \tag{3.59}$$

$$\mathbf{Q}_{43} = -B_3 i\omega + \frac{1}{B_1}\sum_{k=1}^{4}\mathbf{A}^{\vartheta}_{k2} r_k^2 \exp(r_k L) - \frac{A_2}{A_1}\frac{B_2}{B_1} i\omega \sum_{k=1}^{4}\mathbf{A}^{u}_{k4} r_k \exp(r_k L), \tag{3.60}$$

(3.61) $$\mathbf{Q}_{14} = -\frac{A_2}{A_1}\sum_{k=1}^{4} \mathbf{A}^{\vartheta}_{k4} r_k^2,$$

(3.62) $$\mathbf{Q}_{24} = -\frac{A_2}{A_1}\sum_{k=1}^{4} \mathbf{A}^{\vartheta}_{k4} r_k^2 \exp(r_k L),$$

(3.63) $$\mathbf{Q}_{34} = \frac{1}{B_1}\sum_{k=1}^{4} \mathbf{A}^{\vartheta}_{k4} r_k^2,$$

(3.64) $$\mathbf{Q}_{44} = \frac{1}{B_1}\sum_{k=1}^{4} \mathbf{A}^{\vartheta}_{k4} r_k^2 \exp(r_k L).$$

4. The Legendre polynomial and transition matrix

In this section we present the alternative method of calculating the transition matrix through a single layer of thermoelastic material. The method is based on the spectral representation of the matrix and the Legendre interpolation formula.

The transition matrix through a single layer of thickness L is the solution (taken at point L) of the ordinary differential equation of the form

(4.1) $$\frac{d}{dx}\mathbf{T} = \mathcal{A}\mathbf{T},$$

with the initial condition

(4.2) $$\mathbf{T}(0) = \mathbf{Id},$$

where $\mathcal{A}$ is the matrix of the system of equations defined in (2.21). The solution of Eq. (4.1) can be represented by

(4.3) $$\mathbf{T}(L) = \exp(\mathcal{A}L).$$

Since the eigenvalues of the matrix A are known, we can calculate its exponent using the Legendre interpolation polynomial method (see [35]):

(4.4) $$\exp\{\mathcal{A}L\} = \frac{(\mathcal{A} - r_2\mathbf{Id})(\mathcal{A} - r_3\mathbf{Id})(\mathcal{A} - r_4\mathbf{Id})}{(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)} e^{r_1 L} + \frac{(\mathcal{A} - r_1\mathbf{Id})(\mathcal{A} - r_3\mathbf{Id})(A - r_4\mathbf{Id})}{(r_2 - r_1)(r_2 - r_3)(r_2 - r_4)} e^{r_2 L}$$
$$+ \frac{(\mathcal{A} - r_1\mathbf{Id})(\mathcal{A} - r_2\mathbf{Id})(\mathcal{A} - r_4\mathbf{Id})}{(r_3 - r_1)(r_3 - r_2)(r_3 - r_4)} e^{r_3 L} + \frac{(\mathcal{A} - r_1\mathbf{Id})(\mathcal{A} - r_2\mathbf{Id})(\mathcal{A} - r_3\mathbf{Id})}{(r_4 - r_1)(r_4 - r_2)(r_4 - r_3)} e^{r_4 L}.$$

Substituting the matrix $\mathcal{A}$ in Eq. (4.4) and performing calculations, we obtain the explicit expressions for the transition matrix $\mathbf{T}$:

$$\mathbf{T}_{11}(L) = (\operatorname{ch} r_3 L(r_1^2 - i\omega A_2 B_1 + \omega^2 A_1 A_3) + \operatorname{ch} r_1 L(-r_3^2 + i\omega A_2 B_1 - \omega^2 A_1 A_3))/(PD\omega),$$

(4.5) $$\mathbf{T}_{12}(L) = (r_3 \operatorname{sh} r_1 L(-r_3^2 - \omega^2 A_1 A_3 + i\omega A_2 B_1) + r_1 \operatorname{sh} r_3 L(r_1^2 + \omega^2 A_1 A_3 - i\omega A_2 B_1))A_1/(r_1 r_3 PD\omega),$$

$$\mathbf{T}_{13}(L) = (r_3 \operatorname{sh} r_1 L(-r_3^2 - \omega^2 A_1 A_3 + i\omega(A_2 B_1 + B_2 B_3))$$

(4.5) [cont.]
$$+ r_1 \operatorname{sh} r_3L(r_1^2 + \omega^2 A_1A_3 - i\omega(A_2B_1 + B_2B_3)))A_2/(r_1r_3PD\omega),$$

$$\mathrm{T}_{14}(L) = (\operatorname{ch} r_1L - \operatorname{ch} r_3L)A_2B_2/(PD\omega),$$
$$\mathrm{T}_{21}(L) = (r_3 \operatorname{sh} r_1L(r_3^2 + \omega^2 A_1A_3 - i\omega A_2B_1) + r_1 \operatorname{sh} r_3L(-r_1^2 - \omega^2 A_1A_3 + i\omega A_2B_1))A_3\omega/(r_1r_3PD),$$
$$\mathrm{T}_{22}(L) = (\operatorname{ch} r_3L(r_1^2 + \omega^2 A_1A_3) - \operatorname{ch} r_1L(r_3^2 + \omega^2 A_1A_3))/(PD\omega),$$
$$\mathrm{T}_{23}(L) = (-\operatorname{ch} r_1L + \operatorname{ch} r_3L)\omega A_2A_3/PD,$$
$$\mathrm{T}_{24}(L) = (-r_3 \operatorname{sh} r_1L + r_1 \operatorname{sh} r_3L)\omega A_2A_3B_2/(r_1r_3PD),$$
$$\mathrm{T}_{31}(L) = (-r_3 \operatorname{sh} r_1L(ir_3^2 + i\omega^2 A_1A_3 + \omega(A_2B_1 + B_2B_3)) + r_1 \operatorname{sh} r_3L(ir_1^2 + i\omega^2 A_1A_3 + \omega(A_2B_1 + B_2B_3)))B_1/(r_1r_3PD),$$
$$\mathrm{T}_{32}(L) = (\operatorname{ch} r_1L - \operatorname{ch} r_3L)iA_1B_1/PD,$$
$$\mathrm{T}_{33}(L) = (\operatorname{ch} r_3L(r_1^2 - i\omega(A_2B_1 + B_2B_3)) - \operatorname{ch} r_1L(r_3^2 - i\omega(A_2B_1 + B_2B_1)))/(PD\omega),$$
$$\mathrm{T}_{34}(L) = (r_3 \operatorname{sh} r_1L(-r_3^2 + i\omega(A_2B_1 + B_2B_3)) + r_1 \operatorname{sh} r_3L(r_1^2 - i\omega(A_2B_1 + B_2B_3)))B_2/(r_1r_3PD\omega),$$
$$\mathrm{T}_{41}(L) = (-\operatorname{ch} r_1L + \operatorname{ch} r_3L)\omega B_1B_3/PD,$$
$$\mathrm{T}_{42}(L) = (-r_3 \operatorname{sh} r_1L + r_1 \operatorname{sh} r_3L)\omega A_1B_1B_3/(r_1r_3PD),$$
$$\mathrm{T}_{43}(L) = (-r_3 \operatorname{sh} r_1L(r_3^2 i + \omega(A_2B_1 + B_2B_3)) + r_1 \operatorname{sh} r_3L(r_1^2 i + \omega(A_2B_1 + B_2B_3)))B_3/(r_1r_3PD),$$
$$\mathrm{T}_{44}(L) = (\operatorname{ch} r_3L(r_1^2 - i\omega B_2B_3) - \operatorname{ch} r_1L(r_3^2 - i\omega B_2B_3))/(PD\omega),$$

where ch, sh are the hyperbolic cosine and sine functions,

(4.6)
$$\operatorname{ch} x = \frac{e^x + e^{-x}}{2},$$

(4.7)
$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

and

(4.8)
$$PD = \sqrt{a^2\omega^2 - b^2 + 2i\omega d} = \sqrt{A_1^2A_3^2\omega^2 - (A_2B_1 + B_2B_3)^2 + 2i\omega A_1A_3(A_2B_1 - B_2B_3)},$$

r_i, $i = 1,2,3,4$ are defined in (2.24), (2.25), a, b, d are defined in (2.26)–(2.28), and the constants A_i, B_i, $i = 1,2,3$, are defined in (2.11)–(2.16).

5. The layered thermoelastic medium

The transition matrices obtained in the previous section enable us to describe the passage of the thermoelastic wave through a multi-layered medium. In such a case, knowing the transition matrices through individual layers, we can obtain the transition matrix through all the stratified medium as the product of the matrices. The transition matrix

$\mathbf{T}(.)$ enables us to express the wave field $\mathbf{U}$,

(5.1)
$$\mathbf{U} = \begin{bmatrix} \widehat{u} \\ \widehat{\sigma} \\ \widehat{\vartheta} \\ \widehat{\varphi} \end{bmatrix},$$

at any point x in a homogeneous medium, provided the boundary condition $\mathbf{U}_0 = \mathbf{U}(0)$ at $x = 0$ is known, in the following form:

(5.2)
$$\mathbf{U}(x) = \mathbf{T}(x)\mathbf{U}_0.$$

Consider now the multi-layered medium (slab) consisting of N layers of thermoelastic materials, with thicknesses $\triangle_j, j = 1, 2, \ldots, N$. Assume that the stratified medium is surrounded by a homogeneous thermoelastic environment, located at $x < 0$ and $x > d = \sum_{i=1}^{N} \triangle_j$. Since the wave field $\mathbf{U}$ must be continuous at the interfaces of the layers in the stratified medium, we can express the wave satisfying some boundary conditions $\mathbf{U}_0$ at $x = 0$, after it reaches the plane $x = d$:

(5.3)
$$\mathbf{U}(d) = \mathbf{T}_N(\triangle_N)\mathbf{T}_{N-1}(\triangle_{N-1})\ldots\mathbf{T}_2(\triangle_2)\mathbf{T}_1(\triangle_1)\mathbf{U}_0,$$

or, in a more compact form:

(5.4)
$$\mathbf{U}(d) = \prod_{j=1}^{N} \mathbf{T}_j(\triangle_j)\mathbf{U}_0.$$

In the above equation all the material properties of the multi-layered medium are completely described by the 4×4 matrix $\mathcal{T}$, being the product of the transition matrices through the individual layers and interpreted as the transition matrix through the slab built of N layers of homogeneous materials,

(5.5)
$$\mathcal{T} = \prod_{j=1}^{N} \mathbf{T}_j(\triangle_j).$$

Let us remark that the boundary condition $\mathbf{U}_0$ represents jointly the initial wave pulse, going along the positive direction x and measured at $x = 0$, as well as all the pulses generated due to multiple reflections and transmissions of the initial wave pulse at the surfaces of discontinuity inside the slab, going in the opposite direction and also measured at $x = 0$. The vector $\mathbf{U}(d)$ represents all the transmitted wave pulses generated inside the stratified slab, going to plus infinity and measured at $x = d$.

6. The limiting case — homogenization

Assume that the slab is built of $2K$ layers of the thicknesses $l_1(\gamma), l_2(\gamma), \ldots, l_{2K}(\gamma)$, where $l_i(\gamma), i = 1, 2, \ldots, 2K$ are random variables. In the above $\gamma \in \Gamma$ is an elementary event and (Γ, F, P) is the complete probabilistic space (cf. [38]). Assume additionally that the material parameters of the layers and their thicknesses ($\rho_{2j-1}(\gamma)$, $\lambda_{2j-1}(\gamma)$, $\mu_{2j-1}(\gamma)$, $\alpha_{2j-1}(\gamma)$, $\beta_{2j-1}(\gamma)$, $c_{\varepsilon,2j-1}(\gamma)$, $l_{2j-1}(\gamma)$, $\rho_{2j}(\gamma)$, $\lambda_{2j}(\gamma)$, $\mu_{2j}(\gamma)$, $\alpha_{2j}(\gamma)$, $\beta_{2j}(\gamma)$, $c_{\varepsilon,2j}(\gamma)$, $l_{2j}(\gamma)$) are, as the vector random variables, independent and identically distributed for $j = 1, 2, \ldots, K$. Moreover, we assume that the thicknesses of the layers have the following particular property:

$$(l_{2j-1}(\gamma), l_{2j}(\gamma)) = \left(\frac{L_{2j-1}(\gamma)}{2K}, \frac{L_{2j}(\gamma)}{2K}\right), \tag{6.1}$$

for $j = 1, 2, \ldots, K$, are independent, identically distributed two-dimensional random variables with the given mean values:

$$E\{L_{2j-1}(\gamma)\} = L^1, \quad E\{L_2(\gamma)\} = L^2, \tag{6.2}$$

independent of j. In this particular case the periodically repeated segments of the bar are built of the couples of elements with the lengths $l_{2j-1}(\gamma)$, $l_{2j}(\gamma)$, $j = 1, 2, \ldots, K$. For such segments the transition matrices $\mathbf{M}_j(\gamma)$ are the products of the pairs of the transition matrices through the individual layers

$$\mathbf{M}_j(\gamma) = \mathbf{T}_{2j-1}(l_{2j-1}(\gamma))\mathbf{T}_{2j}(l_{2j}(\gamma)), \quad j = 1, 2, \ldots, K, \tag{6.3}$$

and Eq. (5.4) for the Fourier transform of the amplitudes takes the following form ($2K = N$):

$$\mathbf{U}(d) = \prod_{j=1}^{K} \mathbf{M}_j(\gamma)\mathbf{U}_0, \tag{6.4}$$

where $d = d(\gamma) = \sum_{j=1}^{N} l_j(\gamma)$.

To study the asymptotic behavior of the randomized equation for the amplitudes of the waves we apply the law of large numbers for the products of random matrices obtained in [4]. This theorem can be written in the following form:

Consider the sequence of the products of real random matrices

$$\mathbf{P}_K(\gamma) = \prod_{j=1}^{K} \mathbf{M}_{j,K}(\gamma). \tag{6.5}$$

It is assumed that for K tending to infinity the matrices $\mathbf{M}_{j,K}$ can be represented by

$$\mathbf{M}_{j,K}(\gamma) = \mathbf{Id} + \frac{1}{K}\mathbf{B}_{j,K}(\gamma) + \mathbf{R}_j(K, \gamma), \tag{6.6}$$

where $\mathbf{B}_{j,K}(\gamma)$ for $j = 1, 2, \ldots, K$, are independent, identically distributed random matrices, integrable with respect to probability measure $\mathcal{P}$ and $|\mathbf{R}_j(K, \gamma)| = o(K^{-1})$ for large K. Under these conditions the law of large numbers takes place and

$$\lim_{K\to\infty} \mathbf{P}_K(\gamma) = \exp(E\{\mathbf{B}_{j,K}(\gamma)\}), \tag{6.7}$$

in the sense of convergence in distribution of all the vectors obtained by multiplication of the random matrix by an arbitrary deterministic vector.

To analyze the limit case of Eq. (6.4) when K tends to infinity, we decompose, at the beginning, the transition matrix defined in (4.5) under the assumption (6.1) on the thickness of the layers, with respect to the powers of $1/K$:

$$\mathbf{T}_j\left(\frac{L_j}{K}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{L_j}{K}\begin{bmatrix} 0 & A_{1,j} & A_{2,j} & 0 \\ -\omega^2 A_{3,j} & 0 & 0 & 0 \\ i\omega B_{1,j} & 0 & 0 & B_{2,j} \\ 0 & 0 & i\omega B_{3,j} & 0 \end{bmatrix} + o\left(\frac{L_j}{K}\right). \tag{6.8}$$

Multiplying the matrices $\mathbf{T}_1(L_1)$, corresponding to the transition matrices with odd indices and $\mathbf{T}_2(L_2)$ — even indices we obtain that the matrices $\mathbf{B}_j$ required in formula (6.7) are given by (we have changed the numbering of the random variables being the material parameters and the thicknesses of the layers according to the following rule: $p_{2j-1} = p^1_j$, $p_{2j} = p^2_j$ for any parameter p and $j = 1, 2, \ldots, K$, so the parameters with identical superscripts — 1 or 2 — have the identical distribution):

(6.9) $$\mathbf{B}_j = \\ = \begin{bmatrix} 0 & A^1_{1,j}L^1_j + A^2_{1,j}L^2_j & A^1_{2,j}L^1_j + A^2_{2,j}L^2_j & 0 \\ -\omega^2(A^1_{3,j}L^1_j + A^2_{3,j}L^2_j) & 0 & 0 & 0 \\ i\omega(B^1_{1,j}L^1_j + B^2_{1,j}L^2_j) & 0 & 0 & B^1_{2,j}L^1_j + B^2_{2,j}L^2_j \\ 0 & 0 & i\omega(B^1_{3,j}L^1_j + B^2_{3,j}L^2_j) & 0 \end{bmatrix}.$$

The common average value of the matrices $\mathbf{B}_j$ is

(6.10) $$E\{\mathbf{B}_j\} = \\ = \begin{bmatrix} 0 & E\{A^1_1L^1 + A^2_1L^2\} & E\{A^1_2L^1 + A^2_2L^2\} & 0 \\ -\omega^2E\{A^1_3L^1 + A^2_3L^2\} & 0 & 0 & 0 \\ i\omega E\{B^1_1L^1 + B^2_1L^2\} & 0 & 0 & E\{B^1_2L^1 + B^2_2L^2\} \\ 0 & 0 & i\omega E\{B^1_3L^1 + B^2_3L^2\} & 0 \end{bmatrix}.$$

Here the parameters and the thicknesses under the expectation are the random variables with the distribution common for all couples of layers.

The matrix $e^{E\{\mathbf{B}_j\}}$ is of the form analogous to (4.5), where instead of the parameters $A_1(\gamma)$, $A_2(\gamma)$, $A_3(\gamma)$, $B_1(\gamma)$, $B_2(\gamma)$, $B_3(\gamma)$, being random variables, one has the effective constant parameters $A^{\rm eff}_1$, $A^{\rm eff}_2$, $A^{\rm eff}_3$, $B^{\rm eff}_1$, $B^{\rm eff}_2$, $B^{\rm eff}_3$, defined as:

(6.11) $$A^{\rm eff}_1 = \frac{E\{A^1_1(\gamma)L^1(\gamma) + A^2_1(\gamma)L^2(\gamma)\}}{d},$$

(6.12) $$A^{\rm eff}_2 = \frac{E\{A^1_2(\gamma)L^1(\gamma) + A^2_2(\gamma)L^2(\gamma)\}}{d},$$

(6.13) $$A^{\rm eff}_3 = \frac{E\{A^1_3(\gamma)L^1(\gamma) + A^2_3(\gamma)L^2(\gamma)\}}{d},$$

(6.14) $$B^{\rm eff}_1 = \frac{E\{B^1_1(\gamma)L^1(\gamma) + B^2_1(\gamma)L^2(\gamma)\}}{d},$$

(6.15) $$B^{\rm eff}_2 = \frac{E\{B^1_2(\gamma)L^1(\gamma) + B^2_2(\gamma)L^2(\gamma)\}}{d},$$

(6.16) $$B^{\rm eff}_3 = \frac{E\{B^1_3(\gamma)L^1(\gamma) + B^2_3(\gamma)L^2(\gamma)\}}{d},$$

where

(6.17) $$d = L^1 + L^2.$$

Summarizing this section we can say that the effective (homogenized) medium is also thermoelastic, with the material parameters defined in (6.11)–(6.16). It is seen that the statistical relations between the constants inside each layer make the form of the effective material parameters rather complicated.

7. Closing remarks

The formulae for the transition matrices in the thermoelastic wave problem make it possible to analyze certain wave problems. To define them accurately, one must assume the particular kind of excitation at the front boundary of the stratified medium. The excitation is included in the boundary condition $\mathbf{U}_0$, in the matrix evolution equation (5.4). Since the boundary condition contains also the reflected wave pulses, one must separate two waves: the ones going to the right (excitation), and to the left (reflected pulses). This is possible by postulating the specific form of the solution of the equation analogous to (3.1)–(3.2). Then the terms with positive exponents represent the waves going to the left, while the terms with negative ones — that going to the right (excitation). Assuming the known excitation, we can obtain from (5.4) the system of algebraic equations for the coefficients, what makes it possible to calculate the amplitudes of the reflected and transmitted waves. Substituting them into the formulae analogous to (3.1)–(3.2) and calculating the inverse Fourier transforms (using for example the Fast Fourier Transform numerical algorithm — see [33]), one can obtain the shape of the reflected and transmitted waves. The analysis of some particular thermoelastic wave problems will be the subject of our future research.

The wave problem considered in this paper is an example of a wide class of dynamic problems that can be called the coupled field problems. The applied transition matrix method proved to be very effective in the analysis of elastic, thermal and thermoelastic wave problems. The method seems to be applicable to more complicated coupled field problems, for example those including electromagnetic effects.

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