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Global Invariance and Lie-Algebraic Description in the Theory of Dislocations

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The relationships between the Lie-algebraic description of continuously distributed dislocations and the global affine invariance of their tensorial density are studied. The affinely-invariant Lagrange description of static self-equilibrium distributions of dislocations is proposed and field equations describing distributions of internal stresses and couple stresses are formulated. The analogy between the proposed theory of dislocations and "3-fold electrodynamics" is formulated.

1. INTRODUCTION

In this work we will deal with a crystalline body whose crystal structure is a three-dimensional monoatomic and oriented Bravais lattice, distorted by the occurrence of continuously distributed dislocations (Trzęsowski, 1987*a*, p. 317). This crystalline structure is described, in the theory of dislocations, with the help of a physical requirement that the distorted lattice is uniquely defined everywhere (Bilby *et al.*, 1955). This means that the material structure under consideration can be described by assigning to each point of the body an oriented triad of base vectors:

$$\Phi_p = (\mathbf{E}_a(p);\, a = 1, 2, 3), \qquad \mathbf{E}_a(p) \in T_p(\mathfrak{B}) \tag{1}$$

where $\mathfrak{B}$ denotes a body understood as a three-dimensional smooth and oriented differentiable manifold, globally diffeomorphic to an open and simply-connected subset of a three-dimensional Euclidean point space, and $T_p(\mathfrak{B})$ is the space tangent to the body in the point $p \in \mathfrak{B}$ (Trzęsowski, 1987*b*, p. 1059). We will call the smooth field $\Phi : p \in \mathfrak{B} \to \Phi_p$ the *Bravais moving frame.*

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The base vector fields of a Bravais moving frame $\Phi=(\mathbf{E}_a)$ in general do not commute with each other:

$$[\mathbf{E}_a, \mathbf{E}_b] = C^c_{ab}\mathbf{E}_c, \qquad C^c_{ab} \in C^\infty(\mathfrak{B}) \tag{2}$$

where $[\mathbf{E}_a, \mathbf{E}_b] = \mathbf{E}_a \circ \mathbf{E}_b - \mathbf{E}_b \circ \mathbf{E}_a$ is the commutator product (bracket) of vector fields $\mathbf{E}_a$ and $\mathbf{E}_b$ and functions C^c_{ab} define the so-called *object of anholonomity* (Schouten, 1954). The integral curves of the base vector fields are considered as the lattice lines of the distorted crystalline structure (Bilby, 1960).

Let $\Phi^*: p \in \mathfrak{B} \to \Phi^*_p$ denote the Bravais moving coframe:

$$\begin{gathered} \Phi^*_p = (E^a(p);\ a=1,2,3), \qquad E^a(p) \in T^*_p(\mathfrak{B}) \\ \langle E^a(p), \mathbf{E}_b(p)\rangle = \delta^a_b \end{gathered} \tag{3}$$

where $T^*_p(\mathfrak{B})$ is the cotangent space. The triad of exterior differentials of base covector fields belonging to the Bravais moving coframe

$$\tau = (\tau^a;\ a=1,2,3), \qquad \tau^a = dE^a \tag{4}$$

is an infinitesimal counterpart of the systems of Burgers vectors of the distorted lattice (Bilby, 1960). We will call this triad the *Burgers field.* Therefore, the Burgers field is a measure of a distortion of the crystalline structure due to dislocations. The representation of a Burgers field with respect to the Bravais moving coframe is uniquely defined by the object of anholonomity:

$$\tau^a = S^a_{bc}E^b \wedge E^c, \qquad S^a_{bc} = -\tfrac{1}{2}C^a_{bc} \tag{5}$$

where $\wedge$ denotes the exterior product, and defines a tensorial measure of the dislocation density (Bilby, 1960):

$$\mathbf{S}[\Phi] = \mathbf{E}_a \otimes \tau^a = S^a_{bc}\mathbf{E}_a \otimes E^b \otimes E^c \tag{6}$$

It is easy to see that the dependence $\Phi \to \mathbf{S}[\Phi]$ is invariant under the group $GL^+(3)$ of all real 3×3 matrices with positive determinant (the proper full linear group in 3 real dimensions) (Sławianowski, 1985):

$$\forall \mathbf{L} \in GL^+(3), \qquad \mathbf{S}[\Phi\mathbf{L}] = \mathbf{S}[\Phi] \tag{7}$$

where it was taken into account that the group $GL^+(3)$ acts on the Bravais moving frame $\Phi=(\mathbf{E}_a)$ and the Bravais moving coframe $\Phi^*=(E^a)$ according to the following rules:

$$\begin{aligned} \Phi\mathbf{L} &= (\mathbf{E}_a L^a_b;\ b=1,2,3) \\ \Phi^*\mathbf{L} &= (L^{-1a}_b E^b;\ a=1,2,3) \\ \mathbf{L} &= \|L^a_b;\ a,b=1,2,3\|, \qquad \langle E^a, \mathbf{E}_b\rangle = \delta^a_b \end{aligned} \tag{8}$$

This *global invariance* is very important because it is an invariance property of the fundamental physical field describing the distortion of the crystalline structure due to dislocations (see Section 3).

It is well known that the tensorial density $\mathbf{S}[\Phi]$ of dislocations is identical to the torsion tensor of the so-called teleparallelism connection induced on the body $\mathfrak{B}$ by the Bravais moving frame Φ (see Section 3). The covariant derivative ∇^{Φ} corresponding to this connection is uniquely defined by the demand that the Bravais moving frame Φ should be covariantly constant:

$$\nabla^{\Phi}\mathbf{E}_a = 0, \qquad \text{i.e.,} \quad \nabla^{\Phi}E^a = 0 \tag{9}$$

From the purely geometrical as well as from the physical point of view, the simplest and most "homogeneous" case is that for which not only the Bravais moving frame but also the Burgers field is covariantly constant:

$$\nabla^{\Phi}\tau^a = 0, \qquad \text{i.e.,} \quad \nabla^{\Phi}\mathbf{S} = 0 \tag{10}$$

It is equivalent to the condition that the object of anholonomity is a Φ-parallel geometrical object:

$$C^a_{\ bc} = -2S^a_{\ bc} = \text{const} \tag{11}$$

If the Burgers field is covariantly constant, we will say that the Bravais moving frame and its induced teleparallelism are *closed*. In this case we will also say that the distribution of dislocations is *uniformly dense* (Trzęsowski, 1987*b*, p. 1059). In crystals with uniformly dense distributions of dislocations, all the possible types of the crystal structure distortions can be described by the Bianchi classification of three-dimensional real Lie algebras (Trzęsowski, 1987*b*, p. 1059; see Section 2). Thereby, the question appears of how to construct field equations describing distributions of dislocations in the crystalline body, and admitting a closed Bravais moving frame as their solution. In this work the such field equations, corresponding to the closed Bravais moving frames of an orthogonal type (Section 2), are proposed in the case of static self-equilibrium distributions of dislocations (Section 3).

2. LIE-ALGEBRAIC DESCRIPTION

The closed Bravais moving frame Φ (Section 1) spans a three-dimensional real Lie algebra $\mathfrak{g}[\Phi]$ of Φ-parallel vector fields on the body:

$$\mathfrak{g}[\Phi] = \{\mathbf{v} = v^a\mathbf{E}_a : v^a = \text{const}\} \tag{12}$$

The constants of anholonomity $C^a_{\ bc}$ are the structure constants of this Lie algebra [equations (2) and (11)]. Therefore, in crystals with uniformly dense distributions of dislocations, all the possible types of lattice line systems (see Section 1) can be described by the well-known Bianchi classification of three-dimensional real Lie algebras (e.g., Barut and Rączka, 1977). Thereby, we obtain the classification of types of crystal structure distortions due to uniformly dense distributions of dislocations (Trzęsowski, 1987*b*, p. 1059). In this sense we can talk about the classification of the *basic types of dislocations distributions.* For example, in the case of an *Abelian* Lie algebra

$$C^{a}_{bc}=0, \qquad \text{i.e.,} \quad \mathbf{S}=0 \tag{13}$$

the distortion is removable, i.e., the distortion of lattice lines is induced by a global deformation of a body (Trzęsowski, 1987*a*, p. 317).

In this work we will consider the case of a *simple Lie algebra* characterized by the condition that the so-called Killing metric tensor

$$\begin{gathered}\mathbf{c}[\Phi]=c_{ab}[\Phi]E^{a}\otimes E^{b}\\ c_{ab}[\Phi]=C^{c}_{ad}C^{d}_{bc}=4S^{c}_{ad}S^{d}_{bc}\end{gathered} \tag{14}$$

uniquely defined by the torsion tensor, is nonsingular:

$$\det\|c_{ab}[\Phi]\|\neq 0 \tag{15}$$

There are only two types of simple three-dimensional real Lie algebras. These are Lie algebras of a (nonsingular) orthogonal type represented by Lie algebra $so(3)$ of the Euclidean rotation group $SO(3)$ {tensor $\mathbf{c}[\Phi]$ of signature $(---)$} and Lie algebra $so(2, 1)$ of the three-dimensional Lorentz rotation group $SO(2, 1)$ {tensor $\mathbf{c}[\Phi]$ of signature $(++-)$}. The dislocations of Euclidean rotation type can be called *disclinations* (Trzęsowski, 1987*a*, p. 317; 1987*b*, p. 1059). Therefore, according to this definition, disclinations are rather a type of distribution of dislocations than a separate kind of line defect. It should be stressed that the above definition of (continuously distributed) disclinations is not generally accepted in the literature (e.g., de Witt, 1973). The dislocations of Lorentz rotation type are *dislocations of shear type* because a Lorentz rotation can be considered as the shear deformation changing a square into a rhomb (Trzęsowski, 1987*a*, p. 317).

Let us observe that the Killing metric tensor $\mathbf{c}[\Phi]$ is invariant under the proper full linear group:

$$\forall \mathbf{L}\in GL^{+}(3), \qquad \mathbf{c}[\Phi\mathbf{L}]=\mathbf{c}[\Phi] \tag{16}$$

This suggests that the existence of a metric tensor proportional to the nonsingular Killing-like metric tensor,

$$\begin{gathered}\mathbf{C}[\Phi]=-b^{2}\mathbf{c}[\Phi]=C_{ab}[\Phi]E^{a}\otimes E^{b}\\ C_{ab}[\Phi]=-4b^{2}S^{c}_{ad}S^{d}_{bc}, \qquad S^{a}_{bc}\in C^{\infty}(\mathfrak{B}), \qquad b=\text{const}\end{gathered} \tag{17}$$

where sgn $\mathbf{C}[\Phi] = (+++)$ or $(--+)$, can be considered as a definition of such a class of Bravais moving frames that is closed under the action of the proper full linear group. We will call this metric tensor *associated* with the Bravais moving frame Φ. If the associated metric tensor exists, then we will say that the distribution of dislocations corresponding to it is the distribution of a (nonsingular) *orthogonal type.* Because

$$[\mathbf{S}] = [1], \qquad [\mathbf{C}] = \mathrm{cm}^2, \qquad [\mathbf{E}_a] = [E^a] = [1] \tag{18}$$

where $[\cdot]$ is the designation of an absolute dimension of a tensor (Schouten, 1951) and the cgs unit system is used, we will put $[b] = \mathrm{cm}$. Therefore, the constant b is a characteristic length of the distortion of the lattice and its interpretation as the characteristic length of Burgers vectors seems the most natural.

3. STATIC FIELD EQUATIONS

In our description of a crystalline body with dislocations, the Bravais moving frame is a fundamental physical field describing *locally* the material structure as well as a current configuration of that body. Therefore, in the case of Bravais moving frames noninteracting with other physical fields, the equations describing locally a static equilibrium configuration of the crystalline body can be formulated in the form of Euler–Lagrange equations for variations of an action functional of the self-interacting Bravais moving frame Φ:

$$I(\Omega, \Phi) = \int_\Omega \mathscr{L}[\Phi] \tag{19}$$

where $\Omega \subset \mathfrak{B}$ is a three-dimensional regular region (with a regular closed boundary) and $\mathscr{L}[\Phi]$ is a differential 3-form on $\mathfrak{B}$ (a Lagrangian) functionally depending on Φ.

The global invariance (7) means that homogeneous deformations of a body with dislocations do not influence their tensorial density $\mathbf{S}[\Phi]$—the fundamental physical field describing the distortion of the crystalline structure due to dislocations. Therefore, we can anticipate the same invariance of Euler–Lagrange equations describing a static and self-equilibrium distribution of dislocations. This means that the functional dependence $\Phi \to \mathscr{L}[\Phi]$ should be invariant under the proper full linear group:

$$\forall \mathbf{L} \in GL^+(3), \qquad \mathscr{L}[\Phi\mathbf{L}] = \mathscr{L}[\Phi] \tag{20}$$

Further on we will assume that the distribution of dislocations is an orthogonal type (Section 2). In this case, the existence of the metric tensor $\mathbf{C}[\Phi]$ associated with the Bravais moving frame Φ [equation (17)] enables us to factorize the Lagrangian onto the dynamical scalar $F[\Phi]$: $\mathfrak{B} \to R$ and the natural Riemannian or pseudo-Riemannian volume form:

$$d\mu_\Phi = V[\Phi]\varepsilon[\Phi], \qquad \varepsilon[\Phi] = E^1 \wedge E^2 \wedge E^3$$
$$V[\Phi] = (\det\|C_{ab}[\Phi]\|)^{1/2} = b^3(\det\|4S^c_{ad}S^d_{cb}\|)^{1/2} \tag{21}$$

where $V[\Phi]$ is the local characteristic dimensionless volume of the distortion of the lattice and $\varepsilon[\Phi]$ is the oriented volume induced on $\mathfrak{B}$ by the Bravais moving frame Φ, such that

$$\mathscr{L}[\Phi] = F[\Phi]\, d\mu_\Phi \tag{22}$$

Because

$$\forall \mathbf{L} \in GL^+(3), \qquad d\mu_{\Phi\mathbf{L}} = d\mu_\Phi \tag{23}$$

therefore from the condition (20) it follows that the dynamical scalar $F[\Phi]$ should be invariant under the proper full linear group:

$$\forall \mathbf{L} \in GL^+(3), \qquad F[\Phi\mathbf{L}] = F[\Phi] \tag{24}$$

If the Lagrangian $\mathscr{L}[\Phi]$ is the first-order local, i.e., $\mathscr{L}[\Phi]$ is dependent pointwise on algebraic values of Φ and of its first derivatives, then from equations (7) and (24) it follows that the scalar $F[\Phi]$ can be taken in the form of a function of the torsion tensor:

$$F[\Phi] = f(\mathbf{S}[\Phi]) \tag{25}$$

Let us take the so-called geometric dimensional frame reference $(X, \partial) = (X^A, \partial_B)$ characterized by (Post, 1980)

$$[X^A] = \text{cm}, \qquad [dX^A] = \text{cm}, \qquad [\partial_A] = \text{cm}^{-1} \tag{26}$$

and oriented compatibly with the Bravais moving frame Φ. Let us write in this chart

$$\mathbf{E}_a = \underset{a}{e}^A \partial_A, \qquad E^a = \overset{a}{e}_A\, dX^A, \qquad \tau^a = \overset{a}{\tau}_{AB}\, dX^A \wedge dX^B$$
$$\mathbf{C} = C_{AB}\, dX^A \otimes dX^B, \qquad \mathbf{S} = S^A{}_{BC} \partial_A \otimes dX^B \otimes dX^C \tag{27}$$

Then

$$S^A_{BC} = \underset{a}{e}^A \overset{a}{\tau}_{BC}, \qquad \overset{a}{\tau}_{BC} = \tfrac{1}{2}(\overset{a}{e}_{C,B} - \overset{a}{e}_{B,C})$$
$$C_{AB} = \overset{a}{e}_A \overset{b}{e}_B C_{ab} = 4b^2 S^C_{AD} S^D_{CB}, \qquad \overset{a}{e}_{A,B} = \partial_B \overset{a}{e}_A \tag{28}$$

and from equations (21), (22), and (25) it follows that

$$\begin{aligned} \mathscr{L}[\Phi] &= L(\overset{a}{e}_A, \overset{a}{e}_{A,B})\, dX^1 \wedge dX^2 \wedge dX^3, \qquad [\mathscr{L}] = \mathrm{kg}\cdot\mathrm{cm} \\ L(\overset{a}{e}_A, \overset{a}{e}_{A,B}) &= f(S^A_{BC})\, V(S^A_{BC}), \qquad [L] = \mathrm{kg/cm^2} \\ V(S^A_{BC}) &= b^3(\det\|4S^C_{AD}S^D_{CB}\|)^{1/2}, \qquad [V] = [1] \end{aligned} \tag{29}$$

where f is an unspecified dynamical factor.

The Euler–Lagrange equations corresponding to the Lagrangian (29) have the form of the system of balance equations (Sławianowski, 1985)

$$\partial_B \underset{a}{M}^{AB} = \underset{a}{\sigma}^A, \qquad a = 1, 2, 3 \qquad (A, B = 1, 2, 3) \tag{30}$$

for two kinds of nontensorial geometric objects:

$$\begin{aligned} &\boldsymbol{\sigma}_a = \underset{a}{\sigma}^A \partial_A, \qquad \mathbf{M}_a = \underset{a}{M}^{AB} \partial_A \otimes \partial_B, \qquad \underset{a}{\sigma}^A = \partial L/\partial \overset{a}{e}_A \\ &\underset{a}{M}^{AB} = \partial L/\partial \overset{a}{e}_{A,B} = -\underset{a}{M}^{BA}, \qquad [\boldsymbol{\sigma}_a] = [\mathbf{M}_a] = \mathrm{kg/cm^2} \end{aligned} \tag{31}$$

where $\boldsymbol{\sigma}_a$ are vector densities of weight 1 and $\mathbf{M}_a$ are skew-symmetric, twice-contravariant tensor densities of weight 1, transforming under $GL^+(3)$ according to the following rules:

$$\forall \mathbf{L} \in GL^+(3), \qquad \boldsymbol{\sigma}_a[\Phi\mathbf{L}] = \boldsymbol{\sigma}_b[\Phi] L^b_a, \qquad \mathbf{M}_a[\Phi\mathbf{L}] = \mathbf{M}_b[\Phi] L^b_a \tag{32}$$

The Euler–Lagrange equations can be written down equivalently in the covariant form

$$\nabla^{\mathbf{C}}_B \underset{a}{M}^{AB} = \underset{a}{\sigma}^A \tag{33}$$

where $\nabla^{\mathbf{C}}$ denotes covariant differentiation in the sense of the Levi-Civita connection $\Gamma^A_{BC}[\mathbf{C}]$ built of the associated metric tensor $\mathbf{C} = \mathbf{C}[\Phi]$:

$$\Gamma^A_{BC}[\mathbf{C}] = \tfrac{1}{2} C^{AE}(\partial_C C_{EB} + \partial_B C_{EC} - \partial_E C_{BC}) \tag{34}$$

From the densities $\boldsymbol{\sigma}_a$ and $\mathbf{M}_a$ we can build, in a standard way, two kinds of differential forms:

$$\begin{aligned} &\sigma_a = \underset{a}{\sigma}^1\, dX^2 \wedge dX^3 - \underset{a}{\sigma}^2\, dX^1 \wedge dX^3 + \underset{a}{\sigma}^3\, dX^1 \wedge dX^2 \\ &M^a_b = M^{a1}_b\, dX^2 \wedge dX^3 - M^{a2}_b\, dX^1 \wedge dX^3 + M^{a3}_b\, dX^1 \wedge dX^2 \\ &M^{aA}_b = \overset{a}{e}_B \underset{b}{M}^{BA}, \qquad [\sigma_a] = [M^a_b] = \mathrm{kg}\cdot\mathrm{cm} \end{aligned} \tag{35}$$

The invariance properties of the Lagrangian under consideration lead to two kinds of conservation laws (Sławianowski, 1985). Namely, we obtain the conservation laws describing the *local translational invariance* of the Lagrangian, as a consequence of its invariance with respect to the parallel transport along the base fields $\mathbf{E}_a$:

$$d\sigma_a = 0, \qquad a = 1, 2, 3 \tag{36}$$

and we obtain the conservation laws describing the *global affine invariance* [the invariance of L under $GL^+(3)$]:

$$dM^a_b = 0, \qquad a, b = 1, 2, 3 \tag{37}$$

Let us observe that from the densities appearing in the Euler–Lagrange equations we can also build two tensorial objects having absolute dimensions of stresses:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a \otimes E^a = \sigma^A_B \partial_A \otimes dX^B, \qquad \sigma^A_B = \underset{a}{\sigma}{}^A \overset{a}{e}_B$$

$$\mathbf{M} = \mathbf{M}_a \otimes E^a = M^{AB}_C \partial_A \otimes \partial_B \otimes dX^C, \qquad M^{AB}_C = \underset{a}{M}{}^{AB} \overset{a}{e}_C \tag{38}$$

$$[\boldsymbol{\sigma}] = [\mathbf{M}] = \mathrm{kg/cm^2}$$

From equations (36) and (37) it follows that the tensor $\boldsymbol{\sigma}$ represents the translational invariance, whereas the tensor $\mathbf{M}$ represents the affine invariance of the considered Lagrangian. This suggests that we interpret $\boldsymbol{\sigma}$ and $\mathbf{M}$ as the tensors of internal stresses and couple stresses, respectively. The *tensor of internal stresses* $\boldsymbol{\sigma}$ is the quantity analogous to the tensor of force stresses in elastic continua, but not describing a reaction of the material body on external forces. In the case under consideration internal stresses are caused by the self-interaction of dislocations and reflect the nonlinearity of the theory. It can be shown that the Euler–Lagrange equations are equivalent to the covariant field equations for the internal stresses:

$$\begin{gathered} \nabla^C_A \sigma^A_B + \Theta^C_{AB} \sigma^A_C = 0 \\ \Theta^C_{AB} = \Gamma^C_{AB}[\mathbf{C}] - \Gamma^C_{AB}[\Phi] \end{gathered} \tag{39}$$

where $\Gamma^C_{AB}[\mathbf{C}]$ and ∇^C correspond to the associated metric tensor $\mathbf{C} = \mathbf{C}[\Phi]$ [see equations (33) and (34)] and $\Gamma^C_{AB}[\Phi]$ is the teleparallelism connection corresponding to the Bravais moving frame:

$$\Gamma^A_{BC}[\Phi] = \underset{a}{e}{}^A \overset{a}{e}_{C,B} \tag{40}$$

The *tensor of internal couple stresses* $\mathbf{M}$ is the quantity analogous to the tensor of couple force stresses considered in the theory of polar continua (but caused here by self-interaction of dislocations). In this theory the couple stresses are a consequence of assuming that the mechanical action of one part of a body on another across a surface is equivalent to a force and moment distribution. It is consistent with asymmetry of the contravariant representation $\sigma^{AB} = C^{BC}\sigma^{A}_{C}$ of the internal stress tensor. The Euler–Lagrange equations are also equivalent to covariant field equations for the couple stresses (Sławianowski, 1985):

$$\nabla^{\Phi}_{C} M_{A}^{CB} + \vartheta_{C} M_{A}^{CB} = 0 \qquad \vartheta_{C} = 2S^{D}_{DC} \tag{41}$$

where ∇^{Φ} denotes covariant differentiation in the sense of the teleparallelism connection (40). Both kinds of internal stresses are directly related (Sławianowski, 1985)

$$\sigma^{A}_{B} = S^{A}_{CD} M^{CD}_{B} \tag{42}$$

Closed fields, describing uniformly dense distributions of dislocations (Section 1), form a very narrow and special class of Bravais moving frames. Nevertheless, they are very important because they provide a set of universal solutions of our affinely-invariant equations. Namely, translating the theorem which has been formulated in Sławianowski (1986) into the language of this work, we obtain the following theorem.

Theorem. Closed Bravais moving frames, describing distributions of dislocations of an orthogonal type, satisfy the field equations (41) for any choice of the dynamical factor f in the definition (29) of the Lagrangian.

This theorem means that the uniformly dense distribution of dislocations can be treated as the *fundamental state* of the distorted Bravais structure. Therefore, a fundamental state here is not the state of an ideal crystal structure, but the state of its elementary distortion due to dislocations (cf. Trzęsowski, 1987*b*, p. 1059).

4. FINAL REMARKS

The modern formulation of nonlinear field equations describing continuously distributed dislocations is based on a gauge procedure applied to the Lagrange description of crystalline elastic bodies; this was done in the static case (e.g., Turski, 1966; Gairola, 1981) as well as in the dynamic case (e.g., Kadić and Edelen, 1983). In consequence these theories are based on the idea of a local invariance. In contrast to this approach, our theory is based on the global invariance [under the proper full linear group $GL^{+}(3)$]; the internal couple stresses **M** are connected with this invariance. However, our theory is in line with the "gauge approach," according to which the breaking of global translational symmetries describe the occurrence of dislocations in an elastic crystalline body (e.g., Gairola, 1981; cf. also Trzęsowski, 1987*a*, p. 317), because the Lagrangian under consideration

Table I

Potentials:	
$\overset{a}{e}_A$ Bravais moving coframe	e_A vector potential
Field strengths, i.e., field acting upon other systems:	
$2\overset{a}{\tau}_{AB}=\overset{a}{e}_{B,A}-\overset{a}{e}_{A,B}$ Burgers field	$F_{AB}=e_{B,A}-e_{A,B}$ electromagnetic field strength
Currents:	
$-\underset{a}{j}^A=\underset{a}{\sigma}^A=\partial L/\partial \overset{a}{e}_A$ self-interaction currents (force stresses)	$-j_A=\partial L/\partial e_A$ electric currents
Field momenta, i.e., field produced by currents:	
$\underset{a}{M}^{AB}=\partial L/\partial \overset{a}{e}_{A,B}$ couple force stresses	$M^{AB}=\partial L/\partial e_{A,B}$ field produced directly by sources
Field equations:	
$\nabla^C_B \underset{a}{M}^{AB}=-\underset{a}{j}^A$ balance of stresses and couple stresses	$\nabla_B M^{AB}=-j^A$ Maxwell equations

satisfies additionally the postulate of the local translational invariance; the internal stresses $\boldsymbol{\sigma}$ are connected with this invariance.

Let us observe that the field equations (33) have a form which resembles the equations of electrodynamics (Sławianowski, 1986). Thus, from the formal point of view, our theory resemble "3-fold electrodynamics." Its basic objects are linear defects, described by pairs $(\mathbf{E}_a, \tau^a)$, $a=1,2,3$, interacting mutually with each other. They are massless, because the Lagrange function L does not involve any term built algebraically of the field Φ alone. Table I provides a brief "dictionary" of this analogy with classical electrodynamics (Sławianowski, 1985).

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