PRL **93,** 234101 (2004) PHYSICAL REVIEW LETTERS week ending 3 DECEMBER 2004

Finite-Space Lyapunov Exponents and Pseudochaos

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(Received 15 March 2004; published 1 December 2004)

We propose a definition of finite-space Lyapunov exponent. For discrete-time dynamical systems, it measures the local (between neighboring points) average spreading of the system. We justify our definition by showing that, for large classes of chaotic maps, the corresponding finite-space Lyapunov exponent approaches the Lyapunov exponent of a chaotic map when $M \to \infty$, where M is the cardinality of the discrete phase space. In analogy with continuous systems, we say the system has pseudochaos if its finite-space Lyapunov exponent tends to a positive number (or to $+\infty$), when $M \to \infty$.

DOI: 10.1103/PhysRevLett.93.234101 PACS numbers: 05.45.Mt, 05.45.Pq

Introduction.—The question has been asked by physicists on several occasions, What is chaos in discrete-space systems? First, when discussing the role of chaos in quantum mechanics and the correspondence of classical and quantum dynamics [1,2]. Second, when chaotic systems are implemented on finite-state machines (digital computers), for example, as in the numerical simulation of difference and/or differential equations.

The concept of pseudochaos has been introduced in attempts to interpret quantum chaos, and to understand its mechanism and physical meaning [3]. The pseudochaos is defined as a statistical behavior of the dynamical system with discrete energy and/or frequency spectrum [3]. Chirikov wrote [4]: "From the viewpoint of fundamental physics, the pseudochaos is the only kind of chaos principally possible in physical systems of finite dimensions.... So, in the common philosophy of the universal quantum mechanics the pseudochaos is the only true dynamical chaos. The classical chaos is but a limiting pattern which is, nevertheless, very important both in the theory to compare with the real (quantum) chaos and in applications as a very good approximation in macroscopic domain as is the whole classical mechanics." The main aim of this Letter is to propose a plausible definition of pseudochaos using similar tools as for (classical) chaos. The existence of the horseshoe is a fingerprint of chaos in continuous-space systems. In discrete-space systems, however, the existence of a set, on which the map is injection, and for which all periodic orbits are unstable, is a fingerprint of pseudochaos.

The most important particular case of pseudochaos is, of course, quantum chaos. Nevertheless, pseudochaos occurs in classical mechanics as well. The digital computer is a very specific classical "dynamical system": it is an "over-quantized" system [3], meaning that any quantity is discrete, while in quantum mechanics only the product of two conjugated variables are so. Owing to the discreteness, any dynamical trajectory in the computer becomes eventually periodic, the effect well known in the theory and practice of pseudorandom number generators. Periodic approximations in dynamical systems are also considered in the ergodic theory [5], apparently without any relation to pseudochaos.

Chaos is a particular characteristic of motion in continuous spaces. Among many indicators of chaotic motion, positivity of the largest Lyapunov exponent is perhaps the most significant, both in theory [6] and applications [7]. Trajectories in discrete phase spaces are always eventually periodic. In this Letter we first propose a definition of the finite-space Lyapunov exponent. It measures local (between neighboring points) average spreading of the discrete-time dynamical system. Let M be a cardinality of the discrete phase space. We justify our definition by showing that, for large classes of chaotic maps, the corresponding finite-space Lyapunov exponent approaches the Lyapunov exponent of a chaotic map when $M \to \infty$. Then we define pseudochaos in terms of the finite-space Lyapunov exponent in a similar way as for continuous systems: the system is said to be pseudochaotic if its finite-space Lyapunov exponent approaches a positive number (or $+\infty$), when $M \to \infty$.

Finite-space Lyapunov exponent.—Let us consider a map

$$F_M: \{0, 1 \ldots, M-1\} \to \{0, 1 \ldots, M-1\}. \qquad (1)$$

We assume that the map F_M is an injection. Clearly, all trajectories of F_M are periodic; let α_j be a periodic orbit of F_M with period T_j. Since F_M is a bijection, it follows that $\cup_j \alpha_j = \{0, 1 \ldots, M-1\}$ and $\sum_j T_j = M$.

We define the Lyapunov exponent of the map F_M as

$$\lambda_{F_M} = \frac{1}{M}\left[\sum_{i=0}^{M-2} \ln| F_M(i+1) - F_M(i) |\right] + \frac{1}{M} \ln| F_M(M-2) - F_M(M-1) |,$$

where we adapt that the distance between two elements of the set $\{0, 1 \ldots, M-1\}$ is the Euclidean distance between two integers $d_i = | F_M(i+1) - F_M(i) |$. We say that $i \pm 1$ are neighboring points of i. In the above equation all terms measure the divergence of two trajectories evolving in one iteration from two "slightly" different initial conditions: an initial point i and its neighbor $i+1$. Note that in the last term the neighbor of $M-1$ is the point $M-2$. Moreover, if we formally write $F_M(M) \equiv F_M(M-2)$ the last equation can be rewritten in more compact form in the following way:

$$\lambda_{F_M} = \frac{1}{M} \sum_{i=0}^{M-1} \ln | F_M(i+1) - F_M(i) | . \tag{2}$$

Thus, the Lyapunov exponent measures average spreading of the map F_M.

Let

$$\alpha_j = \{a_0^{(j)}, a_1^{(j)} = F_M(a_0^{(j)}), \ldots a_{T_j-1}^{(j)} = F_M(a_{T_j-2}^{(j)})\}$$

be a periodic orbit with period T_j; in another words let $a_0^{(j)} \neq a_1^{(j)} \neq \ldots \neq a_{T_j-1}^{(j)}$ and $F_M^{T_j}(a_0^{(j)}) = a_0^{(j)}$. We define the Lyapunov exponent of the map F_M for the periodic orbit α_j as

$$\lambda_{(F_M, \alpha_j)} = \frac{1}{T_j} \sum_{k=0}^{T_j-1} \ln | F_M(a_k^{(j)} + 1) - F_M(a_k^{(j)}) | . \tag{3}$$

Observe that the Lyapunov exponent of the map F_M can also be rewritten as a weighted sum of the Lyapunov exponents of all periodic orbits:

$$\lambda_{F_M} = \sum_j \frac{T_j}{M} \lambda_{(F_M, \alpha_j)}. \tag{4}$$

Clearly, $0 \leq \lambda_{F_M} \leq \ln(M-1)$. The map with null Lyapunov exponent is $F_M(x) = x$ for each $x \in \{0, 1, \ldots, M-1\}$. The set of all different maps F_M can be divided into equivalent classes, each class having the same Lyapunov exponent.

Example 1: The map F_M defined as $F_M(0) = 0, F_M(1) = 3, F_M(2) = 1, F_M(3) = 2$, and $F_M(x) = x$ for $x = 4, 5, \ldots M-1$ and the map G_M defined as $G_M(35) = 37, G_M(36) = 35, G_M(37) = 36$, and $G_M(x) = x$ for $x = 1, 2, \ldots 34, 38, 39, \ldots M-1$ have both same Lyapunov exponent: $\lambda_{F_M}^{(1)} = \ln 2/M$.

Example 2: Let M be an even number. We define F_M as $F_M(0) = 0$, $F_M(1) = M/2$, $F_M(2) = 1$, $F_M(3) = M/2 + 1, \ldots, F_M(M-2) = M/2 - 1$, $F_M(M-1) = M-1$. The Lyapunov exponent of this map is equal to

$$\lambda_{F_M}^{(2)} = \frac{M/2+1}{M} \ln(M/2) + \frac{M/2-1}{M} \ln(M/2-1).$$

Chaotic maps.—Let us consider a chaotic map

$$f \colon \mathcal{A} \to \mathcal{A}, \tag{5}$$

where for simplicity only we assume that $\mathcal{A} = [0, 1]$. We consider here only one-dimensional maps; however, the generalization to high-dimensional maps will be discussed in extended version of this Letter. Any chaotic map when implement on a computer (finite-state machine) becomes a transformation from a finite set into itself. Therefore, we consider the map

$$f_M \colon \left\{\frac{0}{M}, \frac{1}{M}, \ldots, \frac{M-1}{M}\right\} \to \left\{\frac{0}{M}, \frac{1}{M}, \ldots, \frac{M-1}{M}\right\}, \tag{6}$$

which can be viewed as a discrete-space approximation of the map f [8]. We assume that f_M is an injection. Although one-dimensional chaotic maps are not 1:1, we will see later that one can construct a 1:1 f_M which is an approximation of a chaotic map f. Moreover, as we will show in the extended version of this Letter, the main results of this communication can also be generalized for an arbitrary map f_M.

Consider a permutation $F_M \colon \{0, 1, \ldots, M-1\} \to \{0, 1, \ldots, M-1\}$ induced by f_M in a natural way: $F_M(i) = M f_M(i/M)$. We define the Lyapunov exponent of the map f_M as

$$\lambda_{f_M} = \frac{1}{M} \sum_{i=0}^{M-1} \ln \left| \frac{f_M(\frac{i}{M} + \frac{1}{M}) - f_M(\frac{i}{M})}{\frac{1}{M}} \right|.$$

It is easy to see that $\lambda_{f_M} = \lambda_{F_M}$.

Main theorem.—Let $f \colon [0, 1] \to [0, 1]$ be a piecewise monotonic map with a finite number of pieces. We assume that the map is mixing with an invariant measure μ equivalent to Lebesgue measure with density ρ. We consider a set of M points $\{x_0, x_1, \ldots x_{M-1}\}$, $x_0 < x_1 < \ldots < x_{M-1}$, being a typical trajectory of the map, that is for each $i = 0, 1, \ldots M-1$, we have $f(x_i) = x_j$ for some $j = 0, 1, \ldots, M-1$. We define a permutation $f_M \colon \{\frac{0}{M}, \frac{1}{M}, \ldots, [(M-1)/M]\} \to \{\frac{0}{M}, \frac{1}{M}, \ldots, [(M-1)/M]\}$, as $f_M(\frac{i}{M}) = \frac{j}{M}$ only if $f(x_i) = x_j$.

Assume that for large M the following three conditions hold:

(i) For each $i = 0, 1, \ldots, M-1$, we have

$$\left|\left|\frac{f_M(\frac{i}{M} + \frac{1}{M}) - f_M(\frac{i}{M})}{\frac{1}{M}}\right| - \left|\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right|\right| \leq \varepsilon_1, \tag{7}$$

where $\varepsilon_1 = C_1/M$ and $C_1 > 0$.

(ii) The derivative f' is uniformly continuous at each monotonic interval, i.e., for each $\varepsilon_2 > 0$, there exists $\delta > 0$ such that for each x, y belonging to the same monotonic interval and satisfying $|x - y| < \delta$, we have

$$|f'(x) - f'(y)| < \varepsilon_2. \tag{8}$$

We assume that for $\varepsilon_2 = C_2/M$, $\delta = 1/M$. To satisfy this assumption it is enough to have that $f'(x) < C_2$.

(iii) $f'(x) > 1$ for all $x \in [0, 1]$.

Then $\lim_{M\to\infty} \lambda_{f_M} = \lambda_f$, where λ_f is the Lyapunov exponent of the map f.

We give only sketch of the proof. First observe that due to Lagrange theorem for each i, there exists $x_i < \xi_i < x_{i+1}$, such that

$$\left| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| = | f'(\xi_i) | \equiv A_i.$$

Then from condition (i) it follows that

$$A_i\left(1 - \frac{\varepsilon_1}{A_i}\right) < \left| \frac{f_M(\frac{i}{M} + \frac{1}{M}) - f_M(\frac{i}{M})}{\frac{1}{M}} \right| < A_i\left(1 + \frac{\varepsilon_1}{A_i}\right).$$

Since the invariant measure μ is equivalent to Lebesgue measure and $\{x_i\}_{i=0}^{M-1}$ is a typical trajectory, we have

$$\lim_{M\to\infty} \max_{0\le i\le M-1} |x_{i+1} - x_i| = 0.$$

From last equation, for sufficiently large M, it follows $|x_i - x_{i+1}| \le \frac{1}{M}$, for all $0 \le i \le M-1$. Consequently, $||A_i| - |f'(x_i)|| < \varepsilon_2$. Now we have

$$|f'(x_i)|\left(1 - \frac{\varepsilon_2}{|f'(x_i)|}\right) < A_i < |f'(x_i)|\left(1 + \frac{\varepsilon_2}{|f'(x_i)|}\right).$$

Since $|f'(x)| > 1$ for each $x \in [0, 1]$, we arrive to

$$|f'(x_i)|(1 - \varepsilon_1)(1 - \varepsilon_2) < \left| \frac{f_M(\frac{i}{M} + \frac{1}{M}) - f_M(\frac{i}{M})}{\frac{1}{M}} \right| < |f'(x_i)|(1 + \varepsilon_1)(1 + \varepsilon_2).$$

Consequently, we have

$$\begin{aligned}\frac{1}{M}\sum_{i=0}^{M-1} \ln|f'(x_i)| + \frac{1}{M}\sum_{i=0}^{M-1} \ln(1-\varepsilon_1) + \frac{1}{M}\sum_{i=0}^{M-1} \ln(1-\varepsilon_2) &< \frac{1}{M}\sum_{i=0}^{M-1} \ln\left| \frac{f_M(\frac{i}{M} + \frac{1}{M}) - f_M(\frac{i}{M})}{\frac{1}{M}} \right| \\ &< \frac{1}{M}\sum_{i=0}^{M-1} \ln|f'(x_i)| + \frac{1}{M}\sum_{i=0}^{M-1} \ln(1+\varepsilon_1) + \frac{1}{M}\sum_{i=0}^{M-1} \ln(1+\varepsilon_2).\end{aligned}$$

Since $\{x_i\}_{i=0}^{M-1}$ is a typical trajectory and we assume the system is mixing, it follows

$$\lim_{M\to\infty} \frac{1}{M}\sum_{i=0}^{M-1} \ln|f'(x_i)| = \int \ln f'(x)\rho(x)dx = \lambda_f,$$

where ρ is the density of the invariant measure μ. Furthermore, since $\varepsilon_1 = C_1/M$, $\varepsilon_2 = C_2/M$, and

$$\lim_{M\to\infty} \frac{1}{M}\sum_{i=0}^{M-1} \ln\left(1 \pm \frac{C}{M}\right) = 0,$$

the proof of our main theorem is completed [9].

Remark 1: The main theorem is stated under three assumptions—if $1 < f'(x) < C_2$ then the second and third assumptions are clearly satisfied. Note that condition (7) is always satisfied when f is a piecewise linear map. For general nonlinear maps, if the variation of the derivative of f is small and/or the density of the invariant measure is close to uniform, then it is easy to find a map f_M, which is an approximation of a chaotic map, such that the condition (7) is satisfied.

Example 3: Let us consider the tent map

$$f(x) = \begin{cases} \frac{x}{a}, & 0 \le x \le a, \\ \frac{x-1}{a-1}, & a < x \le 1, \end{cases} \tag{9}$$

where $0 < a < 1$ is a parameter and $x \in [0, 1]$ [10]. This map is mixing, its invariant measure is the Lebesgue measure restricted to $[0, 1]$, and the Lyapunov exponent of the map for a typical trajectory is equal to $\lambda_f = -a\ln a - (1-a)\ln(1-a)$.

For the tent map, the map f_M and consequently the map F_M can be constructed analytically [11]. One has

$$F_M(x) = \begin{cases} 1, & x = 0, \\ \mathrm{Cl}\left(\frac{M}{A}x\right), & 0 < x \le A, \\ \mathrm{Fl}\left[\frac{M}{M-A}(M-x)\right] + 1, & A < x \le M-1, \end{cases} \tag{10}$$

where Cl(z) and Fl(z) denote ceiling and floor of z, respectively, and $x \in \{0, 1, \ldots, M-1\}$.

We now present two numerical examples. In the first example we choose $a = 0.5$. The convergence of λ_{F_M} to $\lambda_f = \ln 2$ is shown in the Table I. For the second example, let $M = 128$ and $A = 35$; thus $a = A/M = 0.2734$. In this case the map F_{128} has two periodic trajectories with periods 102 and 26, respectively. Their Lyapunov exponents are 0.525 and 0.633, respectively. The Lyapunov exponent for the map F_{128} is $\lambda_{F_M} = 0.547$, while the Lyapunov exponent of the tent map is $\lambda_f = 0.5866$. Note that in this case $|\lambda_{F_M} - \lambda_f| \approx 0.039$.

Remark 2: The Lyapunov exponent of the permutation F_M may approach the Lyapunov exponent of the chaotic map f as $M \to \infty$, as in Example 3. In general, however, the Lyapunov exponent λ_{F_M} may tend to zero or infinity. For example, it is easy to see that $\lim_{M\to\infty}\lambda^{(1)}_{F_M} = 0$ and $\lim_{M\to\infty}\lambda^{(2)}_{F_M} = \infty$, where $\lambda^{(1)}_{F_M}$ and $\lambda^{(2)}_{F_M}$ are Lyapunov exponents computed in Examples 1 and 2, respectively.

Pseudochaos.—Let us now consider the case when F_M: $\{0, 1, \ldots, M-1\} \to \{0, 1, \ldots, M-1\}$ is an arbitrary

TABLE I. Values of λ_{F_M} and $\lambda_f - \lambda_{F_M}$ for different M for the tent map with $a = 0.5$.

M	λ_{F_M}	$\lambda_f - \lambda_{F_M}$
256	0.690 439 574 38	$2.707\,606\,17 \times 10^{-3}$
300	0.690 836 689 95	$2.310\,490\,60 \times 10^{-3}$
400	0.691 414 312 60	$1.732\,867\,95 \times 10^{-3}$
500	0.691 760 886 19	$1.386\,294\,36 \times 10^{-3}$
1000	0.692 454 033 37	$6.931\,471\,80 \times 10^{-4}$
1024	0.692 470 279 01	$6.769\,015\,43 \times 10^{-4}$
2600	0.692 880 585 49	$2.665\,950\,69 \times 10^{-4}$
7224	0.693 051 229 95	$9.595\,060\,63 \times 10^{-5}$
22 444	0.693 116 297 15	$3.088\,340\,66 \times 10^{-5}$
34 012	0.693 126 801 07	$2.037\,948\,89 \times 10^{-5}$

map (not necessarily injection). In this case, the map may have eventually periodic orbits. We say that i is a stable fixed point for the map F_M if $F_M(i) = i$ and at least one of its neighbor points $i \pm 1$ is its eventually fixed point; in other words, when $F_M(i+1) = i$ and/or $F_M(i-1) = i$. In a similar way, one can define stable periodic orbits. Let

$$\mathcal{A} = \{F_M | F_M \text{ is a bijection and } F_M \neq \mathrm{Id}\},$$

where Id denotes the identity, be a set of all bijections different from identity. It is clear that for all $F_M \in \mathcal{A}$, the Lyapunov exponent of F_M [defined with Eq. (2)] is always a positive number. This also reflects the fact that all periodic orbits of F_M are unstable (we say that the orbit is unstable if it is not stable).

The above discussion and Remark two motivate the following definition of pseudochaos. Let F_M be an arbitrary map (not necessarily a bijection). Let $C \subset \{0, 1, \ldots, M-1\}$ be a set invariant under the action of the map F_M such that the map F_M restricted to the set C is a bijection. Assume further that the points of $\{0, 1, \ldots, M-1\}$, which are not in C and are mapped with F_M in C, are not neighboring points of the periodic orbits of C. This makes all periodic orbits of C unstable. The main theorem can also be extended for the set C. Let us write G_M for the map F_M restricted to the set C. We say that the map F_M is *pseudochaotic* on the set C if $\lim_{M\to\infty} \lambda_{G_M} > 0$. The existence of the horseshoe is a fingerprint of chaos in continuous-space systems. In discrete-space systems, however, the existence of a set, on which F_M is injection, and for which all periodic orbits are unstable, is a fingerprint of pseudochaos.

Conclusion.—We have suggested an answer to the question What is chaos in discrete-space systems? by proposing definitions of finite-space Lyapunov exponent and pseudochaos. In analogy to continuous systems, the concepts of attractor, pseudochaotic attractor, basin boundary, and so on, can also be rigorously defined for finite discrete-space systems, which will be a subject of our future research topic.

We are grateful to Cvetko Mitrovski for stimulating discussions. This research is supported in part by the NSF. J. S. was partially supported by KBN Grant No. 8T07A 045 20.

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[8] Here we assume regular grid. However, for a chaotic map, with density of the invariant measure not close to the uniform, irregular grids may be more appropriate for the approximation of the map.

[9] One can also define Lyapunov exponent as

$$\begin{aligned}\tilde{\lambda}_{F_M} &= \frac{1}{M} \ln| F_M(1) - F_M(0) | \\ &\quad + \frac{1}{M} \sum_{i=1}^{M-2} \ln| F_M(i \pm 1) - F_M(i) | \\ &\quad + \frac{1}{M} \ln| F_M(M-2) - F_M(M-1) |,\end{aligned}$$

with randomly choosing, from two neighboring points $i+1$ and $i-1$, the neighbor of i. There exist 2^{M-2} such Lyapunov exponents. It is easy to see that if conditions of the main theorem are satisfied, then $\tilde{\lambda}_{F_M}$ approaches λ_f as $M \to \infty$.

[10] Because of space limitation we consider here only the tent map. Other examples of chaotic maps, including the logistic map, will be presented in the extended version of this Letter.

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