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ON THE BASIS OF STATISTICAL MECHANICS. THE LIOUVILLE EQUATION FOR SYSTEMS WITH AN INFINITE COUNTABLE NUMBER OF DEGREES OF FREEDOM

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In this paper statistical properties of physical systems with a phase space that is an infinite-dimensional separable Hilbert space are considered. For such systems it is possible to define the basic concepts of classical mechanics as well as statistical mechanics. In particular as an initial probability measure we assume a quasi-invariant measure. Based on this idea we derived the Liouville equation in an infinite-dimensional case. The obtained Liouville equation contains an additional term dependent also on the assumed initial measure which vanishes when we pass to a system with finite number of degrees of freedom but which is not equal to zero when we study a system with an infinite number of degrees of freedom. This term is in a sense a compensation for avoiding complications due to boundary conditions.

The formalism given in this paper is still valid even in more general situations, when the phase space is not Hilbert space. This fact may play an essential role in statistical mechanics.

1. Introduction

Statistical mechanics is the study of macroscopic systems in terms of their microscopic properties. This viewpoint holds that statistical mechanics is the study of systems having a large number of degrees of freedom, or even that statistical mechanics is valid in the limiting case of an infinite number of degrees of freedom. Thus, according to this viewpoint, statistical mechanics is merely the mechanics of many-body systems.

The state of a system implies a complete description of the system at one instant of time, i.e. a description which is as complete and detailed as permitted by the laws of physics. In classical mechanics, the state is determined by measuring the coordinates and momenta of all the degrees of freedom, and is represented by a point $x = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ in a $2n$-dimensional phase space, where n is the number of degrees of freedom.

Mechanics is the study of evolution of the state of a system. It is a scheme for predicting the precise state at time t_1, provided only that the precise state is known at some other time t_0. Thus, mechanics allows us to predict with certainty the state at t_1.

Statistical mechanics is the study of mechanical systems in situations where the description is incomplete – and as a rule this occurs when we study large systems. Prediction about incompletely specified systems cannot in general be deductive predictions; for instance, it is impossible to predict with certainty the precise number of molecules in a given sub-volume of a container, given only the total energy, total volume, and total number of molecules in the container. Statistical mechanics predictions contain an element of uncertainty. This is the reason that the probability theory forms the mathematical language of statistical mechanics.

In this paper the probability theory is applied to the study systems with infinite countable number of degrees of freedom. As a phase space the infinite-dimensional separable Hilbert space is taken [1]. Observe that contrary to the most common approaches to study the statistical properties of mechanical systems [2–9] – when one first considers systems with finite number of degrees of freedom and next tends this number to infinity with some parameters held fixed and finite, i.e. in the so-called thermodynamic limit or Boltzmann–Grad limit or hydrodynamic limit – in our approach we study explicitly systems with an infinite number of degrees of freedom. We emphasize that we are considering the theory of infinite systems for its own sake as well as due to the fact that this is the only precise way of removing inessential complications due to boundary effects, etc.

The Liouville equation derived in this paper is similar to the classical Liouville equation except that it contains an additional term depending on the assumed initial statistical state (initial probability measure). This term vanishes when we pass on to a system with finite-dimensional phase space but it is not equal to zero when we consider a system with an infinite countable number of degrees of freedom.

The Liouville equation is one of the fundamental differential equations of statistical mechanics [2, 9]. In the classical case the solution of this equation is a scalar function $w(x, t)$, called the probability density defined for every time t on the phase space X of the considered physical system. The Lebesque integral of the function $w(x, t)$ over a measurable subset A of X gives us the probability that at the given moment t the system is in one of the microscopic states of the set A. Knowing the time evolution of the phase density of probability we can compute the mean statistical value as well as the variance of any arbitrary physical quantity. Moreover, it is well known from the theory of the differential equations that the solution of a first-order partial differential equation of

Liouville type is equivalent to the solution of a system of first-order ordinary differential equation of Hamilton type [10].

The paper is organized as follows. In section 2 we recall briefly the standard derivation of the Liouville equation in the finite-dimensional case and then we derive a different form of the Liouville equation which can be generalized to the infinite-dimensional case. In section 3 we derive the most general version of the Liouville equation. In section 4 we discuss the properties of the phase space of systems with infinite countable number of degrees of freedom considering conditions to introduce the Hilbert space structure on this phase space. In section 5 for the sake of completeness we recall the basic concepts of the measure theory (the probability theory is a part of it [11]) and then we discuss the generalization of the fundamental concepts of the measure theory as applied to the infinite-dimensional separable Hilbert space. In section 6 we derive the Liouville equation in infinite-dimensional separable Hilbert space. We use it to describe the system of infinitely many point-particles interacting by a symmetric superstable pair potential. In section 7 we derive the formula of the Radon–Nikodym derivative in the case when phase flux is given by the linear operator for which the generalized divergence exists and then with the help of this formula we derive the Liouville equation. We adapt this Liouville equation for the description of the electromagnetic field.

2. The classical Liouville equation

In order to provide a convenient basis for the discussion of the Liouville equation it is useful to remind briefly the derivation of the classical Liouville equation. However, we present two ways of deriving the Liouville equation: first one is the standard way which leads to the classical Liouville equation, and the second one gives the Liouville equation with an additional term depending on the initial conditions. As we see later on these two formulations are equivalent. Moreover the advantage of the second equation is that it can be generalized to systems with infinite countable number of degrees of freedom with their phase space being an infinite-dimensional separable Hilbert space. Such possibility exists owing to the fact that the concepts which appear in the second formulation have their counterparts in an infinite-dimensional separable Hilbert space. On the other hand it is worthwhile to note that the classical Liouville equation has an essential property, namely it does not contain any term depending on the initial conditions. The elimination of this term is possible due to the existence of translation invariant measure, i.e. Lebesque measure (see section 5 below).

We begin our considerations by introducing the notations. Denote by $w(x, t)$

the probability density, $x \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$, $n < +\infty$. Let the mapping

$$x_t: \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad t \in \mathbb{R}, \tag{2.1}$$

be the phase flux; throughout this section we shall assume that it is a Hamiltonian phase flux, i.e. $x_t = (q_t, p_t)$ and

$$\dot{q}_t = \frac{\partial \mathscr{H}}{\partial p}, \qquad \dot{p}_t = -\frac{\partial \mathscr{H}}{\partial q}, \tag{2.2}$$

where $\mathscr{H}$ is a Hamiltonian. Then the probability P that at time t the system is in one of the microscopic states of set A is equal to

$$P(A) = \int_A w(x, t)\,\mathrm{d}x, \quad A \subset \mathbb{R}^{2n}. \tag{2.3}$$

Based on physical premises, we conclude that

$$P(A_t) = \int_{A_t} w(x, t)\,\mathrm{d}x = \int_A w(x, 0)\,\mathrm{d}x \tag{2.4}$$

for $A_t := x_t(A)$. Whence

$$\frac{\mathrm{d}}{\mathrm{d}t} P(A_t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{A_t} w(x, t)\,\mathrm{d}x = 0. \tag{2.5}$$

Changing the domain of the integration we have

$$\int_A \frac{\mathrm{d}}{\mathrm{d}t} w(x_t(x), t)|\mathrm{D}x_t(x)|\,\mathrm{d}x = 0, \tag{2.6}$$

where $|\mathrm{D}x_t(x)|$ is the Jacobian of the phase flux x_t at point $x \in A$. In view of the arbitrariness of domain A we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} w(x_t(x), t) + w(x_t(x), t)\frac{\mathrm{d}}{\mathrm{d}t} \ln|\mathrm{D}x_t(x)| = 0. \tag{2.7}$$

Taking into account that the phase flux is Hamiltonian we get the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln|\mathrm{D}x_t(x)| = \operatorname{div} v(x_t(x), t) = 0. \tag{2.8}$$

Substituting (2.8) into (2.7) we obtain the Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, w(x_t(x), t) = 0\,. \tag{2.9}$$

Eq. (2.9) can be written in the form

$$\frac{\partial}{\partial t}\, w + \{w, \mathcal{H}\} = 0\,; \tag{2.10}$$

the symbol $\{\ ,\ \}$ denotes the Poisson brackets [2]. We shall denote the initial conditions for this equation by $p_0(x)$.

Now we derive the Liouville equation in a form which can be generalized to the infinite-dimensional separable Hilbert space. For this purpose we take as an initial measure on the phase space $\mathbb{R}^{2n}$ the measure μ_0 defined as follows:

$$\mu_0(A) := \int_A p_0(x)\,\mathrm{d}x\,, \tag{2.11}$$

for any measurable set $A \subset \mathbb{R}^{2n}$.

Remark 1. Measure μ_0 is quasi-invariant (see section 5 below).

Then proceeding as before we have

$$\int_{x_t(A)} w(x, t)\mu_0(\mathrm{d}x) = \int_A w(x, 0)\mu_0(\mathrm{d}x)\,; \tag{2.12}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_A w(x_t(x), t)p_0(x_t(x))|\mathrm{D}x_t(x)|\,\mathrm{d}x = 0\,. \tag{2.13}$$

Now performing differentiation, after simple transformations we obtain the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\, w(x_t(x), t) + w(x_t(x), t)\,\frac{\mathrm{d}}{\mathrm{d}t}\ln p_0(x_t(x))\\&\quad + w(x_t(x), t)\,\frac{\mathrm{d}}{\mathrm{d}t}\ln|\mathrm{D}x_t(x)| = 0\,.\end{aligned} \tag{2.14}$$

Observe that if w is a solution of eq. (2.10) and w_1 is a solution of eq. (2.14) with the initial conditions p_0 and 1, respectively, then

$$w = w_1 \cdot p_0 \,. \tag{2.15}$$

It means that eqs. (2.10) and (2.14) are equivalent.

We emphasize that the existence of the translation invariant measure, namely Lebesque measure, plays a fundamental role in the finite-dimensional case. Moreover, this measure is invariant under phase flux x_t due, of course, to the assumption that the phase flux is Hamiltonian. The last fact is known as the Liouville theorem.

3. The general form of the Liouville equation

The state of an infinite system in statistical mechanics may be described by a probability measure on the space of infinite configurations. In connection with this fact let X be a measurable space and let μ_0 be a probability measure on X. (The space X can be equipped with some structures, in particular it may be a Hilbert space.) Moreover, assume that h_t, $t \in \mathbb{R}$, is

$$h_t\colon X \to X \,, \tag{3.1}$$

such that for every set $A \in \mathcal{M}$ and every t

$$\mu_0(A) = 0 \quad \text{iff } \mu_0(h_t(A)) = 0 \,. \tag{3.2}$$

Denote by μ_t the measure

$$\mu_t(A) := \mu_0(h_t(A)) \,, \quad A \in \mathcal{M} \,. \tag{3.3}$$

Then according to the Radon–Nikodym theorem [11] (see also section 5 below) we conclude that there exists the function (counterpart of Jacobian)

$$g\colon X \times \mathbb{R} \to \mathbb{R}_+ \,, \tag{3.4}$$

such that

$$\int_{h_t(A)} \mu_0(\mathrm{d}x) = \mu_0(h_t(A)) = \mu_t(A) = \int_A g(x, t)\mu_0(\mathrm{d}x) \,. \tag{3.5}$$

As before denote by $w(x, t)$, $x \in X$, the probability density of a certain physical system for which X is its phase space.

Remark 2. This probability density is taken with respect to the measure μ_0.

Assume that $w(x,0) = w_0(x)$ is an initial value for w. Hence

$$P_t(A) = \int_A w(x, t)\mu_0(\mathrm{d}x) \tag{3.6}$$

means the probability that at the moment t the system is in one of the states of domain A. We assume that

$$P_t(A_t) = P(A) = \int_A w_0(x)\mu_0(\mathrm{d}x)\,, \tag{3.7}$$

where $A_t := h_t(A)$. From (3.6) and (3.7) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P_t(A_t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{A_t} w(x, t)\mu_0(\mathrm{d}x) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_A w(h_t(x), t)g(x, t)\mu_0(\mathrm{d}x) = 0\,. \end{aligned} \tag{3.8}$$

According to the arbitrariness of domain A, after differentiation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} w(h_t(x), t) + w(h_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \ln g(x, t) = 0\,. \tag{3.9}$$

Finally, making use of the identity

$$\frac{1}{|\mathrm{D}h_t^{-1}(h_t(x))|} = g(x, t)\,, \tag{3.10}$$

eq. (3.9) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} w(h_t(x), t) + w(h_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{1}{|\mathrm{D}h_t^{-1}(h_t(x))|} = 0\,, \tag{3.11}$$

where $|\mathrm{D}h_t^{-1}(h_t(x))|$ is the Jacobian of the mapping h_t^{-1} at point $h_t(x)$. Eq. (3.11) is the general form of the Liouville equation.

4. Phase space of a system with infinite countable number of degrees of freedom

It is clear that as a phase space of a system with an infinite countable number of degrees of freedom one should take the Cartesian product $\mathbb{R}^\infty$. The space $\mathbb{R}^\infty$

is usually endowed with the product topology $\mathscr{B}$. For this topology the Daniell–Kolmogorov theorem states [11] that if given finite-dimensional probability densities satisfy certain consistent conditions, it is possible to find a probability measure having these densities. In other words this theorem enables to define initial probability measure by its finite-dimensional probability distributions.

Let

$$l_a^2 := \left\{ (x_i)_{i=1}^{+\infty} \in \mathbb{R}^\infty : \sum_{i=1}^{+\infty} a_i x_i^2 < +\infty \right\} , \tag{4.1}$$

where $a = (a_i)_{i=1}^{+\infty}$ is a sequence of fixed positive real numbers. In the set l_a^2, a Hilbert space structure is endowed naturally. As a scalar product $\langle \, | \, \rangle$ one can take

$$\langle x | y \rangle := \sum_{i=1}^{+\infty} a_i x_i y_i , \tag{4.2}$$

where $x = (x_i)_{i=1}^{+\infty} \in l_a^2$, $y = (y_i)_{i=1}^{+\infty} \in l_a^2$.

We shall prove that under certain assumptions on initial probability measure there exists a Hilbert space $l_a^2 \subset \mathbb{R}^\infty$ which may be considered as the phase space. To this end it is sufficient to prove that the support of our initial measure μ_0 is concentrated in some space l_a^2. From a physical point of view it means that some configurations of $\mathbb{R}^\infty$ are impossible.

Theorem 1. For a certain physical system let μ_0 be its initial probability measure on the space $\mathbb{R}^\infty$. If the following mean values and variances are finite:

$$\begin{aligned} \alpha_i &:= \int_{\mathbb{R}^\infty} x_i \mu_0(\mathrm{d}x) < +\infty , \\ \beta_i &:= \int_{\mathbb{R}^\infty} |x_i - \alpha_i|^2 \mu_0(\mathrm{d}x) < +\infty , \end{aligned} \tag{4.3}$$

$i = 1, 2, \ldots$, $x = (x_i)_{i=1}^{+\infty} \in l_a^2$, then there exists a weight sequence $a = (a_i)_{i=1}^{+\infty}$ such that $\mu_0(l_a^2) = 1$.

Proof. Let us consider the events

$$A_j = \{ (x_i)_{i=1}^{+\infty} \in \mathbb{R}^\infty : |x_j - \alpha_j| > |j| \beta_i \} \subset \mathbb{R}^\infty . \tag{4.4}$$

According to the Chebyshev inequality [11] for any $j = 1, 2, \ldots$ we have

$$\mu_0(\{(x_i)_{i=1}^{+\infty} \in \mathbb{R}^\infty : |x_j - \alpha_j| > |j|\beta_j\}) \leq \frac{\beta_j}{j^2\beta_j^2} . \tag{4.5}$$

Thus, the probability of the event that the jth degree of freedom will differ from the mean value by at most $|j|\beta_j$ is less than $\beta_j/j^2\beta_j^2$. Since

$$\sum_{j=1}^{+\infty} \frac{1}{j^2\beta_j} < +\infty \tag{4.6}$$

(if $\beta_j \to 0$ then we have to consider the sets $B_j = \{(x_i)_{i=1}^{+\infty} \in \mathbb{R}^\infty : |x_j - \alpha_j| > j\}$ rather than A_j), in view of the Borel–Cantelli lemma [11] the probability μ_0 of an event A that there will occur an infinite number of events A_j is equal to zero. Thus, it is sufficient to take as the weight coefficients $a = (a_i)_{i=1}^{+\infty}$ a sequence which would ensure that the inequality

$$\sum_{i=1}^{+\infty} a_i(\alpha_i + |\mathrm{i}|\beta_i)^2 < +\infty \tag{4.7}$$

is satisfied and then $\mu_0(l_a^2) = 1$.

To finish the proof it remains to show that l_a^2 is a measurable set. Let us denote

$$K(n) = \left\{ (x_i)_{i=1}^{+\infty} \in \mathbb{R}^\infty : \sum_{i=1}^{+\infty} a_i x_i^2 < n \right\} . \tag{4.8}$$

The ball $K(n)$ is an element of the σ-field generated by $\mathscr{B}$. Since $l_a^2 = \cup_n K(n)$ then the set l_a^2 is also an element of the σ-field generated by $\mathscr{B}$, and the measure $\bar{\mu}_0$ restricted to the space l_a^2, i.e.

$$\bar{\mu}_0(A) := \mu_0(A \cap l_a^2) , \quad A \in \sigma(\mathscr{B}) , \tag{4.9}$$

is well defined. This ends the proof.

5. The fundamental concepts of measure theory [11, 12]

For the sake of completeness, in this section we recall the basic concepts of measure theory.

Let X be a non-empty set.

Definition 1. A σ-field is a non-empty class S of subsets of X such that
a. $\emptyset \in S$,

b. $E \in S$ then $X \setminus E \in S$,
c. if $E_i \in S$, $i = 1, 2, \ldots$, then $\cup_{i=1}^{+\infty} E_i \in S$.

Throughout this paper we shall consider the σ-field of Borel sets, i.e. a σ-field generated by the open sets.

Definition 2. A measure μ is a set function defined on a σ-field S,

$$\mu: S \rightarrow \mathbb{R}_+ ,$$

such that
a. $\mu(\emptyset) = 0$,
b. if $(E_n)_{n=1}^{+\infty}$ is a sequence of disjoint members of S, then

$$\mu\left(\bigcup_{n=1}^{+\infty} E_n\right) = \sum_{n=1}^{+\infty} \mu(E_n) .$$

Definition 3. A measure μ is called σ-finite (finite) if there exists a sequence $(E_n)_{n=1}^{+\infty}$ of members of σ-field S such that

$$X = \bigcup_{n=1}^{+\infty} E_n \quad \text{and} \quad \mu(E_n) < +\infty , \quad n = 1, 2, \ldots ,$$

$$(\mu(X) < +\infty) .$$

Definition 4. We say that a measure μ_1 is equivalent to a measure μ_2 and write $\mu_1 \sim \mu_2$ when

$$\mu_1(A) = 0 \quad \text{iff } \mu_2(A) = 0 , \quad A \in S , \tag{5.1}$$

or the equivalent definition for finite measures

$$\lim \mu_1(A_n) = 0 \quad \text{iff } \lim \mu_2(A_n) = 0 , \quad A_n \in S , \quad n = 1, 2, \ldots \tag{5.2}$$

Let us assume that X is a Hilbert space H.
For every $h \in H$ we denote by μ_h the measure on H defined as follows:

$$\mu_h(A) := \mu(A - h) , \quad A \in \sigma(H) , \tag{5.3}$$

where $A - h := \{g \in H\colon g + h \in A\}$.
The set of all vectors $h \in H$ such that the statement $\mu_h \sim \mu$ holds is denoted by M_μ.

Now we recall:

The Radon–Nikodym theorem. Let measures μ_1 and μ_2 be equivalent; then there exists a measurable function g such that

$$\mu_1(A) = \int_A g \, d\mu_2 . \tag{5.4}$$

The function g of the Radon–Nikodym theorem is called the Radon–Nikodym derivative of μ_1 with respect to μ_2 and is denoted by

$$g = d\mu_1 / d\mu_2 . \tag{5.5}$$

The Radon–Nikodym derivative of measure μ_h with respect to measure μ at point x is denoted by

$$\rho_\mu(h, x) = \frac{d\mu_h}{d\mu}(x) . \tag{5.6}$$

Remark 3. If m is Lebesque measure then

$$\rho_m(h, x) = 1 \quad \text{for every } h \in H . \tag{5.7}$$

Remark 4. If a measure μ is equivalent to Lebesque measure with density $p(x)$ then

$$\rho_\mu(h, x) = \frac{p(x-h)}{p(x)} . \tag{5.8}$$

Definition 5. A measure μ is called translation invariant if

$$\mu(A) = \mu(A - h) \quad \text{for every } h \in H \text{ and } A \in \sigma(H) . \tag{5.9}$$

Definition 6. A measure μ is called quasi-invariant if the set M_μ contains a linear subspace dense in H.

In table I we present the basic properties of translation invariant and quasi-invariant measures both in finite- and infinite-dimensional spaces [13]. In section 7 we shall use a Gaussian measure [14].

Definition 7. A measure μ on a Hilbert space H is called Gaussian measure if its characteristic function

Table I

Finite-dimensional space $\mathbb{R}^n$	Infinite-dimensional separable Hilbert space H
There is a σ-finite translation invariant measure on $\mathbb{R}^n$. It is the Lebesque measure.	There does not exist a σ-finite translation invariant measure on H.
All σ-finite translation invariant measures differ at most by a fixed factor.	
There exists a σ-finite quasi-invariant measure on $\mathbb{R}^n$.	There exists a σ-finite quasi-invariant measure on H.
All σ-finite quasi-invariant measures on $\mathbb{R}^n$ are equivalent to Lebesque measure, whence all such measures are equivalent themselves and have the form $p_0(x)\,\mathrm{d}x$; $p_0(x)$ is a certain function defined on $\mathbb{R}^n$.	There exist σ-finite quasi-invariant measures on H which are not equivalent.

$$\theta(z) := \int_H \exp\{\mathrm{i}\langle z|x\rangle\}\mu(\mathrm{d}x) \tag{5.10}$$

is of the form

$$\theta(z) = \exp\{\mathrm{i}\langle b|z\rangle - \tfrac{1}{2}\langle Az|z\rangle\}\,, \tag{5.11}$$

where $b \in H$ and A is a bounded symmetric non-negative operator acting on H into itself.

Remark 5. It can be shown [15] that the set M_{μ_0} of Gaussian measure μ_0 is equal to $A^{1/2}(H)$.

6. The Liouville equation in infinite-dimensional separable Hilbert space

We start by briefly discussing the mechanics of systems with an infinite countable number of degrees of freedom. A full version of classical mechanics in Hilbert space is given in ref. [1].

Let us consider the continuously differentiable mapping

$$x_t = x(\ ,t)\colon H \to H\,, \quad t \in [0, T]\,, \tag{6.1}$$

being for every $t \in [0, T]$ a homeomorphism. In our considerations this mapping plays the role of phase flux. By

$$v(X, t) = \frac{\partial}{\partial t} x(X, t) \tag{6.2}$$

we denote the velocity of the point $X \in H$. In view of the homeomorphism of mapping $x(\, , t)$, the velocity may also be considered as a function of variables x, t, and then

$$\mathrm{D}v = \mathrm{D}_x v \circ \mathrm{D}x \, , \tag{6.3}$$

where $\mathrm{D}x$, $\mathrm{D}v$, $\mathrm{D}_x v$ denote the Fréchet derivatives of the mappings $x(X, t)$, $v(X, t)$, $v(x, t)$, respectively.

Let us consider the operator $v(x(X, t), t) = v(x, t)$ without changing the symbol of function v. For a space with a finite number of dimensions n denoted by $(\Phi_k)_{k=1}^{n}$ a system of orthogonal and normalized base vectors and by $\langle \, | \, \rangle$ the scalar product, we have

$$\operatorname{div} v = \sum_{k=1}^{n} \langle \Phi_k | \mathrm{D}_x v \, \Phi_k \rangle \, . \tag{6.4}$$

For the infinite-dimensional separable Hilbert space, $n = +\infty$.

Let $(\Phi_k)_{k=1}^{+\infty}$ be an orthonormal system complete in H. We then define

$$\operatorname{div} v := \sum_{k=1}^{+\infty} \langle \Phi_k | \mathrm{D}_x v \, \Phi_k \rangle \, . \tag{6.5}$$

In order to make the above definition meaningful, we assume that the continuous operator $\mathrm{D}_x v$ is a nuclear operator. It was proved [1] that

$$\operatorname{div} v = \frac{\partial}{\partial t} \ln \prod_{k=1}^{+\infty} \lambda_k \, , \tag{6.6}$$

where $(\lambda_k)_{k=1}^{+\infty}$ are the eigenvalues of the operator $\mathrm{D}x := (\mathrm{D}x^* \circ \mathrm{D}x)^{1/2}$. We denote the quantity $\Pi_{k=1}^{+\infty} \lambda_k$ by $\det \mathrm{D}x$. Observe that the above relations are analogous to those for systems with a finite number of degrees of freedom. These relations will be used later in this section.

Now we are in a position to proceed to the Liouville equation. Let μ_t be a probability measure on an infinite-dimensional separable Hilbert space H such that $\mu_t(G_0)$ is the probability that at time t the system is in one of the microscopic states of the domain $G_0 \subset H$. We assume that for every $t \in [0, T]$ the measure μ_t is equivalent to the measure μ_0; then in view of the Radon–Nikodym theorem, there exists a probability density $w(x, t)$ on the phase space

H such that

$$\mu_t(G_0) = \int_{G_0} w(x, t)\mu_0(\mathrm{d}x) \quad \text{for every } G_0 \in \sigma(H) . \tag{6.7}$$

Let G_t be the range of region G_0 at time t given by the mapping x_t, i.e.

$$G_t := x_t(G_0) . \tag{6.8}$$

On the basis of physical premises we assume that the probability of the appearance of the phase point in domain G_t at time t does not change, i.e.

$$\mu_t(G_t) = \mu_0(G_0) \quad \text{for every } t \in [0, T] . \tag{6.9}$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{G_t} w(x, t)\mu_0(\mathrm{d}x) = 0 . \tag{6.10}$$

Theorem 2. Assume that:
1. The measure μ_0 is quasi-invariant.
2. For every finite-dimensional space $L \subset M_{\mu_0}$ the measure of a set consisting of such x that $\rho_{\mu_0}(b, x)$ exists, and which is continuous with respect to $b \in L$, equals 1.
3. There exists a sequence of finite-dimensional subspaces $L_n \subset H$ and a set $B \subset H$ ($\mu_0(B) = 0$) such that for $x \not\in B$ and for all $b \in H$ the following is true:

$$\lim_n \rho_{\mu_0}(P_n b, x) = \rho_{\mu_0}(b, x) , \tag{6.11}$$

   where P_n is an orthogonal projection on the space L_n.
4. For every $t \in [0, T]$ there exists $\det \mathrm{D}x_t^{-1}(x)$, a.e. μ_0.
5. For every $t \in [0, T]$ there is

$$x - x_t(x) \in M_{\mu_0} \quad \text{and} \quad x - x_t^{-1}(x) \in M_{\mu_0}, \quad \text{a.e. } \mu_0 , \tag{6.12}$$

   and the functions

$$\rho_{\mu_0}(x - x_t(x), x) \quad \text{and} \quad \rho_{\mu_0}(x - x_t^{-1}(x), x) \tag{6.13}$$

   are positive.

Then, if

$$\mu_t(G) = \int_G w(x, t)\mu_0(\mathrm{d}x) \tag{6.14}$$

is the probability that at time t the system will be in a domain $G \subset H$, the Liouville equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} w(x_t(x), t) &+ w(x_t(x), t) \operatorname{div} v(x_t(x), t) \\ &+ w(x_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \ln \rho_{\mu_0}(x - x_t(x), x) = 0\end{aligned} \tag{6.15}$$

holds.

Proof. Let us consider the probability measure $\bar{\mu}_t$ defined as follows:

$$\bar{\mu}_t(A) = \mu_0(x_t(A)), \quad A \in \sigma(H). \tag{6.16}$$

Given the above assumptions the measure $\bar{\mu}_t$ is equivalent to the measure μ_0 and the Radon–Nikodym derivative is of the form [15]

$$\frac{\mathrm{d}\bar{\mu}_t}{\mathrm{d}\mu_0} = \det \mathrm{D}x_t^{-1}(x_t(x))\rho_{\mu_0}(x - x_t(x), x). \tag{6.17}$$

Substituting (6.17) into (6.10), performing differentiation and making use of the arbitrariness of the domain G_0, after simple transformations we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} w(x_t(x), t) &+ w(x_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} |\det \mathrm{D}x_t^{-1}(x_t(x))| \\ &+ w(x_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \rho_{\mu_0}(x - x_t(x), x) = 0.\end{aligned} \tag{6.18}$$

From (6.6) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln|\det \mathrm{D}x_t^{-1}(x_t(x))| = \operatorname{div} v(x_t(x), t). \tag{6.19}$$

Substituting (6.19) into (6.18) we obtain the Liouville equation in the form (6.15). This ends the proof of the theorem.

This Liouville equation may be transformed into

$$w(x_t(x), t)\rho_{\mu_0}(x - x_t(x), x) = \exp\left\{ -\int_{t_0}^{t} \operatorname{div} v(x_s(x), s)\, \mathrm{d}s \right\}. \tag{6.20}$$

Observe that, if $\operatorname{div} v = 0$ – and such is the case, for example, when $v = \delta \circ \mathrm{d}\mathscr{H}$ where $\mathscr{H}\colon H \times H \to \mathbb{R}$ is a function of class C^2 and $\delta\colon H \times H \to H \times H$, $\delta(x, y) = (-y, x)$ – then after simple transformations making use of the identity

$$\rho_{\mu_0}(-a, x) = [\rho_{\mu_0}(a, x + a)]^{-1}, \tag{6.21}$$

we have

$$w(x, t) = w(x_t^{-1}(x), 0)\rho_{\mu_0}(x - x_t^{-1}(x), x). \tag{6.22}$$

From the above equation follows a conclusion:

If the form of the operator x_t is known then the form of the function w will be known also.

Remark 6. It was proved [16] that the expression $\mathrm{d} \ln \rho_{\mu_0}(x - x_t(x), x)/\mathrm{d}t$ in eq. (6.15) has a sense of divergence of measure μ_0, where the divergence of measure μ_0 means the "volume" with respect to measure μ_0 which will flow out of a domain at time t per unit time and per unit "volume".

Example 1. Let

$$x = (q, p) = ((q_i)_{i=1}^{+\infty}, (p_i)_{i=1}^{+\infty}) \tag{6.23}$$

be an element of the phase space $l_a^2 \times l_a^2$. The q_i and p_i denote the position and momentum of the ith point, respectively. Assume that the interaction between points is given by a potential such that:

1. ϕ is a superstable potential [7, 17] which means the following: for every $k = 0, \pm 1, \pm 2, \ldots$ let $n_k(x)$ denote the number of points present in the segment $[k - \frac{1}{2}, k + \frac{1}{2}]$; then there exist constants A and B such that $A > 0$ and

$$U(q) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \phi(|q_i - q_j|) \geqslant -Bn + A \sum_{i=-n}^{n} n_i(x) \tag{6.24}$$

   for every n and $q_1, \ldots, q_n$. This assumption ensures that the potential energy tends to infinity when too many points occur in a small segment.
2. ϕ is of class C^1.
3. There exists a bounded nonincreasing function g such that

$$|\phi(r)| \leqslant g(|r|), \quad |\operatorname{grad} \phi(r)| \leqslant g(|r|) \tag{6.25}$$

for every $r \in \mathbb{R}$ and

$$\int_{0}^{+\infty} g(r)\,\mathrm{d}r < +\infty\,. \tag{6.26}$$

Let μ_0 be a quasi-invariant Gibbs measure [18] with potential ϕ; then it can be proved that the phase flux

$$x_t\colon l_a^2 \times l_a^2 \to l_a^2 \times l_a^2 \tag{6.27}$$

given by the system of Newton equations

$$\dot{q}_i = p_i\,, \qquad \dot{p}_i = -\sum_{j\neq i} \operatorname{grad} \phi(|q_i - q_j|)\,, \qquad i = 0, 1, 2, \ldots, \tag{6.28}$$

together with the measure μ_0 satisfy the assumptions of theorem 2.

7. The Liouville equation of the system of oscillators describing electromagnetic field

In this section we derive the Liouville equation of the system for the phase flux x_t: $H \to H$ which is given by the linear operators $A(t)$, $t \in \mathbb{R}$. Moreover, we shall assume that for every t the generalized divergence (see definition 8 below) of the operator $A(t)$ exists. As an initial measure we take a quasi-invariant Gaussian measure μ_0. The support of μ_0 has to be sufficiently large in order to contain all the possible configurations of the considered physical system. Besides, we assume that for almost every x with respect to μ_0 all vectors of the form $x - x_t(x)$ are in the set M_{μ_0}; this property simply implies that the expression $\rho_{\mu_0}(x - x_t(x), x)$ is well defined. Later we will show (lemma 3) that there exist Gaussian measures possessing the above properties.

Consider the phase flux

$$A(t)\colon H \to H\,, \quad t \in \mathbb{R}\,. \tag{7.1}$$

According to the polar decomposition theorem [19] there exists a unitary (linear and continuous) operator $U(t)$ and a self-adjoint positive operator $T(t)$ such that

$$A(t) = U(t)T(t)\,. \tag{7.2}$$

In addition, assume that the operator $T(t)$ has only a point spectrum $(\lambda_i(t))_{i=1}^{+\infty}$ and there exists a sequence of natural numbers $(n_i)_{i=1}^{+\infty}$ for which the infinite product

$$\prod_{i=1}^{+\infty}\left(\prod_{k=n_i+1}^{n_{i+1}}\lambda_k(t)\right)=\delta(t) \tag{7.3}$$

is finite.

Definition 8. The expression (7.3) will be called the generalized divergence of a phase flux $A(t)$, $t\in\mathbb{R}$.

Now for every t we define the measure ν_t on Hilbert space H:

$$\nu_t(B):=\mu_0(A(t)^{-1}(B))\,,\quad B\in\sigma(H)\,. \tag{7.4}$$

Below we prove that the measure ν_t is equivalent to measure μ_0; moreover, we derive an explicit form of the Radon–Nikodym derivative $g(x,t)$, i.e.

$$g(x,t):=\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu_0}(x)\,. \tag{7.5}$$

We start with two lemmas.

Lemma 1. Let μ_0 be a quasi-invariant measure (in particular a Gaussian measure) and, for every t, let $A(t)$ be a self-adjoint positive operator for which the generalized divergence exists. Then the Radon–Nikodym derivative of the measure ν_t with respect to measure μ_0 is of the form

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu_0}(x)=\delta(t)\rho_{\mu_0}(x-T(t)^{-1}x,x)\,. \tag{7.6}$$

Proof. Let $(e_k)_{k=1}^{+\infty}$ be a sequence of eigenvectors of the operator $A(t)$ corresponding to eigenvalues $(\lambda_k)_{k=1}^{+\infty}$. Assume that for every $n\in N$

$$L_n:=\operatorname{span}\{e_1,e_2,\ldots,e_n\}\subset M_{\mu_0}. \tag{7.7}$$

Let $\phi(x)$ denote a cylindrical function of the form

$$\phi(x)=g(P_nx)\,, \tag{7.8}$$

where $g(x)$ is a bounded Borel function on L_n and P_n is the orthogonal projection on L_n. Let the measure ν_{tL_n} be the orthogonal projection of the measure ν_t on L_n. Then taking $n = n_i$ we have

$$\begin{aligned}\int \phi(x)\nu_t(\mathrm{d}x) &= \int g(x)\nu_{tL_{n_i}}(\mathrm{d}x) \\ &= \int g(P_{n_i}T(t)x)\mu_0(\mathrm{d}x) = \int g(T(t)x)\mu_{0L_{n_i}}(\mathrm{d}x) \\ &= \int g(x) \prod_{k=1}^{n_i} \lambda_k(t) \frac{p_{n_i}(P_{n_i}T(t)^{-1}x)}{p_{n_i}(P_{n_i}(x))} \mu_0(\mathrm{d}x) ,\end{aligned} \tag{7.9}$$

where $p_{n_i}(x)$ is the Radon–Nikodym derivative of the measure $\mu_{0L_{n_i}}$ with respect to Lebesgue measure in L_{n_i}. On the other hand, the expression

$$\prod_{k=1}^{n_i} \lambda_k(t) \frac{p_{n_i}(P_{n_i}T(t)^{-1}x)}{p_{n_i}(P_{n_i}x)} \tag{7.10}$$

is the Radon–Nikodym derivative of measure $\mu_{L_{n_i}T(t)}$ with respect to measure $\mu_{L_{n_i}}$, where

$$\mu_{L_{n_i}T(t)}(B) := \mu_{L_{n_i}}(T(t)^{-1}(B)) , \quad B \in \sigma(H) . \tag{7.11}$$

From (7.9) we have

$$\int g(P_m x)\nu_t(\mathrm{d}x) = \int g(P_m x) \prod_{k=1}^{n_i} \lambda_k(t) \frac{p_{n_i}(P_{n_i}T(t)^{-1}x)}{p_{n_i}(P_{n_i}x)} \mu_0(\mathrm{d}x) . \tag{7.12}$$

Passing to the limit with i according to the Fatou lemma [20] we obtain

$$\int g(P_m x)\nu_t(\mathrm{d}x) \geqslant \int g(P_m x)\rho_{\mu_0}(x - T(t)^{-1}x, x)\delta(t)\mu_0(\mathrm{d}x) . \tag{7.13}$$

Therefore for every non-negative Borel function ϕ we have

$$\int \phi(x)\nu_t(\mathrm{d}x) \geqslant \int \phi(x)\rho_{\mu_0}(x - T(t)^{-1}x, x)\delta(t)\mu_0(\mathrm{d}x) . \tag{7.14}$$

Hence the measure μ_0 is equivalent to the measure ν_t. Then

$$\begin{aligned}\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu_0}(x) &= \lim_i \frac{\mathrm{d}v_{tL_{n_i}}}{\mathrm{d}\mu_{0L_{n_i}}}(P_{n_i}x) \\ &= \lim_i \prod_{k=1}^{n_i} \lambda_k(t) \frac{p_{n_i}(P_{n_i}T(t)^{-1}x)}{p_{n_i}(P_{n_i}x)} = \delta(t)\rho_{\mu_0}(x - T(t)^{-1}x, x) ,\end{aligned} \tag{7.15}$$

where we used the relation

$$\lim_i \frac{P_{n_i}(P_{n_i}T(t)^{-1}x)}{p_{n_i}(P_{n_i}x)} = \rho_{\mu_0}(x - T(t)^{-1}x, x) . \tag{7.16}$$

This completes the proof of lemma 1.

Lemma 2. Applying the same technique as in the proof of lemma 1 it can be shown that if $A(t) = U(t)$ is a unitary operator then the measure ν_t is equivalent to the measure μ_0 and the Radon–Nikodym derivative is of the form

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}\nu_0}(x) = \rho_{\mu_0}(x - U(t)^{-1}x, x) , \tag{7.17}$$

where

$$\nu_t(B) := \mu_0(U(t)^{-1}(B)) , \quad B \in \sigma(H) . \tag{7.18}$$

Theorem 3. Assume that $A(t)$: $H \to H$ is a linear operator and $A(t) = U(t)T(t)$ is its polar decomposition. Denote by $\delta(t)$ the generalized divergence of $A(t)$. Then the measure ν_t is equivalent to measure μ_0 and the Radon–Nikodym derivative is

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu_0}(x) = \delta(t)\rho_{\mu_0}(x - A(t)^{-1}x, x) .$$

Here $\nu_t(B) := \mu_0(A(t)^{-1}(B))$, $B \in \sigma(H)$.

Proof. From lemmas 1 and 2 for any non-negative continuous function f we have

$$\begin{aligned}
&\int f(U(t)T(t)x)\mu_0(\mathrm{d}x) \\
&\quad = \int f(U(t)x)\delta(t)\rho_{\mu_0}(x - T(t)^{-1}x, x)\mu_0(\mathrm{d}x) \\
&\quad = \int f(x)\rho_{\mu_0}(x - U(t)^{-1}x, x) \\
&\qquad \times \delta(t)\rho_{\mu_0}(U(t)^{-1}x - T(t)^{-1}U(t)^{-1}x, U(t)^{-1}x)\mu_0(\mathrm{d}x) .
\end{aligned} \tag{7.19}$$

Making use of the identity

$$\rho_{\mu_0}(a + b, x) = \rho_{\mu_0}(a, x)\rho(b, x - a) \tag{7.20}$$

we obtain

$$\rho_{\mu_0}(x - U(t)^{-1}x, x)\rho_{\mu_0}(U(t)^{-1}x - T(t)^{-1}U(t)^{-1}x, U(t)^{-1}x)$$
$$= \rho_{\mu_0}(x - T(t)^{-1}U(t)^{-1}x, x) . \tag{7.21}$$

Hence according to (7.19) and (7.21) we have

$$\int f(U(t)T(t)x)\mu_0(\mathrm{d}x)$$
$$= \int f(x)\delta(t)\rho_{\mu_0}(x - A(t)^{-1}x, x)\mu_0(\mathrm{d}x) . \tag{7.22}$$

Finally, in view of the arbitrariness of function f we obtain the explicit form of the Radon–Nikodym derivative:

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}\mu_0}(x) = \delta(t)\rho_{\mu_0}(x - A(t)^{-1}x, x) . \tag{7.23}$$

The proof is complete.

Theorem 4 *. Let μ_0 be a quasi-invariant measure on H corresponding to the initial probability density of a certain physical system and let $A(t)$: $H \to H$ be a phase flux of this system.

Then, if μ_0 and $A(t)$ satisfy the assumptions of theorem 3, then the Liouville equation of such a system is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} w(x_t(x), t) + w(x_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \ln \delta(t)$$
$$+ w(x_t(x), t) \frac{\mathrm{d}}{\mathrm{d}t} \ln \rho_{\mu_0}(x - x_t(x), x) = 0 . \tag{7.24}$$

Proof. The proof of this theorem is the same as the proof of theorem 2 except that the expression (7.23) should be used instead of the expression (6.17).

Example 2. We shall verify now that the assumptions of theorem 4 are satisfied for the system of infinite countable number of oscillators which describe the behaviour of an electromagnetic field.

Let us consider a set of four equations [21, 22]:

* Theorem 4 can be generalized to the case when for every t a phase flux is given by affine mappings $x_t(x) = A(t)x + b(t)$, $b(t) \in H$ (the basic outline of the proof is the same as for theorem 4), or even to a nonlinear case [23].

$$\operatorname{rot} B = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t}\,, \quad \operatorname{div} E = 4\pi q\,, \tag{7.25}$$

$$\operatorname{rot} E = -\frac{1}{c}\frac{\partial B}{\partial t}\,, \qquad \operatorname{div} B = 0\,, \tag{7.26}$$

which forms the basis of all electromagnetic phenomena. In the above q, j, E, B denote the densities of charge, current, electric field and magnetic induction, respectively.

Since $\operatorname{div} B = 0$ holds, we can define B in terms of a vector potential:

$$B = \operatorname{rot} A\,. \tag{7.27}$$

In addition we assume $\operatorname{div} A = 0$. Expanding A into Fourier series we obtain

$$A(r, t) = \sqrt{\frac{4\pi c^2}{V}} \sum_k e_k A_k(t)\, \mathrm{e}^{\mathrm{i}kr}\,. \tag{7.28}$$

Summation is carried over all $k = (k_1, k_2, k_3)$, $k_l = 2\pi/L$, $n_l \in \mathbb{Z}$, $l = 1, 2, 3$, e_k denotes a unit vector (from $\operatorname{div} A = 0$ it follows that $ke_k = 0$), V is the volume of a cube with the length of the edge equal to L and $r \in \mathbb{R}^3$.

Then for every k the quantities $A_k(t)\,\mathrm{e}^{\mathrm{i}kr}$ and $A_k^*\,\mathrm{e}^{\mathrm{i}kr}$ can be written in the form

$$\begin{aligned} A_k &= Q_k^2 - \mathrm{i}Q_k^1\,, \quad Q_k^2 = \tfrac{1}{2}(A_k + A_k^*) = Q_{-k}^2\,, \\ A_k^* &= Q_k^2 - \mathrm{i}Q_k^1\,, \quad Q_k^1 = \frac{\mathrm{i}}{2}(A_k - A_k^*) = -Q_{-k}^1\,, \end{aligned} \tag{7.29}$$

where

$$\begin{aligned} Q_k^1 &= \frac{1}{\sqrt{4\pi c^2 V}} \int e_k A(r, t) \sin kr \,\mathrm{d}r\,, \\ Q_k^2 &= \frac{1}{\sqrt{4\pi c^2 V}} \int e_k A(r, t) \cos kr \,\mathrm{d}r\,. \end{aligned} \tag{7.30}$$

From (7.28) and (7.30) we have

$$A(r, t) = \sqrt{\frac{4\pi c^2}{V}} \sum_{k,\alpha} e_k Q_k^\alpha(t) \begin{matrix} \sin kr \\ \cos kr \end{matrix}\,. \tag{7.31}$$

$\sin kr$, $\cos kr$ correspond to $\alpha = 1$ and $\alpha = 2$, respectively.

It is known [21, 22] that the system of equations (7.25), (7.26) is equivalent to the system of equations

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi q\,, \tag{7.32}$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c}\,j\,, \tag{7.33}$$

where the scalar potential ϕ satisfies the equation

$$-\operatorname{grad}\phi = E + \frac{1}{c}\frac{\partial A}{\partial t}\,. \tag{7.34}$$

Substituting (7.31) into (7.33) we obtain the equations of motion for oscillators Q_k^1, Q_k^2:

$$\ddot{Q}_k^\alpha + c^2k^2Q_k^\alpha = \sqrt{\frac{4\pi}{V}}\int e_k j(r,t)\,\frac{\sin kr}{\cos kr}\,\mathrm{d}r\,. \tag{7.35}$$

Let us expand

$$j_k^\alpha(t) := \sqrt{\frac{4\pi}{V}}\int e_k j(r,t)\,\frac{\sin kr}{\cos kr}\,\mathrm{d}r$$

into Fourier series

$$j_k^\alpha(t) = \sum_{n=0}^{+\infty} a_n^{k\alpha}\sin nt + \sum_{n=0}^{+\infty} b_n^{k\alpha}\cos nt\,. \tag{7.36}$$

Of course,

$$\sum_{n=0}^{+\infty}(a_n^{k\alpha})^2 < +\infty\,, \qquad \sum_{n=0}^{+\infty}(b_n^{k\alpha})^2 < +\infty\,. \tag{7.37}$$

Then

$$\ddot{Q}_k^\alpha + c^2k^2Q_k^\alpha = \sum_{n=0}^{+\infty} a_n^{k\alpha}\sin nt + \sum_{n=0}^{+\infty} b_n^{k\alpha}\cos nt\,. \tag{7.38}$$

The solution of eq. (7.38) corresponding to initial conditions $Q_k^\alpha(0) = Q_{k0}^\alpha$, $P_k^\alpha(0) = P_{k0}^\alpha$, $P_k^\alpha(t) := \dot{Q}_k^\alpha(t)$ is of the form

$$\begin{aligned} Q_k^\alpha(t) = {} & Q_{k0}^\alpha\cos ckt + P_{k0}^\alpha\frac{1}{ck}\sin ckt \\ & - \sum_{n=0}^{+\infty}\frac{a_n^{k\alpha}}{c^2k^2-n^2}\frac{n}{ck}\sin ckt + \sum_{n=0}^{+\infty}\frac{a_n^{k\alpha}}{c^2k^2-n^2}\sin nt \\ & - \sum_{n=0}^{+\infty}\frac{b_n^{k\alpha}}{c^2k^2-n^2}\cos ckt + \sum_{n=0}^{+\infty}\frac{b_n^{k\alpha}}{c^2k^2-n^2}\cos nt\,, \end{aligned} \tag{7.39}$$

$$P_k^\alpha(t) = -Q_{k0}^\alpha ck \sin ckt + P_{k0}^\alpha \cos ckt$$
$$-\sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} n \cos ckt + \sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} n \cos nt$$
$$+\sum_{n=0}^{+\infty} \frac{b_n^{k\alpha}}{c^2k^2 - n^2} ck \sin ckt - \sum_{n=0}^{+\infty} \frac{b_n^{k\alpha}}{c^2k^2 - n^2} n \sin nt . \tag{7.40}$$

Now denoting $x = ((Q_k^\alpha), (P_k^\alpha))$ we can write $x_t(x) = ((Q_k^\alpha(t)), (P_k^\alpha(t)))$ as follows:

$$x_t(x) = ((\mathbb{P}_{k1}^\alpha A_k^\alpha(t)(Q_k^\alpha, P_k^\alpha))_{\alpha,k}, (\mathbb{P}_{k2}^\alpha A_k^\alpha(t)(Q_k^\alpha, P_k^\alpha))_{\alpha,k}) + B_k^\alpha(t) , \tag{7.41}$$

where for every pair of indexes α, k:

1) $A_k^\alpha(t)$ is a linear operator, $A_k^\alpha(t)$: $\mathbb{R}^2 \to \mathbb{R}^2$,

$$A_k^\alpha(t)(Q_k^\alpha, P_k^\alpha) = \begin{bmatrix} \cos ckt & \frac{1}{ck} \sin ckt \\ -ck \sin ckt & \cos ckt \end{bmatrix} (Q_k^\alpha, P_k^\alpha)$$
$$= \left(Q_k^\alpha \cos ckt + \frac{1}{ck} P_k^\alpha \sin ckt, -Q_k^\alpha ck \sin ckt + P_k^\alpha \cos ckt \right) . \tag{7.42}$$

2) $\mathbb{P}_{k1}^\alpha$: $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{P}_{k2}^\alpha$: $\mathbb{R}^2 \to \mathbb{R}$ are the projections on the first and second coordinate, respectively, i.e.

$$P_{k1}^\alpha(Q_k^\alpha, P_k^\alpha) = Q_k^\alpha , \qquad \mathbb{P}_{k2}^\alpha(Q_k^\alpha, P_k^\alpha) = P_k^\alpha . \tag{7.43}$$

3) $B_k^\alpha(t) \in \mathbb{R}^2$ and

$$B_k^\alpha(t) = \Big(-\sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} \frac{n}{ck} \sin ckt$$
$$+\sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} \sin nt - \sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} \cos ckt$$
$$+\sum_{n=0}^{+\infty} \frac{b_n^{k\alpha}}{c^2k^2 - n^2} \cos nt, -\sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} n \cos kt$$
$$+\sum_{n=0}^{+\infty} \frac{a_n^{k\alpha}}{c^2k^2 - n^2} n \cos nt + \sum_{n=0}^{+\infty} \frac{b_n^{k\alpha}}{c^2k^2 - n^2} ck \sin ckt$$
$$-\sum_{n=0}^{+\infty} \frac{b_n^{k\alpha}}{c^2k^2 - n^2} n \sin nt \Big) . \tag{7.44}$$

Let us fix a sequence of real positive numbers $a = (a_k^\alpha)_{k,\alpha}$ such that

$$\sum_{\alpha,k} a_k^\alpha < +\infty . \tag{7.45}$$

As a phase space for our system of oscillators we take the Hilbert space

$$L_a^2 := l_a^2 \times l_a^2$$
$$= \left\{ ((Q_k^\alpha)_{\alpha,k}, (P_k^\alpha)_{\alpha,k}) : \sum_{\alpha,k} a_k^\alpha (Q_k^\alpha)^2 < +\infty, \sum_{\alpha,k} a_k^\alpha (P_k^\alpha)^2 < +\infty \right\} . \tag{7.46}$$

For every $t \in \mathbb{R}$ we have a mapping,

$$x_t : L_a^2 \to L_a^2 . \tag{7.47}$$

Now we shall prove the existence of Gaussian measure [14] μ_0 on L_a^2 possessing properties which assure that the relations

$$x - x_t(x) \in M_{\mu_0} , \quad x - x_t^{-1}(x) \in M_{\mu_0} \tag{7.48}$$

are valid.

Lemma 3. Let $\lambda := (\lambda_k^\alpha)$ be a sequence of positive real numbers λ_k^α such that

$$\sum_{\alpha,k} \lambda_k^\alpha < +\infty , \quad \sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} k^2 < +\infty . \tag{7.49}$$

Then, there exists Gaussian measure μ_0 on L_a^2 satisfying the following condition:

$$\mu_0(l_{a/\lambda}^2 \times l_{a/\lambda}^2) = 1 , \tag{7.50}$$

where

$$l_{a/\lambda}^2 \times l_{a/\lambda}^2 := \left\{ ((Q_k^\alpha)_{\alpha,k}, (P_k^\alpha)_{\alpha,k}) : \sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (Q_k^\alpha)^2 < +\infty , \right.$$
$$\left. \sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (P_k^\alpha)^2 < +\infty \right\} . \tag{7.51}$$

Proof. Let $\tilde{\mu}_0$ be a Gaussian measure on the space $l_{a/\lambda}^2 \times l_{a/\lambda}^2$ and let $\theta(z)$ be its characteristic function. Since $\Sigma_{\alpha,k} \lambda_k^\alpha < +\infty$ the space $l_a^2 \times l_a^2$ is the completion of the space $l^2_{a/\lambda} \times l^2_{a/\lambda}$ in the norm defined by the formula

$$\|x\| = \langle x | Cx \rangle_{a/\lambda}\,, \quad x \in l^2_{a/\lambda} \times l^2_{a/\lambda}\,. \tag{7.52}$$

$\langle\ |\ \rangle_{a/\lambda}$ is the scalar product in the space $l^2_{a/\lambda} \times l^2_{a/\lambda}$ and C is the nuclear operator with eigenvalues $(\lambda^\alpha_k)_{k,\alpha}$, i.e.

$$C(((Q^\alpha_k)_{\alpha,k}, (P^\alpha_k)_{\alpha,k})) = ((\lambda^\alpha_k Q^\alpha_k)_{\alpha,k}, (\lambda^\alpha_k P^\alpha_k)_{\alpha,k})\,. \tag{7.53}$$

According to the Bochner theorem [11, 15] measure $\bar{\mu}_0$ can be extended to the measure μ_0 on the space $l^2_a \times l^2_a$. This completes the proof of the lemma.

Remark 6. The measure μ_0 obtained in the above lemma may be taken as an initial measure.

Now we shall prove the validity of relations (7.48). For this purpose it is sufficient to show (remark 5) that $x - x_t(x) \in C^{1/2}(l^2_a \times l^2_a)$, a.e. μ_0, where $C^{1/2}$ is the Hilbert–Schmidt operator and whose eigenvalues are equal to $(\sqrt{\lambda^\alpha_k})_{\alpha,k}$.

We have

$$x - A(t)x = ((Q^\alpha_k), (P^\alpha_k)) - \left(\left(Q^\alpha_k \cos ckt + P^\alpha_k \frac{1}{ck} \sin ckt\right),\right.$$
$$\left.(-Q^\alpha_k ck \sin ckt + P^\alpha_k \cos ckt)\right)\,. \tag{7.54}$$

Hence we should prove that

$$\sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} \left(Q^\alpha_k - Q^\alpha_k \cos ckt - P^\alpha_k \frac{1}{ck} \sin ckt\right)^2 < +\infty\,, \qquad \text{a.e. } \mu_0\,, \tag{7.55}$$

$$\sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} (P^\alpha_k + Q^\alpha_k ck \sin ckt - P^\alpha_k \cos ckt)^2 < +\infty\,, \qquad \text{a.e. } \mu_0\,. \tag{7.56}$$

The above inequalities are implied by the following facts which are immediate consequences of lemma 3, inequality (7.49) and Cauchy–Schwarz inequality:

$$\sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} (Q^\alpha_k \cos ckt)^2 \le \sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} (Q^\alpha_k)^2 < +\infty\,, \qquad \text{a.e. } \mu_0\,, \tag{7.57}$$

$$\sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} \left(P^\alpha_k \frac{1}{ck} \sin ckt\right)^2 \le \sum_{\alpha,k} \frac{a^\alpha_k}{\lambda^\alpha_k} (P^\alpha_k)^2 < +\infty\,, \qquad \text{a.e. } \mu_0\,, \tag{7.58}$$

$$\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} Q_k^\alpha P_k^\alpha \frac{1}{ck} \sin ckt$$

$$\leqslant \sqrt{\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (Q_k^\alpha)^2} \sqrt{\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} \left(P_k^\alpha \frac{1}{ck} \sin ckt\right)^2} < +\infty\,, \quad \text{a.e. } \mu_0\,, \tag{7.59}$$

$$\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (Q_k^\alpha ck \sin ckt)^2$$

$$\leqslant \sqrt{\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (Q_k^\alpha c \sin ckt)^2} \sqrt{\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} k^2} < +\infty\,, \quad \text{a.e. } \mu_0\,, \tag{7.60}$$

$$\sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (P_k^\alpha \cos ckt)^2 \leqslant \sum_{\alpha,k} \frac{a_k^\alpha}{\lambda_k^\alpha} (P_k^\alpha)^2 < +\infty\,, \quad \text{a.e. } \mu_0\,. \tag{7.61}$$

It remains to prove the existence of generalized divergence of the operator $A(t)$. For this purpose it is sufficient to make the following observation:

For every α, k the determinant of the mapping $A_k^\alpha(t)$: $\mathbb{R}^2 \to \mathbb{R}^2$,

$$A_k^\alpha(t)(Q_k^\alpha, P_k^\alpha) = \begin{bmatrix} \cos ckt & \frac{1}{ck} \sin ckt \\ -ck \sin ckt & \cos ckt \end{bmatrix} (Q_k^\alpha, P_k^\alpha)\,,$$

is equal to 1.

8. Final remarks

The obtained mathematical model can be applied to the study of systems having the phase flux x_t such that the measure $\nu_t(\nu_t(A) := \mu_0(x_t^{-1}(A)))$ is equivalent to the initial probability measure μ_0 so that the Radon–Nikodym theorem holds.

The following three questions arise: 1) What is the connection between the solution of the derived Liouville equation and the solution of equations of motion? A partial answer to this question is the expression (6.22). 2) Let us assume that we consider some class of physical systems. What is the most general phase space for this class? It might be interesting to select the space dual to a nuclear space as the phase space. Such a space has the following property: finitely additive cylindrical measures can always be extended to σ-finite measures [13]. 3) How can one extend the deduced mathematical model to systems with continuous spectrum as their phase flux? It seems that the application of appropriate boundary transitions with the use of spectral theorems would be a good idea; one would also have to generalize the concept of phase velocity divergence.

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