GEOMETRICAL AND PHYSICAL GAUGING IN THE THEORY OF DISLOCATIONS

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Geometrical and variational aspects of the gauge theory of static equilibrium distributions of dislocations are analysed. The gauge procedure, based on Kondo and Kröner's Gedanken Experiments for dislocated bodies, is formulated and basal interpretation rules of the theory are proposed. The unimodular invariance of the free gauge field theory is discussed.

1. Introduction

It is well-known that a crystalline solid with a continuous distribution of dislocations can be described as a locally homogeneous body endowed with such non-Euclidean geometry that reduces to the Euclidean one if dislocations are absent (e.g. [2, 10–12 and 28]). It can be illustrated by a simple "Gedanken Experiment", called the *cutting-relaxation procedure*, suggested by Kondo [10, 11] and showing that a body with dislocations can be represented by a collection of small homogeneous pieces of the body, consisting of the same crystalline material but translated and rotated with respect to one another (Section 3). This collection fails to mesh to the form of an Euclidean continuous body, but it can be done if the body is endowed with a non-Euclidean geometry compensating the collection elements discrepancy (Section 4).

On the other hand, the second Gedanken Experiment, called the *continuization* of the crystal and suggested by Kröner [14, 16], shows that for a continuously dislocated crystal with a Bravais lattice, there exists (in a continuous limit — see Section 3) a field of three local crystallographic directions defined up to (acting locally) rotational symmetries of that Bravais lattice. The translational symmetries of the crystal are lost in a continuized crystal but an internal length measurement (in general non-Euclidean) along these crystallographic directions is preserved.

We can think of in order to adjust the cutting-relaxation procedure to the continuization procedure, the rotational discrepancy appearing in the former procedure as a spatial representation of relative rotations of local crystallographic directions appearing in the latter procedure ("Consistency condition" — Section 4). If so, a crystalline solid with continuously distributed dislocations can be described with the help of a local action of spatial rotations induced by the breakage of material rotational symmetries of a homogeneous crystalline solid. We can also think of the non-Euclidean internal length measurement along local crystallographic directions, as the one compensating the translational discrepancy of Kondo Gedanken Experiment. It is the very basis for the definition of geometrical elements of a gauge procedure (Section 4). We will formulate variational elements of this procedure based on the theory of static equilibrium states of elastic solids (Sections 2, 5 and 6).

2. Classical elasticity

In mechanics of continua, a *body* is understood as a three-dimensional smooth and oriented manifold $\mathcal{B}$, globally diffeomorphic to an open subset of a three-dimensional Euclidean point space E^3. These global diffeomorphisms as well as the images of the body under their action are called its (global) *configurations*. If $\varphi, \kappa : \mathcal{B} \to E^3$ are configurations, then a diffeomorphism in E^3 defined by

$$\lambda = \varphi \circ \kappa^{-1} : B_\kappa \to B_\varphi , \tag{2.1}$$

where $\mathcal{B}_\kappa = \kappa(\mathcal{B})$, $\mathcal{B}_\varphi = \varphi(\mathcal{B})$, is called the *deformation* (of the configuration $\mathcal{B}_\kappa$) of the body. If we identify the tangent spaces $T_P(\mathcal{B}_\kappa)$ at points $P \in \mathcal{B}_\kappa$ with the real Euclidean vector space $\underset{\sim}{E}^3$ (space of translations in E^3), then the differential $d\lambda$ of a deformation λ defines the so-called *deformation gradient* $\underset{\sim}{F}_\kappa : \mathcal{B}_\kappa \to \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^{3*}$:

$$\forall\, \underset{\sim}{v} \in \underset{\sim}{E}^3 \; \underset{\sim}{F}_\kappa(P)\underset{\sim}{v} = d\lambda_P(\underset{\sim}{v}), \qquad P \in \mathcal{B}_\kappa . \tag{2.2}$$

If we additionally distinguish a point $0 \in E^3$, then the deformation gradient can be written in the form of the Frechet derivative $D\underset{\sim}{\lambda}$ of the localization $\underset{\sim}{\lambda}$ of the deformation (see [27] — Appendix):

$$\begin{aligned} \underset{\sim}{F}_\kappa(P) &= D\underset{\sim}{\lambda}(\overrightarrow{0P}) , \\ \underset{\sim}{\lambda}(\overrightarrow{0P}) &= \overrightarrow{0\lambda(P)}, \qquad P \in \mathcal{B}_\kappa . \end{aligned} \tag{2.3}$$

By using the differentiation rule for composition of functions, we conclude that the deformation gradients $\underset{\sim}{F}_\kappa$ and $\underset{\sim}{F}_{\widehat{\kappa}}$ which are derived from the same actual configuration φ but correspond to two different reference configurations κ and $\widehat{\kappa}$ respectively, are related by

$$\underset{\sim}{F}_\kappa(P) = \underset{\sim}{F}_{\widehat{\kappa}}(\eta(P))\underset{\sim}{P}_\kappa(P) , \tag{2.4}$$

where $\underset{\sim}{P}_\kappa$ denotes the deformation gradient of the deformation $\eta = \widehat{\kappa} \circ \kappa^{-1} : \mathcal{B}_\kappa \to \mathcal{B}_{\widehat{\kappa}}$.

Let us consider the charts $X = (X^A) : \mathcal{B} \to R^3$ and $x = (x^k) : E^3 \to R^3$ (global or local) of the body $\mathcal{B}$ and the configuration space E^3, respectively. If $\varphi : \mathcal{B} \to E^3$ is a configuration and $p \in \mathcal{B}$ is a body point, then the coordinates $(X^1, X^2, X^3) = X(p)$ of the point p are called its *Lagrange coordinates* while the coordinates $(x^1, x^2, x^3) = x(\varphi(p))$ of the image $\varphi(p)$ of p under φ are called its *Eulerian coordinates*. If κ is a distinguished reference configuration, we can identify the body $\mathcal{B}$ with its configuration $\mathcal{B}_\kappa$ and then the chart $X_\kappa = X \circ \kappa^{-1} : \mathcal{B}_\kappa \to R^3$ is called also Lagrange coordinates. In this and the next section we will differentiate X_κ-coordinates from X-coordinates but in remaining sections we will deal with a fixed reference configuration κ and both Lagrange coordinate systems will be identified. Let $x = (x^k)$ be Cartesian Eulerian coordinates defined by an orthonormal base $(\underset{\sim}{e}_k; k = 1,2,3)$ of the Euclidean vector space $\underset{\sim}{E}^3$ and by a fixed point $0 \in E^3$, i.e. for $p \in \mathcal{B}$:

$$x(\varphi(p)) = (x^k) \Leftrightarrow \overrightarrow{0\varphi(p)} = x^k \underset{\sim}{e}_k . \tag{2.5}$$

If $x^k = \lambda^k(X_\kappa^A)$ is a coordinate description of the deformation λ (eq. (2.1)) in Lagrange coordinates X_κ and in the above Cartesian coordinates, $X_{\hat{\kappa}}^A = \eta^A(X_\kappa^B)$ is a coordinate description (in Lagrange coordinates X_κ and $X_{\hat{\kappa}}$) of the deformation η describing the passage from $\mathcal{B}_\kappa$ to $\mathcal{B}_{\hat{\kappa}}$, then (see eqs. (2.1)–(2.4)):

$$\begin{aligned} \underset{\sim}{F}_\kappa(P) &= \underset{\sim}{F}(X_\kappa(P)), \qquad \underset{\sim}{P}_\kappa(P) = \underset{\sim}{P}(X_\kappa(P)), \\ \underset{\sim}{F}(X_\kappa) &= \underset{\sim}{e}_k \otimes d\lambda^k(X_\kappa) = \lambda^k{}_{,A}(X_\kappa)\underset{\sim}{e}_k \otimes dX^A , \\ \underset{\sim}{P}(X_\kappa) &= \eta^A{}_{,B}(X_\kappa)\widehat{\partial}_A \otimes dX_\kappa^B, \qquad \widehat{\partial}_A = \partial/\partial X_{\hat{\kappa}}^A . \end{aligned} \tag{2.6}$$

The physically equivalent Eulerian coordinates are those defined by Euclidean geometry of the configuration space E^3, i.e. these are Cartesian coordinates transforming according to the rule

$$\begin{aligned} x'^k &= R^k{}_l x^l + a^l , \\ \underset{\sim}{R} = (R^k{}_l; {}^{k\downarrow 1,2,3}_{l\rightarrow 1,2,3}) &\in 0(3), \qquad \underset{\sim}{a} = (a^l; l \downarrow 1,2,3) \in R^3 , \end{aligned} \tag{2.7}$$

where 0(3) denotes the group of all real orthogonal 3×3 matrices. In order to define the notion of physical equivalence of Lagrange coordinates we have to consider relations describing material properties of the body. First of all, in the framework of continuum mechanics, it is admitted that a body $\mathcal{B}$ is a set endowed with a non-negative scalar measure μ, the *mass distribution* of the body, such that for any configuration φ of $\mathcal{B}$ there exists a mass density ϱ_φ of the body in this configuration and for $\mathcal{P} \subset \mathcal{B}$

$$\mu(\mathcal{P}) = \int_{\mathcal{P}_\varphi} \varrho_\varphi \, dV , \tag{2.8}$$

where $\mathcal{P}_\varphi = \varphi(\mathcal{P})$ and the integral is defined by means of the Lebesgue measure in the Euclidean point space E^3. If κ is a distinguished reference configuration, $\varphi = \widehat{\kappa}$ is another reference configuration and $\eta = \widehat{\kappa} \circ \kappa^{-1}$, then from eqs. (2.6) and (2.8) it follows that

$$\varrho_\kappa = |\det \underset{\sim}{P}_\kappa|(\varrho_{\hat{\kappa}} \circ \eta) . \tag{2.9}$$

From eqs. (2.6) and (2.9) it follows that the following transformation of affine Lagrange coordinates:

$$\begin{aligned} X'^A &= P^A{}_B X^B + b^A , \\ \underset{\sim}{P} = (P^A{}_B) \in \mathrm{GL}(3), |\det \underset{\sim}{P}| &= 1, \qquad \underset{\sim}{b} = (b^A) \in R^3 , \end{aligned} \tag{2.10}$$

where GL(3) denotes the group of all real nonsingular 3×3 matrices, represents the relation of physical undistinguishability of two reference configurations defined by their mass density measuring.

Next, we have to define a response of the material structure of the body on its deformation. In the local theory of the *elastic response* of material bodies, this response depends only on the actual configuration of atoms in a macroscopically small ("physically infinitesimal") neighbourhood of each point of the body. In such neighbourhood a deformed material structure of the body (e.g. its crystalline structure) may be considered as homogeneously deformed. In this way the description of the elastic response of a deformed material body may be reduced to the consideration of the influence of deformation gradients on internal forces of this body. If we deal with elasticity as a classical Lagrangian field theory, the equations describing locally a static equilibrium configuration of an elastic body can be formulated in the form of Euler–Lagrange equations for variations of an action I_κ of actual configurations φ defining deformations of a reference configuration $\mathcal{B}_\kappa$:

$$I_\kappa(\mathcal{P};\varphi) = \int_{\mathcal{P}_\kappa} l_\kappa[\varphi], \tag{2.11}$$

where $\mathcal{P}_\kappa = \kappa(\mathcal{P})$, $\mathcal{P} \subset \mathcal{B}$ is a three-dimensional regular region with a regular closed boundary and $l_\kappa[\varphi]$ is a differential 3-form on $\mathcal{B}_\kappa$ (a Lagrangian) functionally depending on φ only through point values of the deformation (2.1) and its deformation gradient, i.e. (see eqs. (2.2), (2.3) and (2.6)):

$$\begin{aligned} l_\kappa[\varphi](P) &= l_\kappa(P, \lambda(P), \underset{\sim}{F}_\kappa(P)) \\ &= L_\kappa(X_\kappa, \underset{\sim}{\lambda}(X_\kappa), \underset{\sim}{F}(X_\kappa))d\mu_\kappa(X_\kappa)|_{X_\kappa = X_\kappa(P)}, \end{aligned} \tag{2.12}$$

where $\underset{\sim}{\lambda}(X_\kappa) = \lambda^k(X_\kappa)\underset{\sim}{e}_k$ is a coordinate description of the localization $\underset{\sim}{\lambda}$ of λ (eq. $(2.3)_2$) and $d\mu_\kappa$,

$$\begin{aligned} d\mu_\kappa(X_\kappa) &= \varrho_\kappa(X_\kappa)dV(X_\kappa), \\ dV(X_\kappa) &= dX^1_\kappa \wedge dX^2_\kappa \wedge dX^3_\kappa \end{aligned} \tag{2.13}$$

denotes the mass density 3-form in the reference configuration $\mathcal{B}_\kappa$. The condition of the action functional independence on the choice of a reference configuration means that one should have (see eqs. (2.6), (2.9), (2.12) and (2.13)):

$$L_\kappa(X_\kappa, \underset{\sim}{\lambda}(X_\kappa), \underset{\sim}{F}(X_\kappa))|\det \underset{\sim}{P}(X_\kappa)| = L_{\hat\kappa}(X_{\hat\kappa}, \underset{\sim}{\widehat\lambda}(X_{\hat\kappa}), \underset{\sim}{F}(X_{\hat\kappa})), \tag{2.14}$$

where $X_{\hat\kappa} = \eta(X_\kappa)$ and $\underset{\sim}{x} = \underset{\sim}{\widehat\lambda}(X_{\hat\kappa})$ are coordinate descriptions of the deformation $\eta = \widehat\kappa \circ \kappa^{-1}$ and the localization $\underset{\sim}{\widehat\lambda}$ of the deformation $\widehat\lambda = \varphi \circ \widehat\kappa^{-1}$ (eq. $(2.3)_2$), respectively. The action functional should also be *frame indifferent*, i.e. it should be invariant under spatial isometries. It is equivalent to the assumption that the Lagrangian l_κ has the following form [26]:

$$\begin{aligned} l_\kappa[\varphi](P) &= l_\kappa(P, \underset{\sim}{C}_\kappa(P)) \\ &= -\sigma_\kappa(X_\kappa, \underset{\sim}{C}(X_\kappa))d\mu_\kappa(X_\kappa)|_{X_\kappa = X_\kappa(P)}, \end{aligned} \tag{2.15}$$

where σ_κ is the stored (elastic) energy function in the reference configuration $\mathcal{B}_\kappa$ and $\underset{\sim}{C}_\kappa(P)$ is the so-called material deformation tensor (or right Cauchy–Green tensor) defined by

$$\begin{aligned} \underset{\sim}{C}_{\kappa}(P) &= (\kappa^{-1*}\underset{\sim}{c}_{\varphi})_P, \quad P \in \mathcal{B}_{\kappa}, \\ \underset{\sim}{c}_{\varphi}(p) &= \varphi^*(\underset{\sim}{\delta})_p, \quad p \in \mathcal{B}, \end{aligned} \tag{2.16}$$

where $\underset{\sim}{\delta} = \delta_{kl}\underset{\sim}{\varepsilon}^k \otimes \underset{\sim}{\varepsilon}^l$ denotes the Euclidean metric tensor on E^3, φ is an actual configuration and ψ^* denotes pull back of tensors by ψ [19]. It is easy to see that

$$\begin{aligned} \underset{\sim}{C}_{\kappa}(P) &= \underset{\sim}{F}_{\kappa}(P)^T \underset{\sim}{F}_{\kappa}(P) = \underset{\sim}{C}(X_{\kappa}(P)), \\ \underset{\sim}{C}(X_{\kappa}) &= C_{AB}(X_{\kappa}) dX^A_{\kappa} \otimes dX^B_{\kappa}, \\ C_{AB}(X_{\kappa}) &= \lambda^k{}_{,A}(X_{\kappa})\lambda^l{}_{,B}(X_{\kappa})\delta_{kl}, \end{aligned} \tag{2.17}$$

where the Eulerian Cartesian coordinates (x^k) and the general Lagrangian coordinates (X^A) have been used. The induced on the body $\mathcal{B}$ flat metric tensor $\underset{\sim}{c}_{\varphi}$ is called just as the metric tensor $\underset{\sim}{C}_{\kappa}$ on $\mathcal{B}_{\kappa}$ [19]. Particularly, if κ is a distinguished (global) reference configuration and $\varphi = \kappa$, then

$$\underset{\sim}{c}_{\kappa} = \delta_{ab} d\xi^a \otimes d\xi^b \tag{2.18}$$

and the Riemannian flat manifold $(\mathcal{B}, \underset{\sim}{c}_{\kappa})$ can be identified with the reference configuration $\mathcal{B}_{\kappa}$ considered as a Riemannian submanifold of the Euclidean point space E^3. The $\underset{\sim}{c}_{\kappa}$-orthonormal Lagrange coordinates $\xi = (\xi^a)$ defined by eq. (2.18) can be then considered as Lagrangian coordinates on $\mathcal{B}_{\kappa}$ defined by an orthonormal base $(\underset{\sim}{c}_a; a = 1, 2, 3)$ of the space $\underset{\sim}{E}^3$ and by a fixed point $0 \in E^3$ according to: for $P \in \mathcal{B}_{\kappa}$,

$$\xi(P) = (\xi^a) \Leftrightarrow \overrightarrow{0P} = \xi^a \underset{\sim}{c}_a. \tag{2.19}$$

The form of the stored energy function σ_{κ} (eq. (2.15)) means that the so-called second Piola–Kirchhoff stress tensor $\underset{\sim}{S}_{\kappa}$ defined by:

$$\begin{aligned} \underset{\sim}{S}_{\kappa} &= 2\varrho_{\kappa}\frac{\partial \sigma_{\kappa}}{\partial \underset{\sim}{C}} = S^{AB}(X_{\kappa})\partial_A \otimes \partial_B, \\ S^{AB} &= 2\varrho_{\kappa}\frac{\partial \sigma_{\kappa}}{\partial C_{AB}}, \quad \partial_A = \partial/\partial X^A \end{aligned} \tag{2.20}$$

is a symmetric tensor ($\underset{\sim}{S}_{\kappa} = \underset{\sim}{S}^T_{\kappa}$). The so-called first Piola–Kirchhoff stress tensor $\underset{\sim}{t}_{\kappa}$ is defined by (see eqs. (2.12), (2.15) and (2.20)):

$$\begin{aligned} \underset{\sim}{t}_{\kappa} &= -\varrho_{\kappa}\frac{\partial L_{\kappa}}{\partial \underset{\sim}{F}} = t^A{}_k \partial_A \otimes \underset{\sim}{\varepsilon}^k, \\ t^A{}_k &= -\varrho_{\kappa}\frac{\partial L_{\kappa}}{\partial F^k_A} = S^{AB}F^l{}_B\delta_{lk}, \quad F^k{}_A = \lambda^k{}_{,A}. \end{aligned} \tag{2.21}$$

The symmetric Cauchy stress tensor $\underset{\sim}{T}_{\varphi}$, appearing in the spatial description of elastic bodies, is defined by

$$\begin{aligned} \underset{\sim}{T}_{\varphi} \circ \lambda &= (\varrho_{\varphi} \circ \lambda/\varrho_{\kappa})\underset{\sim}{F}\,\underset{\sim}{t}_{\kappa} = (T^{kl} \circ \lambda)\underset{\sim}{\varepsilon}_k \otimes \underset{\sim}{\varepsilon}_l, \\ T^{kl} \circ \lambda &= (\varrho_{\varphi} \circ \lambda/\varrho_{\kappa})S^{AB}F^k{}_A F^l{}_B. \end{aligned} \tag{2.22}$$

If external forces are absent then Euler–Lagrange equations have, in Cartesian (Eulerian as well as Lagrangian) coordinate systems, the following form [36]:

$$\partial_A t_k^A = 0 \,. \tag{2.23}$$

If the body is homogeneous, i.e. if (see eqs. (2.12) and (2.15)):

$$\begin{gathered} l_\kappa[\varphi](P) = L_\kappa(\underset{\sim}{F}(X_\kappa))d\mu_\kappa(X_\kappa)|_{X_\kappa = X_\kappa(P)}\,, \\ L_\kappa(\underset{\sim}{F}) = -\sigma_\kappa(\underset{\sim}{F}^T \underset{\sim}{F})\,, \end{gathered} \tag{2.24}$$

then eqs. $(2.6)_1$ and (2.20)–(2.22) define a relation between stresses and deformation gradients of the form:

$$\underset{\sim}{T}(P) = \underset{\sim}{h}_\kappa(\underset{\sim}{F}_\kappa(P)), \qquad P \in \mathcal{B}_\kappa\,, \tag{2.25}$$

where it was denoted $\underset{\sim}{T} = \underset{\sim}{T}_\varphi \circ \lambda$, the domain of the function $\underset{\sim}{h}_\kappa$ is the set $\mathrm{GL}^+(\underset{\sim}{E}^3)$ of all nonsingular and orientation preserving tensors $\underset{\sim}{F} \in \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^{3*} \cong \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^3$ and the values of this function are symmetric tensors $\underset{\sim}{T} \in \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^3$. The function $\underset{\sim}{h}_\kappa$, called the response function (of the considered elastic body), has a structure depending on the reference configuration κ selected. The choice of κ is arbitrary and we must examine the implications of this fact. It is easy to see that if $\widehat{\kappa}$ is a new reference configuration, then we obtain from eqs. (2.4), (2.9), (2.14) and (2.24) that for $P \in \mathcal{B}_\kappa$:

$$\underset{\sim}{h}_{\hat\kappa}(\underset{\sim}{F}_{\hat\kappa}(\eta(P))) = \underset{\sim}{h}_\kappa(\underset{\sim}{F}_\kappa(P))\,, \tag{2.26}$$

which means that the response function defined in such a way characterizes an elastic material independently of the choice of a reference configuration. Thus, although we define the concept of a material with the help of a fixed reference configuration, the property of a material being elastic is an intrinsic property of that material. Therefore, two reference configurations κ and $\widehat{\kappa}$ are physically undistinguishable iff (cf. eqs. (2.4) and (2.9)) for $P \in \mathcal{B}_\kappa$:

$$\begin{gathered} |\det \underset{\sim}{P}_\kappa(P)| = 1\,, \\ \forall\, \underset{\sim}{F} \in \mathrm{GL}^+(\underset{\sim}{E}^3), \qquad \underset{\sim}{h}_{\hat\kappa}(\underset{\sim}{F}) = \underset{\sim}{h}_\kappa(\underset{\sim}{F}\underset{\sim}{P}_\kappa(P)) = \underset{\sim}{h}_\kappa(\underset{\sim}{F})\,. \end{gathered} \tag{2.27}$$

The set of the values of all deformation gradients $\underset{\sim}{P}_\kappa : \mathcal{B}_\kappa \to \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^3$, describing the change of the reference configuration κ and fulfilling the conditions (2.27), constitute a subgroup G_κ of the unimodular group $U(\underset{\sim}{E}^3) \subset \underset{\sim}{E}^3 \otimes \underset{\sim}{E}^3$ called the *symmetry group* with respect to the reference configuration κ of the elastic body $\mathcal{B}$. The elements of this group generate all reference configurations of the elastic body that are physically undistinguishable if we will restrict ourselves to the mass density and stresses measuring only. Note, that the only physically admissible deformations of $\mathcal{B}_\kappa$ are those preserving orientation. Therefore, the physically undistinguishable deformations of a reference configuration are those generated by elements of the (special) unimodular group $\mathrm{SL}(\underset{\sim}{E}^3)$, i.e. it should be $G_\kappa \subset \mathrm{SL}(\underset{\sim}{E}^3)$. In mechanics of continua the *elastic solids* are defined as those for which there exists such reference configuration κ, called an *undeformed state* of the body, that its symmetry group is a subgroup of the special orthogonal group $\mathrm{SO}(\underset{\sim}{E}^3)$. It can be shown [26] that for elastic solids

$$G_\kappa = \{ \underset{\sim}{Q} \in \mathrm{SO}(\underset{\sim}{E}^3) : \forall \underset{\sim}{F} \in \mathrm{GL}^+(\underset{\sim}{E}^3)\ L_\kappa(\underset{\sim}{F}\underset{\sim}{Q}) = L_\kappa(\underset{\sim}{F}) \} . \tag{2.28}$$

In the case of *elastic crystalline solids* the symmetry group G_κ can be considered as the one describing orthogonal point symmetries of an oriented Bravais lattice [26, 27]. The *isotropic elastic solids* are characterized by the condition that $G_\kappa = \mathrm{SO}(\underset{\sim}{E}^3)$ in an undeformed state κ.

Finally, we see that if κ is an undeformed state of a homogeneous elastic solid, $\underset{\sim}{c}_\kappa$ is the metric tensor induced on $\mathcal{B}$ by κ (eq. (2.18)) and $\xi = (\xi^a)$ are $\underset{\sim}{c}_\kappa$-orthonormal Lagrange coordinates on $\mathcal{B}$, then the *material space* of the considered body can be defined as a geometric space with a relation of physical equivalence of all Lagrange coordinates $\xi' = (\xi'^a)$ of the following form (cf. eq. (2.10)):

$$\begin{gathered} \xi'^a = Q^a{}_b \xi^b + q^a , \\ \underset{\sim}{Q} = (Q^a{}_b) \in G \subset \mathrm{SO}(3), \qquad \underset{\sim}{q} = (q^q) \in R^3 , \end{gathered} \tag{2.29}$$

where G is a group of matrices isomorphic with the symmetry group G_κ, and SO(3) denotes the special orthogonal group of 3×3 real matrices. Note, that in the continuum theory of crystalline elastic solids it is admitted that G is a discrete group but the translations $\underset{\sim}{q}$ in the equation (2.29) are assumed to be unrestricted; it is an approximation of symmetries of real crystals revealing their rotational symmetries only [25]. Because of that we will consider the group G as the one describing symmetries of a local material structure of the body, whereas translations appearing in the equation (2.29) will be interpreted as transformations describing the homogeneity of this body (see also the concept of a continuized crystal — Section 3).

3. Dislocated crystalline solid

Assume that a stress-free crystalline solid is loaded by boundary tractions in the elastic regime. The occurrence of crystalline structure defects can be recognized by that unloading does not take the body back to its original configuration. The unloaded state will thus contain residual stress field. On the other hand, we assume that the stored energy is only due to elastic deformations and clearly residual stresses cannot be captured by a deformation gradient because these would model a body that unloads completely [18]. In the case of dislocated bodies we can characterize deformations of that unloaded state based on an assumption that the distorted lattice is uniquely defined everywhere (e.g. [2, 10–12]). Namely, following Kondo, one images removing of infinitely small part of a dislocated crystalline body and allowing it to relax (by removing all boundary tractions) up to an unstrained state called the *natural state*. The discrete material structure of the natural state coincides with a perfect lattice and we can use the difference between these states and the deformed state as a measure of the stored elastic energy. Let us consider, in order to describe this measure explicitly, a point P of a reference configuration $\mathcal{B}_\kappa$ with Lagrange coordinates $X = X_\kappa(\underset{\sim}{P})$, and let dX^A denotes the distance between points $P, Q \in \mathcal{B}_\kappa$. If $\delta x^i(X)$ denotes the distance between these material points in a deformed state $\mathcal{B}_\varphi$ and $\delta\xi^a(X)$ is the same relaxed material element, then the relations

$$\begin{aligned} \delta x^i(X) &= F^i{}_A(X)dX^A\,, \\ \delta\xi^a(X) &= P^a{}_A(X)dX^A\,, \\ \delta x^i(X) &= B^i{}_a(X)\delta\xi^a(X) \end{aligned} \tag{3.1}$$

define the so-called *distortions*: total ($F^i{}_A$), plastic ($P^a{}_A$) and elastic ($B^i{}_a$). The elastic distortion is a measure of the stored elastic energy and from eqs. (3.1) it follows that

$$F^i{}_A(X) = B^i{}_a(X)P^a{}_A(X)\,. \tag{3.2}$$

Repeating the Kondo cutting-relaxation procedure for many small elements of the body, we obtain an amorphous collection of elements of an crystalline solid with a perfect lattice, which are translated and rotated with respect to one another and, therefore, fail to mesh to form a homogeneous Euclidean material continuum. It means, among other things, the discrepancy of relaxed material elements $\delta\xi^a(X_\kappa(P))$, $P \in \mathcal{B}_\kappa$. If we define the moving coframe $\Phi^* = (E^a)$ by

$$\begin{aligned} E^a(X) &= \delta\xi^a(X)\,, \\ E^a(X) &= \overset{a}{e}_A(X)dX^A, \quad \overset{a}{e}_A(X) = P^a{}_A(X)\,, \end{aligned} \tag{3.3}$$

then the *translational discrepancy* of those material elements can be expressed by the nonintegrability condition of Φ^*, i.e. by the condition that, for at least one 1-form E^a, it should be

$$\tau^a = dE^a \neq 0\,. \tag{3.4}$$

Such representation of the translational discrepancy ought to be treated as a continuous limit, neglecting the finitenes of the lattice spacing. We can think, for example, of some limiting process in which lattice constants of a Bravais lattice decrease more and more but the lattice rotational symmetries as well as the mass per unit volume and the content of defects, remain unchanged. The resulting body, called by Kröner *continuized crystal* [14, 16], retains locally the most characteristic properties of the original crystal, namely the existence of three crystallographic directions at each point, the rotational equivalence of triads of these directions and the countability of lattice steps along these directions. Let us consider a moving frame $\Phi = (\underset{\sim}{E}_a; a = 1,2,3)$ of base vectors parallel to local crystallographic directions of the continuized crystal, as the one defining relaxed material line elements of Kondo Gedanken Experiment according to eq. (3.3) and the following duality condition:

$$\underset{\sim}{E}_a(X) = \underset{a}{e}^A(X)\partial_A, \quad \underset{a}{e}^A(X)\overset{b}{e}_A(X) = \delta^b_a\,. \tag{3.5}$$

Then, from the Kröner's Gedanken Experiment, it follows the existence of a *local rotational uncertainty* to select the moving coframe $\Phi^* = (E^a)$, i.e. these are defined up to the following transformation:

$$\begin{aligned} E^a(X) \to E'^a(X) &= Q^a{}_b(X)E^b(X)\,, \\ \underset{\sim}{Q} = (Q^a{}_b) &: \mathcal{B}_\kappa \to G \subset \mathrm{SO}(3)\,, \end{aligned} \tag{3.6}$$

where G is the group of point symmetries of an ideal Bravais reference lattice S_0, defining a discrete crystalline structure of the considered crystalline solid and identified with the material symmetry group of an homogeneous elastic crystalline continuum (see Section 2). A pair (Φ, G) describes the *short-range order* of the dislocated crystalline solid treated as a locally homogeneous body; we will call such pair the *Bravais moving frame*. Moreover, the Kröner's Gedanken Experiment gives also a basis for the definition of a length measurement within the dislocated crystal [16], although (discrete!) translational symmetries of the crystal are losed in the considered limiting process. Namely, because dislocations have no influence on local metric properties of the crystalline body [14], an *internal length measurement* in the body can be defined with the help of a metric tensor g_κ of the form

$$\begin{aligned} g_\kappa(P) &= g_{ab} E^a(X_\kappa(P)) \otimes E^b(X_\kappa(P)), \\ g_{ab} &= \underset{\sim}{C}_a \cdot \underset{\sim}{C}_b = \text{const}, \qquad P \in \mathcal{B}_\kappa, \end{aligned} \tag{3.7}$$

where the constants g_{ab} constitute the metric matrix associated with the base $(\underset{\sim}{C}_a; a = 1,2,3)$ of an ideal Bravais reference lattice S_0 [28]. Note, that although at each point P of the reference configuration $\mathcal{B}_\kappa$ we can define a Bravais lattice S_P, situated in the space $T_P(\mathcal{B}_\kappa) \cong \underset{\sim}{E}^3$ tangent to $\mathcal{B}_\kappa$ at P, with zeroth vector $\underset{\sim}{o}_P \in T_P(\mathcal{B}_\kappa)$ as a point of this lattice and with base vectors $\underset{\sim}{E}_a(X_\kappa(P))$, $a = 1,2,3$, this lattice has only rotational symmetries of the reference lattice S_0. These base vectors do not describe translational symmetries of S_0 although they define internal length measurement scales along local crystallographic directions at each point of the continuized crystal treated as a locally homogeneous body.

The important fact to be noted is that the natural coframe basis $(d\xi^a)$ for the material space of a homogeneous body (Section 2) is replaced by the moving coframe (E^a) for the material space of a locally homogeneous body. This anholonomic moving coframe carries all the information about defects inherent in the material. For example from the condition (3.4) it follows, that circuit integrals of the 1-forms E^a around boundaries of 2-dimensional regions, can be used to define an intrinsic material Burgers vector. Because of that the triad of exterior derivatives $\tau = (\tau^a; a = 1,2,3)$ is an infinitesimal counterpart of a system of Burgers vectors of a dislocated lattice [3] and is called *Burgers field* [22]. The Burgers field is a measure of the *long range distortion* of the crystalline structure due to dislocations. Its representation with respect to the Bravais moving coframe:

$$\tau^a = \tfrac{1}{2}\tau^a_{bc} E^b \wedge E^c, \tag{3.8}$$

where $\wedge$ denotes the exterior product, is uniquely defined by the so-called object of anholonomity $C^a_{bc} \in C^\infty(\mathcal{B}_\kappa)$ of this coframe:

$$\begin{aligned} \tau^c_{ab} &= -C^c_{ab}, \\ [\underset{\sim}{E}_a, \underset{\sim}{E}_b] &= C^c_{ab} \underset{\sim}{E}_c, \end{aligned} \tag{3.9}$$

where $[\underset{\sim}{E}_a, \underset{\sim}{E}_b] = \underset{\sim}{E}_a \circ \underset{\sim}{E}_b - \underset{\sim}{E}_b \circ \underset{\sim}{E}_a$ is the commutator (bracket) of vector fields $\underset{\sim}{E}_a$ and $\underset{\sim}{E}_b$ considered as first order differential operators (eq. (3.5)). It defines the tensorial measure

$\underset{\sim}{S}[\Phi]$ of the *dislocation density* by [3]:

$$\underset{\sim}{S}[\Phi] = \underset{\sim}{E}_a \otimes \tau^a = S^a{}_{bc}\underset{\sim}{E}_a \otimes E^b \otimes E^c , \quad S^a{}_{bc} = \tfrac{1}{2}\tau^a{}_{bc} . \tag{3.10}$$

Let us observe that the global rescaling of the internal length measurement defined by

$$\Phi \to \Phi\underset{\sim}{L} = (\underset{\sim}{E}_a L^a{}_b) , \quad \underset{\sim}{L} = (L^a{}_b) \in \mathrm{GL}^+(3) \tag{3.11}$$

does not change this tensorial measure [29]:

$$\forall\, \underset{\sim}{L} \in \mathrm{GL}^+(3)\ \underset{\sim}{S}[\Phi\underset{\sim}{L}] = \underset{\sim}{S}[\Phi] \tag{3.12}$$

and, therefore, we can assume without loss of generality that (see eq. (3.7)):

$$g_\kappa = \delta_{ab}E^a \otimes E^b \tag{3.13}$$

but, according to the continuized crystal concept, the group G ought to be still the general point symmetries group.

The integral curves of base vectors $\underset{\sim}{E}_a$ are interpreted as *lattice lines* of the distorted crystalline structure [3]. Note, that if the dislocations are absent, i.e. if

$$\tau^a = 0 \tag{3.14}$$

then, up to a global deformation of the body, such defined lattice lines coincide with geodesics of Euclidean parallelism (i.e. these are straight lines). But in general, lattice lines are not geodesics of the parallelism defined by the integral length measurement (i.e. these are not g_κ-geodesics). It can be shown [35] that lattice lines are g_κ-geodesics iff the base vectors $\underset{\sim}{E}_a$ are Killing vectors with respect to g_κ, i.e. iff

$$L_{\underset{\sim}{E}_a} g_\kappa = 0 , \tag{3.15}$$

where L denotes the Lie derivative operator (e.g. [6]). It is equivalent to the condition that the moving frame $\Phi = (\underset{\sim}{E}_a)$ spans a three-dimensional real Lie algebra $\mathfrak{g}[\Phi]$ of Φ-parallel vector fields:

$$\mathfrak{g}[\Phi] = \{\underset{\sim}{v} = v^a\underset{\sim}{E}_a : v^a = \mathrm{const}\} \tag{3.16}$$

isomorphic with the Lie algebra so(3) of three-dimensional rotations, i.e. having the following commutation rules (see Appendix):

$$[\underset{\sim}{E}_a, \underset{\sim}{E}_b] = \varepsilon_a{}^c{}_b\underset{\sim}{E}_c , \tag{3.17}$$

where $\varepsilon_a{}^c{}_b = \delta^{cd}\varepsilon_{adb}$ and ε_{abc} is the permutation symbol. The lattice lines system defined by the condition (3.17) possesses rotational symmetries only and thereby can be considered as the one being a continual counterpart of discrete *disclinations* (on classification of discrete line defects see [22]). Therefore, according to this definition, disclinations are rather a type of a distribution of dislocations than a separate kind of line defects. It should be stressed that the above definition of continuously distributed disclinations is not generally accepted in the literature (e.g. [4, 8]). More generally, a continuous distribution of dislocations defined by

$$C^a_{bc} = \text{const}, \text{ i.e. } S^a{}_{bc} = \text{const} \tag{3.18}$$

is called *uniformly dense* [28]. From the theory of Lie algebras it then follows that there exists a finite number of types of uniformly dense distributions of dislocations [28]; these are labelled by types of non-isomorphic three-dimensional real Lie algebras (see e.g. [1]). For example, it follows from this classification that there exists also another one *orthogonal type* of uniformly dense distribution of dislocations, corresponding to the Lie algebra SO(2,1) of the three-dimensional Lorentz rotations group SO(2,1). These are dislocations of a shear type because a Lorentz rotation can be considered as the shear deformation changing a square onto a rhomb (see [27] — Appendix). An Abelian Lie algebra (i.e. with $C^a_{bc} = 0$) describes the case when dislocations are absent.

4. Geometrical gauging

In Section 3 it was shown that the *translational discrepancy* appearing in Kondo Gedanken Experiment can be described, in terms of Kröner's Gedanken Experiment, with the help of a Bravais moving frame (Φ, G) (eqs. (3.3)–(3.6)) and an internal length measurement tensor g_κ associated with this moving frame (eqs. (3.7) and (3.13)). Let us consider, in order to describe the *rotational discrepancy* appearing in Kondo Gedanken Experiment, infinitesimal relative rotations of local crystallographic directions appearing in Kröner's Gedanken experiment. These relative rotations can be described with the help of a Cartan connection ∇ assigning to a fixed moving frame $\Phi = (\underset{\sim}{E}_a)$ a matrix of 1-forms $\omega^a{}_b$ with values in the Lie algebra so(3) of infinitesimal three-dimensional rotations (e.g. [33]). Namely, if

$$\begin{aligned} \nabla \underset{\sim}{E}_a &= \omega^b{}_a \otimes \underset{\sim}{E}_b\,, \\ \omega^a{}_b &= \omega_c{}^a{}_b E^c = \omega_A{}^a{}_b dX^A\,, \end{aligned} \tag{4.1}$$

then (see Appendix):

$$\begin{aligned} \underset{\sim}{\omega} &= (\omega^a{}_b) = \omega^c \otimes \underset{\sim}{\gamma}_c \in \wedge^1(\mathcal{B}_\kappa) \otimes \text{so}(3)\,, \\ \omega^c &= \overset{c}{\omega}_A dX^A, \qquad (\underset{\sim}{\gamma}_c)^a{}_b = \varepsilon_c{}^a{}_b = \delta^{ad}\varepsilon_{cdb}\,, \end{aligned} \tag{4.2}$$

where $\wedge^1(\mathcal{B}_\kappa)$ denotes the space of 1-forms on $\mathcal{B}_\kappa$, $\underset{\sim}{\gamma}_a$, $(a = 1, 2, 3)$ are infinitesimal generators of the rotation group SO(3) and ε_{abc} is the permutation symbol. It means that

$$\begin{aligned} \omega^a{}_b &= \varepsilon^a{}_{bc}\omega^c = -\omega^b{}_a\,, \\ \omega_A{}^a{}_b &= \varepsilon^a{}_{bc}\overset{c}{\omega}_A, \qquad \omega_c{}^a{}_b = -\omega_c{}^b{}_a \end{aligned} \tag{4.3}$$

and $\omega_c{}^a{}_b$ are the so-called Ricci coefficients of rotation describing the above mentioned infinitesimal relative rotations. The antisymmetry condition $(4.3)_1$ is equivalent to the metric compatibility condition:

$$\nabla g_\kappa = 0 \tag{4.4}$$

stating that ∇ is a Riemann–Cartan connection associated with g_κ. Note, that there exist two kinds of such connections. Namely, these corresponding to the only ones non-trivial Lie subgroups of SO(3): 3-parameters group $G = \mathrm{SO}(3)$ and 1-parameter abelian group $G = G(\underset{\sim}{n})$ — the Lie group of all rotations about a fixed axis parallel to a versor $\underset{\sim}{n} = n^a \underset{\sim}{\delta}_a$ (see Appendix). A Cartan connection assigning to the moving frame $\Phi = (\underset{\sim}{E}_a)$ a matrix $\underset{\sim}{\omega}$ of 1-forms with values in the Lie algebra $\mathfrak{g}(\underset{\sim}{n}) \subset \mathrm{so}(3)$ of the group $G(\underset{\sim}{n})$ is defined by eqs. (4.1)–(4.3) with

$$\begin{aligned} &\omega^c = n^c\omega, \qquad \underset{\sim}{\omega} = \omega \otimes \underset{\sim}{\gamma}_{\underset{\sim}{n}}, \qquad \omega = \omega_A dX^A, \\ &\underset{\sim}{\gamma}_{\underset{\sim}{n}} = n^c \underset{\sim}{\gamma}_c, \qquad \delta_{ab}n^a n^b = 1, \qquad n^c = \mathrm{const.} \end{aligned} \tag{4.5}$$

The elastic distortion $B^k{}_a$ (Section 3) defines an elastic deformation of the Bravais moving frame according to

$$F^k(X) = B^k{}_a(X)E^a(X) = F^k{}_A(X)dX^A, \tag{4.6}$$

where $F^k{}_A$ denotes the total distortion defined by eqs. (3.2) and $(3.3)_2$. Let us define an isometry between material and configurational Euclidean spaces by (cf. eqs. (2.5) and (2.19)):

$$\begin{aligned} \underset{\sim}{c}_a &= L^k{}_a \underset{\sim}{\xi}_k, \\ \delta_{ab} &= L^k{}_a L^l{}_b \delta_{kl}, \qquad \underset{\sim}{L} = (L^k{}_a) \in \mathrm{SO}(3). \end{aligned} \tag{4.7}$$

For transformations $x^k \to x'^k$ and $\xi^a \to \xi'^a$ of Cartesian coordinates defined by eqs. (2.7) and (2.29) respectively, we have

$$x^k = L^k{}_a \xi^a, \qquad x'^k = L^k{}_a \xi'^a \tag{4.8}$$

iff

$$\underset{\sim}{R} = T(\underset{\sim}{Q}) = \underset{\sim}{L}\underset{\sim}{Q}\underset{\sim}{L}^{-1}, \qquad \underset{\sim}{a} = \underset{\sim}{L}\underset{\sim}{q}. \tag{4.9}$$

Therefore, if the elastic distortion covers with an isometry (i.e. if $B^k{}_a = L^k{}_a$), then the Bravais moving coframe (Φ^*, G) can be identified with its spatial representation (Φ^*_L, G_L) defined by

$$\begin{aligned} \Phi^*_L &= (e^k), \qquad e^k = L^k{}_a E^a, \\ G_L &= T(G) = \underset{\sim}{L}\underset{\sim}{G}\underset{\sim}{L}^{-1} \subset \mathrm{SO}(3). \end{aligned} \tag{4.10}$$

The local rotational uncertainty transformation (3.6) takes then the form of the following transformation of the moving coframe Φ^*_L:

$$\begin{aligned} &e^k(X) \to e'^k(X) = R^k{}_l(X)e^l(X), \\ &\underset{\sim}{R} = (R^k{}_l) : \mathcal{B}_\kappa \to G_L \subset \mathrm{SO}(3). \end{aligned} \tag{4.11}$$

Now, we can formulate, as the basic postulate of the proposed theory — the following consistency condition of Kondo and Kröner's Gedanken Experiments:

Consistency condition. The rotational discrepancy, appearing in the Kondo Gedanken Experiment, can be represented by a spatial representation of Ricci coefficients of rotation defined by (see eqs. (4.2)–(4.10)):

$$\underset{\sim}{\kappa} = (\kappa^k{}_l) = T(\underset{\sim}{\omega}) = \omega^a \otimes \underset{\sim}{T}_a \in \wedge^1(\mathcal{B}_\kappa) \otimes \mathfrak{g}_L\,,$$
$$\underset{\sim}{T}_a = \underset{\sim}{L}\underset{\sim}{\gamma}_a\underset{\sim}{L}^{-1} = (T_a{}^k{}_l), \qquad \kappa^k{}_l = T_a{}^k{}_l\omega^a\,, \tag{4.12}$$

where $\mathfrak{g}$ and $\mathfrak{g}_L = \underset{\sim}{L}\mathfrak{g}\underset{\sim}{L}^{-1}$ are Lie algebras of Lie groups G and G_L, respectively.

The relation of physical equivalence of triads of crystallographic directions defined by (3.6) induces the following transformation of the connection 1-forms:

$$\omega^a{}_b \to \omega'^a{}_b = Q^a{}_c(\omega^c{}_d Q_b{}^d + dQ_b{}^c)\,, \tag{4.13}$$

where $Q_a{}^b = (\underset{\sim}{Q}^{-1})^b{}_a$, and induces, according to the representation (4.10), the following uncertainty of the representation (4.12) of the rotational discrepancy:

$$\kappa^k{}_l \to \kappa'^k{}_l = R^k{}_n(\kappa^n{}_m R_l{}^m + dR_l{}^n)\,, \tag{4.14}$$

where $\underset{\sim}{R} = (R^k{}_l)$ defines the transformation (4.11). Let $(\underset{\sim}{\Phi}_L, G_L)$, $\Phi_L = (\underset{\sim}{e}_k)$ denotes the spatial representation of the Bravais moving frame (Φ, G) defined by eq. (4.10) and the duality condition $\langle e^k, \underset{\sim}{e}_l\rangle = \delta^k_l$, and let us consider the coordinate description (λ^k) of the localization $\underset{\sim}{\lambda}$ (eq. $(2.3)_2$) defined by the moving frame Φ_L:

$$\underset{\sim}{\lambda}(X) = \lambda^k(X)\underset{\sim}{e}_k(X)\,. \tag{4.15}$$

The transformation (4.11) induces then the following transformation:

$$\lambda^k(X) \to \lambda'^k(X) = R^k{}_l(X)\lambda^l(X) \tag{4.16}$$

being a local version of the transformation (2.7) (with $a^l = 0$). Therefore, the differentials $d\lambda^k$ of coordinates λ^k transform according to

$$d\lambda^k \to d\lambda'^k = R^k{}_l(d\lambda^l + R_n{}^l dR^n{}_m\lambda^m)\,. \tag{4.17}$$

The above "contamination" of the deformation gradient $\underset{\sim}{F}$ (cf. eqs. (2.3) and (2.6)) by local spatial rotations is a mathematical expression of the physical reasoning from Section 3, stating that we cannot evaluate the stored elastic energy of a dislocated body by using that local measure of deformations. Because of that we have to formulate such local description of deformations, that will compensate for those contaminating influence of local spatial rotations. It means a replacement of the deformation gradient with a geometric object $h^k{}_A$ defining the total distortion $F^k{}_A$:

$$F^k{}_A(X) = h^k{}_A(X, \lambda^l(X), \lambda^l{}_{,A}(X)) \tag{4.18}$$

and transforming under the local action of the group G_L according to following rule:

$$F^k{}_A(X) \to F'^k{}_A(X) = R^k{}_l(X)F^l{}_A(X) \tag{4.19}$$

being a local version of the deformation gradient transformation rule under spatial isometries (see Section 2). For example, the 1-form F^k of the total distortion (eq. (4.6)) defined as

$$F^k = \nabla\lambda^k = d\lambda^k + \kappa^k{}_l\lambda^l \tag{4.20}$$

transforms according to the rule (4.19) because from eqs. (4.14)–(4.17) it follows that

$$\nabla\lambda^k \to \nabla'\lambda'^k = R^k{}_l \nabla\lambda^l . \tag{4.21}$$

Note, that in this example, the deformation gradient $\underset{\sim}{F}$ is replaced for the tensor field $\underset{\sim}{F}_\Phi$ of the form

$$\underset{\sim}{F}_\Phi(X) = \underset{\sim}{e}_k(X) \otimes \nabla\lambda^k(X) = \nabla \underset{\sim}{\lambda}(X) . \tag{4.22}$$

Therefore, a compensation for the contaminating influence of local spatial rotations, can be realized by means of the replacement of the Frechet derivative operation with the Riemann–Cartan covariant differentiation.

Each Bravais moving frame determines an associated internal length measurement tensor g_κ: namely, the one in which this particular frame is orthonormal (eq. (3.13)). The corresponding Riemann–Cartan connection (eq. (4.4)) defines both the curvature 2-form

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \tfrac{1}{2} R^a{}_{bCD} dX^C \wedge dX^D \tag{4.23}$$

and the torsion 2-form

$$\begin{aligned} T^a &= dE^a + \omega^a{}_b \wedge E^b = \tfrac{1}{2} T^a{}_{BC} dX^B \wedge dX^C \\ &= \tfrac{1}{2} T^a{}_{bc} E^b \wedge E^c . \end{aligned} \tag{4.24}$$

The matrix $\underset{\sim}{\Omega}$ of curvature 2-forms of the Riemann–Cartan connection has values in the Lie algebra $\mathfrak{g} \subset \mathrm{so}(3)$:

$$\begin{gathered} \underset{\sim}{\Omega} = (\Omega^a{}_b) = R^c \otimes \underset{\sim}{\gamma}_c \varepsilon \wedge^2 (\mathcal{B}_\kappa) \otimes \mathfrak{g} , \\ R^a = d\omega^a - \tfrac{1}{2} \varepsilon^a{}_{bc} \omega^b \wedge \omega^c = \tfrac{1}{2} R^a{}_{CD} dX^C \wedge dX^D , \\ R^a = n^a d\omega \text{ if } G = G(\underset{\sim}{n}) , \end{gathered} \tag{4.25}$$

where $\wedge^2(\mathcal{B}_\kappa)$ denotes the space of 2-forms on $\mathcal{B}_\kappa$. From eqs. (4.23) and (4.25) it follows that

$$\Omega^a{}_b = \varepsilon^a{}_{bc} R^c, \text{ i.e. } R^a{}_{bCD} = \varepsilon_c{}^a{}_b R^c{}_{CD} . \tag{4.26}$$

The Riemann–Cartan torsion has the following representation:

$$T^a = dE^a - \varepsilon^a{}_{bc} \omega^b \wedge E^c . \tag{4.27}$$

The eqs. (4.25) and (4.27) are Cartan structure equations for the Riemann–Cartan geometry. The integrability conditions of these equations, called *Bianchi identities*, have the following form:

$$\begin{aligned} \nabla R^a &= dR^a + \omega^a{}_b \wedge R^b = 0 , \\ \nabla T^a &= dT^a + \omega^a{}_b \wedge T^b = \varepsilon^a{}_{bc} R^c \wedge E^b \end{aligned} \tag{4.28}$$

or, equivalently,

$$\begin{aligned} dR^a &= \varepsilon^a{}_{bc} \omega^b \wedge R^c , \\ dT^a &= \varepsilon^a{}_{bc} (\omega^b \wedge T^c - R^b \wedge E^c) . \end{aligned} \tag{4.29}$$

If $G = G(\underset{\sim}{n})$ Bianchi identities reduce to

$$\begin{aligned} dR^a &= 0 \\ dT^a &= \varepsilon^a{}_{bc} n^b (dE^c \wedge \omega - E^c \wedge d\omega) . \end{aligned} \tag{4.30}$$

Note, that from one coframe $\Phi^* = (E^a)$ orthonormal with respect to g_κ we may obtain others by applying a local rotation (eq. (3.6)) and under such transformation the connection 1-forms transform according to the rule (4.13), while the torsion and curvature are tensorial:

$$\begin{aligned} T'^a &= Q^a{}_b T^b , \\ \Omega'^a{}_b &= Q^a{}_c Q_b{}^d \Omega^c{}_d . \end{aligned} \tag{4.31}$$

A preferred orthonormal coframe may be used also to define a new path independent parallelism. The new parallel transport is defined by the condition that in a preferred orthonormal frame, called *orthoparallel* [20], the new connection 1-forms $\lambda^a{}_b$ and its curvature 2-forms $\bigwedge^a{}_b$ vanish:

$$\lambda^a{}_b \stackrel{*}{=} 0, \qquad \bigwedge^a{}_b \stackrel{*}{=} 0 . \tag{4.32}$$

Evidently, the new connection coefficients are generally nonvanishing (while the new curvature vanishes) in any other orthonormal frame:

$$\lambda'^a{}_b \neq 0, \qquad \bigwedge'^a{}_b = 0 . \tag{4.33}$$

This is tied with the fact that the new parallel transport is path independent. Conversely, if a parallel transport is path independent (it is the so called teleparallelism), the curvature vanishes. There then exists a special orthonormal frame $\Phi = (\underset{\sim}{E}_a)$ in which the connection 1-forms $\omega^a{}_b$ vanish. This orthoparallel frame is uniquely determined up to global (constant) rotations. Therefore, the new parallel transport is characterized by torsion and, in the orthoparallel frame, the torsion 2-form is given by eqs. (3.4), (3.8) and (3.9) (cf. eq. (4.24)). Note, that a teleparallel connection is uniquely defined by the condition of ∇ covariant constancy of the Bravais moving frame Φ (see eqs. (4.1) and (4.32)). Therefore, in this case, the set $\mathfrak{g}[\Phi]$ of all Φ — parallel vector fields (eq. (3.16)) coincides with the set of all ∇ — covariantly constant vector fields, i.e. vector fields $\underset{\sim}{v}$ tangent to $\mathcal{B}_\kappa$ and fulfilling the condition

$$\nabla \underset{\sim}{v} = 0 . \tag{4.34}$$

We see, that the tensorial measure $\underset{\sim}{S}[\Phi]$ of the dislocation density (eq. (3.10)) covers with the torsion tensor corresponding to the path independent geometry defined by Φ. This identification can be preserved in the case of a path dependent Riemann–Cartan geometry. Namely, we can think the occurrence of such additional lattice defects in the dislocated body, that its local homogeneity is not disturbed. For example, point defects caused by intersection of dislocation lines and vacancies with self-interstitial atoms allow their description by means on an internal length measurement tensor [17] and thereby a Riemann–Cartan geometry associated with this metric tensor can be considered. Such *point defects* can be called *rotations generating* because their occurrence can be described by means of Ricci coefficients of rotation, vanishing (in an orthoparallel Bravais moving frame — eq. (4.32)) if dislocations are the only sort of defects appearing in the body. Note, that the case $T^a = 0$, $R^a \neq 0$ means that certain distributions of point defects can annihilate dislocation densities [17].

5. Physical gauging

It follows from our description of dislocated crystalline solids that these can be considered as locally homogeneous bodies whose local material symmetries act in a twofold manner. Firstly, their act in the material space of the body as local transformations of an distinguished moving frame (eqs. (3.3)–(3.6)) — the Bravais moving frame carrying informations about defects inherient in the body material structure. Secondly, their act in the configurational space as local transformations of the elastically deformed Bravais moving frame (eqs. (4.14)–(4.22) with $R^k{}_l(X)$ defined by eqs. $(3.6)_2$ and (4.9)). It can also be formulated in terms of the bundle geometry if we assume that the considered material symmetries (of an homogeneous elastic solid — Section 2) form a Lie group $G \subset \mathrm{SO}(3)$, i.e. that $G = \mathrm{SO}(3)$ (isotropic solids) or $G = G(\underset{\sim}{n})$ (tranversally isotropic solids) (see Appendix). Then, the first manner of that action can be treated as the action of the group G as a structure group of a principal bundle of moving frames on the body (e.g. [7, 9, 28, 34]). This principal bundle is the set of all moving frames carrying the same information about defects inherent in the material (see Section 3). The second manner of that action can be viewed as the local action of the group G on the space $R^3(x^k)$ — the arithmetic space R^3 whose points are by standart denoted by $\underset{\sim}{x} = (x^k) = (x^1, x^2, x^3)$ and which is treated as the typical fibre of a vector bundle associated with the above principal bundle (e.g. [7, 34]). The Lie group G acts locally on $R^3(x^k)$ to the right:

$$
\begin{aligned}
&(\underset{\sim}{x}, \underset{\sim}{Q}) \to T(\underset{\sim}{Q})\underset{\sim}{x}, \qquad \underset{\sim}{Q} : \mathcal{B}_\kappa \to G, \\
&(T(\underset{\sim}{Q})\underset{\sim}{x})^k = T(\underset{\sim}{Q})^k{}_l x^l, \qquad T(\underset{\sim}{Q})(X) = T(\underset{\sim}{Q}(X)),
\end{aligned}
\tag{5.1}
$$

where the representation $T : G \to \mathrm{SO}(3)$ is defined by eq. (4.9) with local rotations $\underset{\sim}{Q}(X)$ appearing in the transformation rule (3.6) and $\mathcal{B}_\kappa$ is a reference configuration (with general Lagrange coordinates $X = (X^A)$) identified with the body itself (see Section 2). Since the space $R^3(x^k)$ can be identified with the configurational space $(E^3, x; E(3))$, where $x : E^3 \to R^3$ is a fixed Cartesian chart (eq. (2.5)) and $E(3)$ denotes the group of Euclidean transformations (2.7) of Cartesian coordinates, the considered associated bundle can be taken as a configurational space of the dislocated body. It is the well known geometric framework for the formulation of a gauge theory (e.g. [7]). We will consider the gauge theory with *matter fields* represented by localizations $\underset{\sim}{\lambda}$ of deformations $\lambda : \mathcal{B}_\kappa \to E^3$ (eq. $(2.3)_2$) and with *local gauges*, defined by Bravais moving frames (Φ, G), interpreted as physical fields describing a short-range ordering of the material structure under consideration (Sections 3 and 4).

An evaluation of the stored elastic energy of a dislocated body ought to be formulated in the form allowing us to describe interactions between Bravais moving frames and matter fields, and reducing to the elastic energy of a homogeneous crystalline solid (Section 2) when lattice defects are absent, i.e., in terms of the geometrical gauging (Section 4), if

$$T^a = 0, \quad R^a = 0. \tag{5.2}$$

The condition (5.2) means the existence of a reference configuration $\widehat{\kappa}$ that coincides with an undeformed state of a homogeneous crystalline elastic solid (Section 2), and the existence of Cartesian Lagrange coordinates $\xi = (\xi^a)$ on $\mathcal{B}_{\hat{\kappa}}$ (eq. (2.19)) such that, for a coordinate description $\xi^a = \eta^a(X)$ of the deformation $\eta = \widehat{\kappa} \circ \kappa^{-1} : \mathcal{B}_\kappa \to \mathcal{B}_{\hat{\kappa}}$, the following equations are valid (see eqs. (3.3), (4.26) and (4.32)):

$$\begin{gathered} \omega^a{}_b \overset{*}{=} 0, \quad \Omega^a{}_b = 0, \\ E^a = d\eta^a, \quad \text{i.e. } \overset{a}{e}_A(X) = \eta^a{}_{,A}(X). \end{gathered} \tag{5.3}$$

As a consequence, if $\lambda : \mathcal{B}_\kappa \to \mathcal{B}_\varphi$ and $\widehat{\lambda} : \mathcal{B}_{\widehat{\kappa}} \to \mathcal{B}_\varphi$ are deformations, $x = (x^k)$ are Cartesian Eulerian coordinates (eq. (2.5)) and $x^k = \lambda^k(X)$, $x^k = \widehat{\lambda}^k(\xi)$ are coordinate descriptions of these deformations, then (see eqs. (4.6), (4.12) and (4.20)):

$$\begin{gathered} F^k \overset{*}{=} d\lambda^k, \\ B^k{}_a(X) = \widehat{\lambda}^k{}_{,a}(\xi)|_{\xi=\eta(X)} \end{gathered} \tag{5.4}$$

and the decomposition (3.2) of the total distortion $F^k{}_A$ reduces to a coordinate description of the relation (2.4):

$$\lambda^k{}_{,A}(X) = \widehat{\lambda}^k{}_{,a}(\eta(X))\eta^a{}_{,A}(X). \tag{5.5}$$

The internal length measurement tensor (eq. (3.13)) reduces then to an Euclidean metric tensor induced on the body by its embedding in E^3 (eq. (2.18)). Moreover, $G_{\hat{\kappa}} = G \subset \mathrm{SO}(3)$ and the undeformed state $\widehat{\kappa}$ is a natural state (Section 3) of a homogeneous body. A Lagrangian of a homogeneous crystalline elastic solid has, with respect to the undeformed state $\widehat{\kappa}$, the following form (see eq. (2.24)):

$$\begin{gathered} l_{\hat{\kappa}}(\xi) = L_{\hat{\kappa}}(\widehat{\lambda}^k{}_{,a}(\xi))\varrho_{\hat{\kappa}}\, dV(\xi), \\ \varrho_{\hat{\kappa}} = \text{const}, \quad dV(\xi) = d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \end{gathered} \tag{5.6}$$

and the Lagrangian $l_{\hat{\kappa}}$ will be invariant with respect to global spatial rotations iff

$$\begin{gathered} L_{\hat{\kappa}}(B^k{}_a) = -\sigma_{\hat{\kappa}}(C_{ab}), \\ C_{ab} = B^k{}_a B^l{}_b \delta_{kl}, \end{gathered} \tag{5.7}$$

where $\sigma_{\hat{\kappa}}$ is a function of the stored elastic energy. If $\mathcal{B}_\kappa$ is the other reference configuration, then

$$l_\kappa(X) = l_{\hat{\kappa}}(\xi)|_{\xi=\eta(X)} = L_\kappa(\lambda^k{}_{,A}(X))\varrho_\kappa(X)dV(X), \tag{5.8}$$

where eqs. (2.9), (2.12)–(2.14) and (5.5) were taken into account and it was denoted:

$$\varrho_\kappa(X) = \varrho_{\hat{\kappa}} e(X), \qquad e(X) = \det(\overset{a}{e}_A(X)) . \tag{5.9}$$

A physical gauging is based on an evaluation of the stored elastic energy of the dislocated body by means of the so-called *minimal replacement* $l_\kappa \to l_m$ defined by

$$\begin{aligned} l_m(X) = l_m(F^k, E^a)(X) &= L_m(F^k(X), E^a(X)) dV(X) \\ &= L_\kappa(\nabla_A \lambda^k(X)) d\mu_\kappa(X) , \end{aligned} \tag{5.10}$$

where F^k (eqs. (4.6) and (4.18)) has the form (4.20), ∇ is the Riemann–Cartan covariant derivative (eqs. (4.4) and (4.12)), 3-form $d\mu_\kappa$ is defined by eqs. (2.13) and (5.9), and

$$\begin{aligned} L_\kappa(F^k{}_A) &= -\sigma_\kappa(C_{AB}) , \\ C_{AB} &= F^k{}_A F^l{}_B \delta_{kl} = \overset{a}{e}_A \overset{b}{e}_B C_{ab} , \end{aligned} \tag{5.11}$$

where the tensor C_{ab} is defined by $(5.7)_2$ with

$$B^k{}_a = \underset{a}{e}^A F^k{}_A = \nabla_a \lambda^k . \tag{5.12}$$

The such defined Lagrangian l_m has still the global spatial and material orthogonal invariance (see Section 2) but it is not (in general) invariant with respect to the global (spatial as well as material) translations. However, when rotations generating point defects (Section 4) are absent, i.e. if

$$R^a = 0, \qquad T^a \neq 0 , \tag{5.13}$$

then the distribution of dislocations is described by a path independent geometry (Section 4) and, in an orthoparallel moving frame (uniquely defined up to global rotations — Section 4):

$$\begin{aligned} \omega^a{}_b \overset{*}{=} 0, \qquad T^a \overset{*}{=} \tau^a &= dE^a , \\ F^k \overset{*}{=} d\lambda^k . \end{aligned} \tag{5.14}$$

The Lagrangian l_m is then invariant under global spatial translations. Note, that in the case (5.13), the Lagrangian l_m considered for a general Bravais moving frame is invariant under local covariantly constant spatial translations (eqs. (3.16) and (4.34) with $\underset{\sim}{E}_a$ replaced for $\underset{\sim}{e}_l$). If rotations generating point defects occur, the above translational invariance (global as well as local) is broken (see further on — Final remarks). The lack of material translational invariance is due to heterogeneity of dislocated bodies. The lack of spatial translational invariance is due to proper stresses produced by lattice defects and interacting with matter fields. Moreover, l_m is additionally invariant under local (rotational) material symmetries, defining physically equivalent Bravais moving frames (eq. (3.6)), and — under local spatial rotations defining energetically equivalent total distortions (eqs. (4.20), (4.21) and (5.11)). Therefore, the minimal replacement $l_\kappa \to l_m$ means a generalization of the frame indifference principle of classical elasticity theory (Section 2) based on the treatment of the material symmetry group $G \subset SO(3)$ as a gauge group. This generalization, although similar to the one formulated by Utiyama [31] in order to describe a gravitational field, differs from it because here not all possible local rotations have to be compensated (i.e. in general $G_L \neq SO(3)$) — eq. (4.10)). Only isotropic materials (Section 2) are those for which the proposed gauge procedure is a three-dimensional spatial analogy of the Utiyama approach; such full analogy has been proposed by Turski [30] but without a clear statement that it is possible for isotropic materials only.

Since in a fixed Bravais moving coframe the internal length measurement tensor is univocally determined (eq. (3.13)), we can assume, without loss of generality, that a first-order local static theory of mutual interactions between continuously distributed dislocations and matter fields is described by a Lagrange 3-form l with $(\lambda^k, E^a, \omega^a)$ as independent variables (see eqs. (4.1)–(4.3), (4.12) and (4.20)) and depending additionally on their exterior derivatives. The so-called *minimal coupling* construction for the gauge group G means an assumption that the total Lagrangian l can be written in the form of a sum of the minimally replaced matter Lagrangian l_m (eqs. (2.13), (5.9)–(5.12)) and a G-invariant Lagrangian l_g of the gauge field (E^a, ω^a):

$$l = l_m(\nabla\lambda^k, E^a) + l_g(E^a, \omega^a, dE^a, d\omega^a)\,. \tag{5.15}$$

It follows from the Cartan structure equations $(4.25)_2$ and (4.27) that the Lagrangian l_g can be assumed in the form

$$\begin{aligned} l_g &= L_g(E^a, \omega^a, T^a, R^a)dV \\ &= L'_\kappa(E^a, \omega^a, T^a, R^a)d\mu_\kappa \end{aligned} \tag{5.16}$$

depending on the curvature (R^a) and the torsion (T^a) 2-forms, and more convenient to physical interpretations. In a fixed Lagrange coordinate system $X = (X^A)$ (see eqs. (3.3), (4.1), (4.3), (4.24), $(4.25)_2$ and (5.10)):

$$\begin{aligned} L_m &= L_m(\nabla_A\lambda^k, \overset{a}{e}_A) = L_\kappa(\nabla_A\lambda^k)\varrho_\kappa\,, \\ L_g &= L_g(\overset{a}{e}_A, \overset{a}{\omega}_A, T^a{}_{AB}, R^a{}_{AB}) = L'_\kappa(\overset{a}{e}_A, \overset{a}{\omega}_A, T^a{}_{AB}, R^a{}_{AB})\varrho_\kappa\,. \end{aligned} \tag{5.17}$$

The variation δl of the Lagrangian l with respect to field variables $(\nabla\lambda^k, \omega^a, E^a, T^a, R^a)$ is defined as (e.g. [24, 32]):

$$\begin{aligned} \delta l = \delta\nabla\lambda^k \wedge \frac{\partial l}{\partial\nabla\lambda^k} + \delta E^a \wedge \frac{\partial l}{\partial E^a} + \delta\omega^a \wedge \frac{\partial l}{\partial\omega^a} \\ + \delta T^a \wedge \frac{\partial l}{\partial T^a} + \delta R^a \wedge \frac{\partial l}{\partial R^a}\,. \end{aligned} \tag{5.18}$$

The Cartan structure equations enable us to write δl in terms of variables $(\lambda^k, E^a, \omega^a)$:

$$\begin{aligned} \delta l = &-\delta\lambda^k \wedge \nabla\left(\frac{\partial l}{\partial\nabla\lambda^k}\right) + \delta E^a \wedge \left(\nabla\left(\frac{\partial l}{\partial T^a}\right) + \frac{\partial l}{\partial E^a}\right) \\ &+ \delta\omega^a \wedge H_a + d\left(\delta\lambda^k \wedge \frac{\partial l}{\partial\nabla\lambda^k} + \delta\omega^a \wedge \frac{\partial l}{\partial R^a} + \delta E^a \wedge \frac{\partial l}{\partial T^a}\right), \end{aligned} \tag{5.19}$$

where it was denoted

$$\begin{aligned}\nabla\left(\frac{\partial l}{\partial \nabla\lambda^k}\right) &= d\left(\frac{\partial l}{\partial \nabla\lambda^k}\right) - \kappa^n_k \wedge \frac{\partial l}{\partial \nabla\lambda^n},\\ \nabla\left(\frac{\partial l}{\partial R^a}\right) &= d\left(\frac{\partial l}{\partial R^a}\right) - \omega^b_a \wedge \frac{\partial l}{\partial R^b},\\ \nabla\left(\frac{\partial l}{\partial T^a}\right) &= d\left(\frac{\partial l}{\partial T^a}\right) - \omega^b_a \wedge \frac{\partial l}{\partial T^b},\\ H_a = T_a{}^k{}_n\lambda^n \frac{\partial l}{\partial \nabla\lambda^k} &+ \frac{\partial l}{\partial \omega^a} + \nabla\left(\frac{\partial l}{\partial R^a}\right) - \varepsilon^c{}_{ab}E^b \wedge \frac{\partial l}{\partial T^c}.\end{aligned} \tag{5.20}$$

It follows from eq. (5.19) that for every three-dimensional regular region $\mathcal{P} \subset \mathcal{B}_\kappa$ (with a regular closed boundary), variations δI_κ of the action integral I_κ (cf. eq. (2.11)):

$$\delta I_\kappa(\mathcal{P}; \lambda^k, E^a, \omega^a) = \int_{\mathcal{P}} \delta l(X) \tag{5.21}$$

vanish for variations $(\delta\lambda^k, \delta E^a, \delta\omega^a)$ vanishing on the boundary of $\mathcal{P}$, iff the following *Euler–Lagrange equations* are fulfilled:

$$\begin{aligned}\nabla\left(\frac{\partial l}{\partial \nabla\lambda^k}\right) &= 0,\\ \nabla\left(\frac{\partial l}{\partial T^a}\right) + \frac{\partial l}{\partial E^a} &= 0,\\ H_a = 0 \quad &\text{if} \quad G = \mathrm{SO}(3),\\ n^a H_a = 0 \quad &\text{if} \quad G = G(\underset{\sim}{n}).\end{aligned} \tag{5.22}$$

The variations of the considered p-forms induced by infinitesimal rotations (Appendix (A8)–(A12)) have the following form (see eqs. (3.6), (4.14) and (4.31)):

$$\begin{aligned}\delta E^a = \varepsilon^a{}_b E^b, \quad \delta T^a &= \varepsilon^a{}_b T^b,\\ \delta\Omega^a{}_b &= \varepsilon^a{}_c \Omega^c{}_b - \varepsilon^c{}_b \Omega^a{}_c,\\ \delta\omega^a{}_b = \nabla\varepsilon^a{}_b &= d\varepsilon^a{}_b + \omega^a{}_c\varepsilon^c{}_b - \omega^d{}_b\varepsilon^a{}_d.\end{aligned} \tag{5.23}$$

Since

$$(\varepsilon^a{}_b) = \varepsilon^c \underset{\sim}{\gamma}_c, \qquad \text{i.e. } \varepsilon^a{}_b = \varepsilon^a{}_{bc}\varepsilon^c, \tag{5.24}$$

we obtain that

$$\begin{aligned}\delta E^a = \varepsilon^c\varepsilon^a{}_{bc}E^b, &\quad \delta T^a = \varepsilon^c\varepsilon^a{}_{bc}T^b,\\ \delta R^a = \varepsilon^c\varepsilon^a{}_{bc}R^b, &\quad \delta\omega^a = \nabla\varepsilon^a = d\varepsilon^a + \omega^a{}_b\varepsilon^b\end{aligned} \tag{5.25}$$

and (see eqs. (4.9), (4.16) and (4.21)):

$$\begin{aligned}\delta\lambda^k = \eta^k{}_l\lambda^l, &\quad \delta\nabla\lambda^k = \eta^k{}_l\nabla\lambda^l,\\ \eta^k{}_l &= -\varepsilon^a T^k_{a\,l}.\end{aligned} \tag{5.26}$$

If $G = G(\underset{\sim}{n})$, then $\varepsilon^c = \varepsilon n^c$ and formulas (5.25) take the form

$$\begin{aligned} \delta E^a &= \varepsilon n^c \varepsilon^a{}_{bc} E^b, \quad \delta\omega^a = n^a d\varepsilon\,, \\ \delta T^a &= \varepsilon n^c \varepsilon^a{}_{bc} T^b, \quad \delta R^a = 0\,. \end{aligned} \tag{5.27}$$

From eqs. (5.19) and (5.25)–(5.27) it follows that the variation $\delta_G l$ of the Lagrangian l induced by infinitesimal rotations can be written in the following form:

$$\begin{aligned} \delta_G l &= \varepsilon^c (\delta_G l)_c + d\left(\varepsilon^c \frac{\partial l}{\partial \omega^c}\right), \\ (\delta_G l)_c &= \Delta_c + \varepsilon^a{}_{bc} R^b \wedge \frac{\partial l}{\partial R^a} \quad \text{if } G = \mathrm{SO}(3)\,, \\ (\delta_G l)_c &= \Delta_c = -T_c{}^k{}_m \nabla\lambda^m \wedge \frac{\partial l}{\partial \nabla\lambda^k} - \nabla\left(\frac{\partial l}{\partial \omega^c}\right) + \\ &+ \varepsilon^a{}_{bc}\left(T^b \wedge \frac{\partial l}{\partial T^a} + E^b \wedge \frac{\partial l}{\partial E^a}\right) \quad \text{if } G = G(\underset{\sim}{n})\,. \end{aligned} \tag{5.28}$$

The condition of the action functional invariance with respect to the gauge group G is equivalent to the condition (e.g. [36]):

$$\delta_G l = 0\,, \tag{5.29}$$

which in particular means that should be

$$\begin{aligned} \frac{\partial l}{\partial \omega^a} &= 0 \quad \text{if } G = \mathrm{SO}(3)\,, \\ n^a \frac{\partial l}{\partial \omega^a} &= 0 \quad \text{if } G = G(\underset{\sim}{n})\,. \end{aligned} \tag{5.30}$$

Let us introduce, in order to formulate the others invariance conditions, the following designations:

$$\begin{aligned} L &= L_m + L_g = (L_\kappa + L'_\kappa)\varrho_\kappa\,, \\ \Sigma^A{}_B &= \sigma^A{}_B - L\delta^A{}_B, \quad \sigma^A{}_B = -\varrho_\kappa \overset{a}{e}_B \frac{\partial L'_\kappa}{\partial \overset{a}{e}_A} \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} \underset{a}{\sigma}^A &= \frac{\partial L}{\partial \overset{a}{e}_A} = \underset{a}{e}^B \Sigma^A{}_B\,, \\ \underset{a}{m}^{AB} &= \frac{\partial L}{\partial T^a_{AB}} = \varrho_\kappa \frac{\partial L'_\kappa}{\partial T^a_{BA}} = -\underset{a}{m}^{BA}\,, \\ \underset{a}{n}^{AB} &= \frac{\partial L}{\partial R^a_{AB}} = \varrho_\kappa \frac{\partial L'_\kappa}{\partial R^a_{AB}} = -\underset{a}{n}^{BA}\,, \\ n^{AB} &= n^a \underset{a}{n}^{AB} = -n^{BA} \end{aligned} \tag{5.32}$$

and let us generalize the first ($t^A{}_k$) and the second (S^{AB}) Piola–Kirchhoff stress tensors and the Cauchy stress tensor (T^{kl}) (see eqs. (2.20)–(2.22), (5.11) and (5.17)):

$$
\begin{aligned}
t^A{}_k &= -\frac{\partial L}{\partial \nabla_A \lambda^k} = -\varrho_\kappa \frac{\partial L_\kappa}{\partial \nabla_A \lambda^k}\,,\\
S^{AB} &= 2\varrho_\kappa \frac{\partial \sigma_\kappa}{\partial C_{AB}} = S^{BA}\,,\\
T^k{}_l &= (\varrho_\varphi \circ \lambda/\varrho_\kappa) t^A{}_l \nabla_A \lambda^k\,,\\
T^{kl} = T^k{}_m \delta^{ml} &= (\varrho_\varphi \circ \lambda/\varrho_\kappa) S^{AB} \nabla_A \lambda^k \nabla_B \lambda^l = T^{lk}\,.
\end{aligned} \tag{5.33}
$$

Then, from eqs. (5.28), (5.30) and from the symmetry of the Cauchy stress tensor (eq. $(5.33)_4$), it follows that the invariance condition (5.29) reduces to the following conditions:

$$
\begin{aligned}
&\Pi^{ab} = \Theta^{ab} + N^{ab} = \Pi^{ba} && \text{if } G = \mathrm{SO}(3)\,,\\
&\varepsilon_{abc}\Theta^{ab} n^c = 0 && \text{if } G = G(\underset{\sim}{n})\,,
\end{aligned} \tag{5.34}
$$

where $A^{ab} = \delta^{bc} A^a{}_c$ and it was denoted

$$
\begin{aligned}
\Theta^{ab} &= M^{ab} + \Sigma^{ab}\,,\\
\Sigma^b{}_a &= \overset{b}{e}_B \underset{a}{e}^A \Sigma^B{}_A = \sigma^b{}_a - L\delta^b{}_a\,,\\
M^b{}_a &= \tfrac{1}{2} \underset{a}{m}^{AB} T^b{}_{AB}\,, \qquad N^b{}_a = \tfrac{1}{2} \underset{a}{n}^{AB} R^b{}_{AB}\,.
\end{aligned} \tag{5.35}
$$

We can rewrite, taking into account conditions (5.30) and designations (5.32), (5.33), the Euler–Lagrange equations (5.22) in the following form:

$$
\begin{aligned}
&\nabla_A t^A{}_k = 0\,,\\
&\nabla_B \underset{a}{m}^{AB} = \underset{a}{\sigma}^A\,,\\
&\nabla_B \underset{a}{n}^{AB} = \underset{a}{j}^A \quad \text{if } G = \mathrm{SO}(3)\,,\\
&\nabla_B n^{AB} = j^A \quad \text{if } G = G(\underset{\sim}{n})\,,
\end{aligned} \tag{5.36}
$$

where it was denoted (see eqs. (4.1) and (4.12)):

$$
\begin{aligned}
\nabla_A t^A{}_k &= \partial_A t^A{}_k - \kappa_A{}^l{}_k t^A{}_l\,,\\
\nabla_B \underset{a}{m}^{AB} &= \partial_B \underset{a}{m}^{AB} - \omega_B{}^b{}_a \underset{b}{m}^{AB}\,,\\
\nabla_B \underset{a}{n}^{AB} &= \partial_B \underset{a}{n}^{AB} - \omega_B{}^b{}_a \underset{b}{n}^{AB}
\end{aligned} \tag{5.37}
$$

and the following currents have been introduced:

$$
\begin{aligned}
\underset{a}{j}^A &= -T_a{}^k{}_l t^A{}_k \lambda^l + \varepsilon^c{}_{ab} \overset{b}{e}_B \underset{c}{m}^{AB}\,,\\
j^A &= n^a \underset{a}{j}^A\,.
\end{aligned} \tag{5.38}
$$

Note, that from eqs. (5.34), $(5.37)_2$ and (5.38) it follows that if we will denote:

$$\nabla_A \underset{a}{j}{}^A = \partial_A \underset{a}{j}{}^A - \omega_A{}^b{}_a \underset{b}{j}{}^A , \tag{5.39}$$

then the following *balance equations*:

$$\begin{aligned} \nabla_A \underset{a}{j}{}^A &= \varepsilon_{abc} N^{bc} \quad &&\text{if } G = \mathrm{SO}(3) , \\ \partial_A j^A &= 0 &&\text{if } G = G(\underset{\sim}{n}) \end{aligned} \tag{5.40}$$

would be fulfilled. If $G = \mathrm{SO}(3)$ (dislocated isotropic solids) then there exist 21 independent variables: 3 variables of the matter field λ^k, 9 independent connection coefficients $\omega_A{}^a{}_b$ (see eq. (4.3)) and 9 components $\overset{a}{e}_A$ of the Bravais moving coframe. The 21 Euler–Lagrange equations $(5.36)_1$ are constrained by 3 invariance conditions $(5.40)_1$ and constitute 18 independent field equations. As a result, a complete specification of the geometric structure of the dislocated isotropic continuum needs 3 more equations. In other words, there remain 3 degrees of freedom to be gauged. The freedom is called gauge freedom and it is specified by *gauge conditions*. For example, the continuity equations

$$\nabla_A \underset{a}{\sigma}{}^A = 0 \tag{5.41}$$

can be taken as such conditions. If $G = G(\underset{\sim}{n})$ (dislocated tranversally isotropic solids) then the number of independent connection coefficients reduces to 3 coefficients ω_A (see eq. (4.5)) and so the number of independent variables is 15. The 13 Euler–Lagrange equations are here constrained by the condition $(5.40)_2$. Therefore, we have also 3 gauge degrees of freedom and (5.41) can be also taken as gauge conditions.

Since (see eq. (4.1)):

$$\nabla_A t^A{}_k = \overset{b}{e}_A \nabla_b (\underset{a}{e}{}^A t^a{}_k) = \nabla_a t^a{}_k + \omega_a{}^a{}_b t^b{}_k , \tag{5.42}$$

where it was denoted

$$\nabla_a t^a{}_k = \partial_a t^a{}_k - \kappa_a{}^l{}_k t^a{}_l \tag{5.43}$$

and components $T^a{}_{bc}$ of the torsion 2-form T^a have the form (see eqs. (3.4), (3.8), (3.9) and (4.24)):

$$\begin{aligned} T^c{}_{ab} &= \omega_a{}^c{}_b - \omega_b{}^c{}_a - C^c_{ab} , \\ \omega_a{}^b{}_b &= \sout{C^b_{ab}} = \sout{-C^b_{ba}} = 0 \end{aligned} \tag{5.44}$$

so, we can rewrite eq. $(5.36)_1$ in the following equivalent form:

$$\nabla_a t^a{}_k + q_k = 0, \qquad q_k = \sout{T^a{}_{ab} t^b{}_k} \; (T^a{}_{ab} - C^a_{ba}) t^b{}_k . \tag{5.45}$$

The equation (5.45) is an equilibrium condition of the elastic body with defects of the dislocation type; the force q_k is called the *inhomogeneity force*. The form (5.45) of the equilibrium condition, suitable for examining the proper stresses produced by dislocations, was first obtained by Turski [30] and next by Noll [21]. The equations $(5.36)_2$ and (5.41) describe the self-interaction of dislocations with $\underset{a}{\sigma}{}^A$ — self-balancing currents and

$\underset{a}{m}^{AB}$— field momenta of self-interacting dislocations. Note, that in general $M^{ab} \neq M^{ba}$ (see eq. (5.35)). This suggest that we interpret M^{ab} as the tensor of *internal couple stresses*, i.e. a quantity analogous to the tensor of couple stresses considered in the theory of polar continua but caused by the self-interaction of dislocations [29]. Namely, in this theory the (asymmetric) couple stresses are a consequence of assuming that the mechanical action of one part of a body on another across a surface is equivalent to a force and moment distribution. The source terms $\underset{a}{j}^A$ in eq. $(5.36)_3$ (or the term j^A in eq. $(5.36)_4$) are due to momentum moments produced by force stresses $t^A{}_k$ (defined by the elastic reaction of the body) and by field momenta $\underset{a}{m}^{AB}$ (defined by the self-interaction of dislocations). The corresponding field momenta $\underset{a}{n}^{AB}$ describe the influence of the curvature tensor and thus — the influence of rotations generating point defects. Therefore, N^{ab} (see eq. (5.35)) can be interpreted as the tensor of *internal double forces stresses*, equivalent to a distribution of double forces with moments (see e.g. [13]). If G = SO(3) then these internal stresses are sources of momentum moments currents $\underset{a}{j}^A$ (eq. $(5.40)_1$). If $G = G(\underset{\sim}{n})$ then we are able to compute the field momenta n^{AB} and the momentum moment current j^A in the $\underset{\sim}{n}$-direction only; moreover, in this case the current j^A is a self balancing quantity (eq. $(5.40)_2$).

6. Final remarks

In Section 5 it was shown that if rotations generating point defects are absent (the condition (5.13)), then the proposed gauge procedure is consistent with the frame indifference principle (see Section 2) globally (eqs. $(5.10)_1$ and $(5.14)_2$) as well as locally (see remarks following eq. (5.14)). Moreover, in this case connection coefficients are univocally defined by the Bravais moving frame (Section 4). In a consequence, the Lagrangian l_g of the free gauge field (eq. (5.16)) can be assumed in the following form (see eqs. (3.4) and (5.14)):

$$l_g = l_g[\Phi] = L_g(E^a, \tau^a)dV\,. \tag{6.1}$$

The gloval invariance (3.12) means that homogeneous deformations of a body with dislocations do not influence their tensorial density $\underset{\sim}{S}[\Phi]$ (eq. (3.10)) — the fundamental physical field describing the distortion of a crystalline structure due to dislocations. Particularly, the tensorial density $\underset{\sim}{S}[\Phi]$ is invariant under unimodular homogeneous deformations of the body, preserving its mass density. Therefore, we can anticipate the same invariance of Euler–Lagrange equations describing a static and self-equilibrium distribution of dislocations [29]. This means that the functional dependence $\Phi \to l_g[\Phi]$ should be invariant under the unimodular group SL(3): (see eq. (3.11)):

$$\forall\, \underset{\sim}{L} \in \mathrm{SL}(3), \qquad l_g[\Phi \underset{\sim}{L}] = l_g[\Phi]\,. \tag{6.2}$$

The Lagrange function L_g fulfilling the condition (6.2) can be taken in the following form (see eqs. (2.9) and (5.9)):

$$L_g(E^a, \tau^a) = f(\underset{\sim}{E}_a \otimes \tau^a, \varrho_\kappa)\,. \tag{6.3}$$

If the interaction of matter fields with rotations generating point defects is taken into accounts, then the translational invariance (global as well as local) of the matter Lagrangian l_m is losed (Section 5). It can be formally removed in the following way (e.g. [8], [18]). The Kondo Gedanken Experiment (Section 3) suggests an expression of the total distortion as a sum of an deformation gradient and infinitesimal rigid body motions:

$$F^k = d\lambda^k + \Delta\lambda^k, \quad \Delta\lambda^k = \kappa^k{}_l\lambda^l + b^k, \tag{6.4}$$

where the term $\kappa^k{}_l\lambda^l$ describes the influence of infinitesimal relative rotations and the term b^k describes the influence of translations on the deformation gradient. Translations b^k may be assumed, according to the proposed representation of the translational discrepancy (Section 3), in the form of spatial representations e^k of covector fields E^a belonging to the Bravais moving coframe (Φ^*, G) (eq. (4.10)). Then, the total distortion 1-form F^k can be written in the following form:

$$\begin{aligned} F^k &= B^k{}_l e^l, \\ B^k{}_l &= D_l\lambda^k = \nabla_l\lambda^k + \delta^k{}_l. \end{aligned} \tag{6.5}$$

The affine covariant derivative $D_l\lambda^k$ (e.g. [3, 9]) is invariant with respect to the local Euclidean transformations

$$\lambda^k(X) \to \lambda'^k(X) = R^k{}_l(X)\lambda^l(X) + a^l(X) \tag{6.6}$$

of the coordinate description (λ^k) of the localization $\underset{\sim}{\lambda}$ of a deformation (eq. (4.15)). Thereby, we are able to define a physical equivalence relation of total distortion 1-forms, based on their form (6.5) and the transformation (6.6) (cf. eq. (5.10)). In a consequence, the minimal replacement $l_\kappa \to l_m$, based on the affine covariant derivative application, is consistent with the frame indifference principle. However, with such defined gauge group, the local material structure of a continuously dislocated body is characterized (see eq. (4.9)) not only by its local material symmetries (rotations belonging to the group G) but also by local translations which are not material symmetries of a continuized crystal (Section 3). Because of that, basing a gauge theory of dislocated bodies on the Euclidean gauge group seems to be a physically incorrect application of the gauge procedure. Note, that if rotations generating point defects occur but the body is undeformed, i.e. if $R^a \neq 0$ and $t^A{}_k = 0$, then the proposed field equations (5.36) can be accepted as those describing a free gauge field.

We know that three-dimensional Lie algebras classify all geometrically permitted types of uniformly dense distributions of dislocations (Section 3). This fact and the invariance condition (6.2) suggest the consideration of such theory of free gauge fields that permits an uniformly dense distribution of dislocations, say of the Lie algebra $\mathfrak{g} \subset \mathrm{sl}(3)$ type (sl(3) — the Lie algebra of the Lie group SL(3)), as a particular solution of field equations [29]. Let us consider, for example, a distribution of dislocations that would not be uniformly dense, but locally, would be everywhere of the type $\mathfrak{g}$ [28]. Such distribution of dislocations can be described with the help of a Cartan connection ∇ assigning to a fixed moving frame a matrix of connection 1-forms $\omega^a{}_b$ with values in $\mathfrak{g}$ (eqs. (4.1) and (4.2) with so(3) replaced by $\mathfrak{g}$). Then, in general

$$\nabla g_{\kappa} = \alpha \neq 0 \tag{6.7}$$

and the coefficients $\omega_c{}^a{}_b$ defined by

$$\omega^a{}_b = \omega_c{}^a{}_b E^c \tag{6.8}$$

can be interpreted as those describing infinitesimal relative unimodular deformations of local crystallographic directions of a continuized crystal (cf. Kröner's Gedanken Experiment — Section 3). For example, the so-called point stacking faults [17] can be considered as point defects producing such relative deformations. Note, that the application here of the Kondo cutting-relaxation procedure reveals an unimodular discrepancy in place of the rotational discrepancy considered in Section 3. Moreover, local rotational uncertainty defined by eq. (3.6) will be preserved only if the considered gauge group (being a Lie subgroup of SL(3)) will contain a material symmetry group (i.e. SO(3) or $G(\underset{\sim}{n})$) as its subgroup.

Appendix

Let us consider the group SO(3) as a 3-parameter Lie group of matrices preserving the metric tensor δ_{ab} of the Euclidean space R^3. Let $\underset{\sim}{\gamma}_a$, $a = 1, 2, 3$, be infinitesimal generators of SO(3), i.e. base vectors of the Lie algebra so(3) of the Lie group SO(3), with commutation rules

$$[\underset{\sim}{\gamma}_a, \underset{\sim}{\gamma}_b] = \gamma^c_{ab} \underset{\sim}{\gamma}_c\,. \tag{A.1}$$

For example, if

$$(\underset{\sim}{\gamma}_c)^a{}_b = \varepsilon_c{}^a{}_b = \delta^{ad} \varepsilon_{cdb}\,, \tag{A.2}$$

where ε_{abc} is the permutation symbol equals to 1 if (a, b, c) is an even permutation of (1,2,3), equal to -1 if it is odd, and 0 otherwise, then [6]:

$$\gamma^c_{ab} = \varepsilon^c_{ab} = \delta^{cd} \varepsilon_{abd}\,. \tag{A.3}$$

A rotation $\underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{k})$ with the rotation vector $\underset{\sim}{k} = k^a \underset{\sim}{\delta}_a$, $\underset{\sim}{\delta}_a = (\delta^b_a;\, b = 1, 2, 2)$, can be represented as

$$\begin{gathered} \underset{\sim}{Q}(\underset{\sim}{k}) = \exp(k^a \underset{\sim}{\gamma}_a)\,, \\ \underset{\sim}{Q}(\underset{\sim}{k})^{-1} = \underset{\sim}{Q}(-\underset{\sim}{k}), \qquad \underset{\sim}{Q}(\underset{\sim}{o}) = \underset{\sim}{I} = (\delta^a{}_b)\,. \end{gathered} \tag{A.4}$$

The tensor $\underset{\sim}{Q}(\underset{\sim}{k})$ represents a rotation about an axis parallel to $\underset{\sim}{k}$ and $k = (\delta_{ab} k^a k^b)^{1/2}$, $0 \leq k \leq \pi$ is the angle of that rotation. The Lie group $G(\underset{\sim}{n})$ of all rotations about a fixed axis parallel to a versor $\underset{\sim}{n} = n^a \underset{\sim}{\delta}_a$ is a 1-parameter Lie group with elements represented as $\underset{\sim}{Q}(\underset{\sim}{k}) = \underset{\sim}{Q}(k\underset{\sim}{n}) = \underset{\sim}{Q}_{\underset{\sim}{n}}(k)$, where

$$\begin{gathered} \underset{\sim}{Q}_{\underset{\sim}{n}}(k) = \exp(k \underset{\sim}{\gamma}_{\underset{\sim}{n}})\,, \\ \delta_{ab} n^a n^b = 1, \qquad \underset{\sim}{\gamma}_{\underset{\sim}{n}} = n^a \underset{\sim}{\gamma}_a\,. \end{gathered} \tag{A.5}$$

The group is abelian:

$$\underset{\sim}{Q}_{\tilde{n}}(k_1)\,\underset{\sim}{Q}_{\tilde{n}}(k_2) = \underset{\sim}{Q}_{\tilde{n}}(k_1 + k_2) = \underset{\sim}{Q}_{\tilde{n}}(k_2)\,\underset{\sim}{Q}_{\tilde{n}}(k_1) \tag{A.6}$$

and, under the action of an internal automorphism $\underset{\sim}{L}$ of the group SO(3), the rotation matrix $\underset{\sim}{Q}(\underset{\sim}{k})$ transforms according to:

$$\underset{\sim}{L}\,\underset{\sim}{Q}(\underset{\sim}{k})\underset{\sim}{L}^{-1} = \underset{\sim}{Q}(\underset{\sim}{L}\,\underset{\sim}{k}). \tag{A.7}$$

The matrices $\underset{\sim}{Q} = (Q^a{}_b)$ of infinitesimal rotations, i.e. belonging to the Lie algebra so(3) of the group SO(3), and their inverses have the form:

$$Q^a{}_b = \delta^a{}_b + \varepsilon^a{}_b\,, \qquad (\underset{\sim}{Q}^{-1})^a{}_b = \delta^a{}_b - \varepsilon^a{}_b\,, \tag{A.8}$$

where

$$\varepsilon^a{}_b = k^c(\underset{\sim}{\gamma}_c)^a{}_b = k^c\varepsilon_c{}^a{}_b = -\varepsilon^b{}_a\,. \tag{A.9}$$

Since the matrices of spatial rotations induced by the group G of material rotations have the form (see eqs. (4.9), (A.4) and (A.7)):

$$\begin{gathered} T(\underset{\sim}{Q}) = \exp(k^a\underset{\sim}{T}_a)\,, \\ \underset{\sim}{T}_a = \underset{\sim}{L}\,\underset{\sim}{\gamma}_a\underset{\sim}{L}^{-1} = (T_a{}^k{}_l), \qquad T_a{}^k{}_l = L^k{}_c(\underset{\sim}{L}^{-1})^d{}_l\varepsilon_a{}^c{}_d \end{gathered} \tag{A.10}$$

the corresponding matrices $\underset{\sim}{R} = (R^k{}_l)$ of infinitesimal spatial rotations and their inverses have the form:

$$\begin{gathered} R^k{}_l = \delta^k{}_l + \eta^k{}_l, \qquad (\underset{\sim}{R}^{-1})^k{}_l = \delta^k{}_l - \eta^k{}_l\,, \\ \eta^k{}_l = -k^aT_a{}^k{}_l = -L^k{}_c(L^{-1})^d{}_l\varepsilon^c{}_d\,. \end{gathered} \tag{A.11}$$

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