Arch. Mech., 36, 2, pp. 185–193, Warszawa 1984

# On constrained size-effect bodies

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ELASTIC size-effect bodies with internal constraints are considered. Investigations have been restricted to the class of so-called homogeneous processes. The model of dynamics, the form of generalized forces describing the response of bodies under consideration and the interpretation rule for these forces have been proposed.

Rozważane są sprężyste ciała z efektem skali i z wewnętrznymi więzami. Rozważania zostały ograniczone do klasy tzw. jednorodnych procesów. Zaproponowano model dynamiki, postać uogólnionych sił opisujących reakcję rozważanych ciał oraz regułę interpretacyjną dla tych sił.

Рассматриваются упругие тела с эффектом масштаба и с внутренними связями. Рассуждения ограничены классом т. наз. однородных процессов. Предложены модель динамики, вид обобщенных сил, описывающих реакцию рассматриваемых тел, а также интерпретационный принцип для этих сил.

## 1. Introduction

THE SUBJECT of this paper is the elastic size-effect body which has internal constraints. The description of such a body, on the ground of the M. E. GURTIN and P. P. GUIDUGLI internal constraints theory [1], has been presented in the paper [2]. From this description it appears that in the "purely mechanical" case (that is in an adiabatic, isothermic and isentropic process) the dissipation inequality for a body with internal constraints has the form

$$\delta = -\dot{E} + W \geqslant 0, \tag{1.1}$$

where $E$ is the total internal energy of this body and $W$ is a net working. In the case of an unconstrained body or, for instance, in the case of so-called "scalar constraints" (see Sect. 2, Eq. (2.2)) the formula (1.1) reduces to $\delta = 0$. It is an open question whether $\delta = 0$ is valid for every form of constraints or not. Hence one should take into account the possibility that in the M. E. Gurtin and P. P. Guidugli internal constraints theory, the law of conservation of energy is neglected. In this paper we postulate the fulfillment of this law for every form of constraints. It has been shown in the paper that for elastic bodies the dipole moment of the momentum balance law, which we stated as a postulate in [2] (cf. also [3] and [11]), can be derived from the definition of the net working. We also depart from the definition of constraint effects as additive corections for quantities independent of constraints.

## 2. The space of mechanical configurations

We will deal with a material body of an immovable center of mass, homogeneously deformed. Spatial configurations of such a body can be identified with the subsets $\mathfrak{B}$ of the Euclidean vector space $E^3$ (the physical space in this case) that have the form $\mathfrak{B} = l(\mathbf{F})(\mathfrak{B}_0)$ where $\mathfrak{B}_0 \subset E^3$ is a distinguished set (called the body reference configuration) and $l(\mathbf{F})$ is a linear mapping of the form

$$
\text{(2.1)} \qquad \begin{gathered} l(\mathbf{F})(\mathbf{X}) = \mathbf{F}\cdot\mathbf{X}, \mathbf{X} \in E^3, \\ \mathbf{F} \in GL^+(E^3) = \{\mathbf{F} \in E^3 \otimes E^3 : \det \mathbf{F} > 0\}. \end{gathered}
$$

We assume that the body reference configuration $\mathfrak{B}_0$ is compact, connected and has a nonempty interior; such sets will be called *"solid figures"*.

Let us denote by $M$ a connected $C^1$ — manifold such that (cf. [1])

(M1) $M \subset GL^+(E^3)$,

(M2) $\mathbf{1} \in M$,

(M3) $\mathbf{Q} \in 0^+(E^3)$, $\mathbf{F} \in M \Rightarrow \mathbf{QF} \in M$,

where $0^+(E^3) \subset GL^+(E^3)$ is the set of all proper orthogonal tensors, $\mathbf{AB}$ denotes the group multiplication (in the shape of the simple contraction) in $GL^+(E^3)$ and $\mathbf{1}$ is the unity of this group. We will call this manifold $M$ *the space of mechanical configurations* and the curve in $M$ — *the homogeneous M-process*. The solid figure $\mathfrak{B}_0$ with the attached mass $m$ and the family of all solid figures of the form $\mathfrak{B} = l(\mathbf{F})(\mathfrak{B}_0)$, $\mathbf{F} \in M$ will be called the *M-body*. A set $\mathfrak{B}$ of the form $\mathfrak{B} = l(\mathbf{F})(\mathfrak{B}_0)$, $\mathbf{F} \in M$ will be called the *admissible* or the *M-admissible* (spatial) configuration. The set $\mathfrak{B}_0$ will be considered as the $M$-body reference configuration. It follows from the condition (M2) that a $M$-body reference configuration is the $M$-admissible configuration, and from the condition (M3) that rigid rotations of $M$-admissible configurations are also $M$-admissible configurations. We say that the $M$-body is "an unconstrained body" if $\dim M = 9$ ($= \dim GL^+(E^3)$) and that the $M$-body has "constraints" if $\dim M < 9$ (cf. [1, 2]). We will say that the $M$-body has "scalar constraints" if

$$
\text{(2.2)} \qquad \begin{gathered} M = \{\mathbf{F} \in GL^+(E^3) : h(\mathbf{F}) = 0\}, \\ h \in C^1(G), M \subset G \in \operatorname{top} GL^+(E^3). \end{gathered}
$$

In this paper we will consider spaces of mechanical configurations with the differential structure induced from the Euclidean linear space $E^3 \otimes E^3$. Let us denote by $T_p(E^3)$ the Euclidean linear space of tensors of valence $p (p = 0, 1, 2, \ldots)$ over $E^3$; in these spaces the full tensor contraction defines the inner product. By the symbol $D$ we will denote the Euclidean covariant derivative in $T_2(E^3) = E^3 \otimes E^3$ (cf. [4]). If $f: G \to T_p(E^3)$, $G \in \operatorname{top} T_2(E^3)$ is a differentiable function (in the Frechet sense), then the derivative $D$ is defined by

$$
\text{(2.3)} \qquad \forall (\mathbf{F}, \mathbf{V}) \in G \times T_p(E^3), \quad D_{\mathbf{V}} f(\mathbf{F}) = \frac{\partial f}{\partial \mathbf{F}}(\mathbf{F}) \cdot \mathbf{V},
$$

where the symbol $\cdot$ denotes the full tensor contraction. We will consider a space mechanical configurations $M$ to be a Riemanian submanifold ("hypersurface") in the space $T_2(E^3)$, namely the tangent space $\dot{M}(\mathbf{F})$ to $M$ at the point $\mathbf{F} \in M$ will be identified with a linear subspace of $T_2(E^3)$ with the inner product induced from $T_2(E^3)$. The Euclidean covariant derivative $D$ and the orthogonal projection operators $P_{\mathbf{F}}: T_2(E^3) \to \dot{M}(\mathbf{F})$, $\mathbf{F} \in M$ induce the Riemanian covariant derivative $\nabla$ on the hypersurface $M$ (cf. [4])., For real-valued functions on $M$ extensible on an open set $G \in T_2(E^3)$, this covariant derivative can be defined as follows:

$$\forall (\mathbf{F}, \mathbf{V}) \in \dot{M}, \quad \nabla_{\mathbf{V}} h(F) = P_{\mathbf{F}}\left(\frac{\partial h}{\partial \mathbf{F}}(\mathbf{F})\right) \cdot \mathbf{V}, \tag{2.4}$$

where $\dot{M}$ is the tangent bundle to $M$ and $h: G \to R$; this means that

$$\forall \mathbf{F} \in M, \quad \nabla h(\mathbf{F}) = P_{\mathbf{F}}\left(\frac{\partial h}{\partial \mathbf{F}}(\mathbf{F})\right). \tag{2.5}$$

If $\dot{M}(\mathbf{F}) = T_2(E^3)$, then $\nabla = D$.

## 3. The elastic-size-effect $M$-body

Let $\mathfrak{B}_0$ be the $M$-body and $\mathbf{F}: I \to M (I \subset R$ — interval) — a homogeneous $M$-process. Let us consider external force fields on $\mathfrak{B}_0$: the body force field $\mathbf{b}(\mathbf{X}, \tau)$, $\mathbf{X} \in \operatorname{Int} \mathfrak{B}_0$, $\tau \in I$ (Int $\mathfrak{B}_0$ — the interior of $\mathfrak{B}_0$) and the surface force field $\mathbf{s}(\mathbf{X}, \tau)$, $\mathbf{X} \in \partial \mathfrak{B}_0$, $\tau \in I$ ($\partial \mathfrak{B}_0$ — the boundary surface of $\mathfrak{B}_0$). The power $P$ of external forces acting on $\mathfrak{B}_0$ and the kinetic energy $K$ of this $M$-body have the following form:

$$\begin{aligned} K(\mathfrak{B}_0; \tau) &= \frac{1}{2} \int_{\mathfrak{P}_0} |\mathbf{v}(\mathbf{X}, \tau)|^2 \mathrm{dm}(\mathbf{X}), \\ P(\mathfrak{B}_0; \tau) &= \int_{\mathfrak{P}_0} \mathbf{b}(\mathbf{X}, \tau) \cdot \mathbf{v}(\mathbf{X}, \tau) dV(\mathbf{X}) + \oint_{\partial \mathfrak{P}_0} \mathbf{s}(\mathbf{X}, \tau) \cdot \mathbf{v}(\mathbf{X}, \tau) dS(\mathbf{X}), \end{aligned} \tag{3.1}$$

where $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$, $dm(\mathbf{X}) = \varrho_0 dV(\mathbf{X})$, $\varrho_0 = m/V(\mathfrak{B}_0)$, $V(\mathfrak{B}_0) = \operatorname{vol} \mathfrak{B}_0$ and, according to the assumption that the $M$-body has an immovable center of mass,

$$\begin{aligned} \mathbf{v}(\mathbf{X}, \tau) &= \dot{\mathbf{F}}(\tau) \cdot \mathbf{X}, \mathbf{X} \in \mathfrak{B}_0, \\ \dot{\mathbf{F}}(\tau) &= \frac{dF}{d\tau}(\tau) \in \dot{M}(\mathbf{F}(\tau)). \end{aligned} \tag{3.2}$$

Let us designate by $W(\mathfrak{B}_0; \tau)$ the net working at the time $\tau \in I$, that is the quantity which in an inertial frame of reference has the form ([5]):

$$W(\mathfrak{B}_0; \tau) = P(\mathfrak{B}_0; \tau) - \dot{K}(\mathfrak{B}_0; \tau) \tag{3.3}$$

and let us denote by $\Phi(\mathfrak{B}_0; \tau)$ the internal energy at the time $\tau \in I$ of the "size-effect $M$-body $\mathfrak{B}_0$" (cf. [2]).

*The size-effect is* formalized here by dependence of the internal energy and the net working on the solid figure $\mathfrak{B}_0$. This can be presented, for instance, by dependence on such global geometrical characteristics as:

width (in any direction) or thickness of the solid figure,
volume of the solid figure

and also by dependence on geometrical characteristics of the boundary surface of this solid figure, such as:

boundary surface area,

total mean curvature or total Gaussian curvature of this surface.

For example, in the classical theory of capillarity a finite internal energy density is attributed not only to the volume but also to the surface measure. This means that we can endow a body with the internal energy functional $\varphi(\mathfrak{B})$ of the form

$$\varphi(\mathfrak{B}) = aV(\mathfrak{B})+bF(\mathfrak{B}), \tag{3.4}$$

where $a$ and $b$ are some constants, $V(\mathfrak{B})$ and $F(\mathfrak{B})$ are the volume and the boundary surface area of an admissible spatial configuration $\mathfrak{B}$ of the $M$-body $\mathfrak{B}_0$. The attempt to generalize the classical theory of capillarity, undertaken in the paper [6], resulted among others in the generalization of the internal energy functional $\varphi(\mathfrak{B})$ in the form (cf. also [7])

$$\varphi(\mathfrak{B}) = aV(\mathfrak{B})+bF(\mathfrak{B})+cH(\mathfrak{B})+d, \tag{3.5}$$

where $c$ and $d$ are some constants, $H(\mathfrak{B})$ is the total mean curvature of the set $\mathfrak{B}$ boundary surface and $\mathfrak{B}$ is a convex solid figure. In our notations

$$\Phi(\mathfrak{B}_0;\tau) = \varphi(\mathfrak{B}_\tau), \quad \mathfrak{B}_\tau = l(\mathbf{F}(\tau))(\mathfrak{B}_0) \tag{3.6}$$

for the convex reference configuration $\mathfrak{B}_0$. This function $\Phi(\mathfrak{B}_0;\tau)$, $\tau\in I$ can be interpreted as the internal energy function (in homogeneous $M$-processes) of the convex $M$-body $\mathfrak{B}_0$ of an elastic incompressible nonlocal fluid (in such a case the space $M$ of mechanical configurations has the form (3.20)).

The principle of conservation of energy for the $M$-body $\mathfrak{B}_0$ means that (cf. Introduction)

POSTULATE 1. In every homogeneous $M$-process

$$\forall\tau\in I, \quad \dot{\Phi}(\mathfrak{B}_0;\tau) = W(\mathfrak{B}_0;\tau). \tag{3.7}$$

The mechanical response of the $M$-body $\mathfrak{B}_0$ to its deformation can be described by

POSTULATE 2. In every homogeneous $M$-process

$$\forall\tau\in I, \quad \exists\mathbf{N}(\mathfrak{B}_0;\tau)\in T_2(E^3), \quad W(\mathfrak{B}_0;\tau) = -\mathbf{N}(\mathfrak{B}_0;\tau)\cdot\dot{\mathbf{F}}(\tau). \tag{3.8}$$

The tensor $\mathbf{N}(\mathfrak{B}_0;\tau)$, describing the mechanical response of the $M$-body $\mathfrak{B}_0$ at the time $\tau\in I$, is called *the generalized force*; this quantity is not at all univocally definite by the homogeneous $M$-process. If $\mathbf{F}: I\to M$ is a certain homogeneous $M$-process and $\mathbf{N}(\mathfrak{B}_0;\tau)$ an arbitrary generalized force at the time $\tau\in I$ in this process, then this force can be represented in the following form:

$$\begin{gathered}\mathbf{N}(\mathfrak{B}_0;\tau) = \mathbf{N}_t(\mathfrak{B}_0;\tau)+\mathbf{N}_n(\mathfrak{B}_0;\tau),\\ \mathbf{N}_t(\mathfrak{B}_0;\tau)\in\dot{M}(\mathbf{F}(\tau)), \quad \mathbf{N}_n(\mathfrak{B}_0;\tau)\in\dot{M}(\mathbf{F}(\tau))^{\perp}\end{gathered} \tag{3.9}$$

where $\dot{M}(\mathbf{F})^{\perp}$ is the orthogonal complement in $T_2(E^3)$ of the tangent space $\dot{M}(\mathbf{F})$. It follows from the definition of generalized forces $\mathbf{N}_t$ and $\mathbf{N}_n$ that they have to satisfy the orthogonality condition

$$\forall\tau\in I \quad \mathbf{N}_t(\mathfrak{B}_0;\tau)\cdot\mathbf{N}_n(\mathfrak{B}_0;\tau) = 0. \tag{3.10}$$

Since in every homogeneous $M$-process

$$(3.11)\qquad \begin{aligned} &W(\mathfrak{B}_0;\tau) = -\mathbf{N}_t(\mathfrak{B}_0;\tau)\cdot\dot{\mathbf{F}}(\tau),\ \mathbf{N}_n(\mathfrak{B}_0;\tau)\cdot\dot{\mathbf{F}}(\tau) = 0,\\ &\mathbf{N}_t(\mathfrak{B}_0;\tau) = P_{\mathbf{F}(\tau)}\big(\mathbf{N}(\mathfrak{B}_0;\tau)\big),\end{aligned}$$

where $P_{\mathbf{F}}\colon T_2(E^3)\to \dot{M}(\mathbf{F})$, $\mathbf{F}\in M$ are orthogonal projection operators, therefore the generalized force $\mathbf{N}_t(\mathfrak{B}_0;\tau)$ can be considered as the so-called "constitutive quantity" (both in the case of an unconstrained $M$-body and in the case of an $M$-body with constraints — cf. [5]). Thus the constitutive equations of the size-effect $M$-body describing its *elastic response* are given by

POSTULATE 3. There are continuous functions $E(\mathfrak{B}_0;\cdot):M\to R$ and $\mathbf{N}_e(\mathfrak{B}_0;\cdot)$: $M\to T_2(E^3)$ such that for every homogeneous $M$-process $\mathbf{F}:I\to M$

$$(3.12)\qquad \forall\tau\in I,\quad \Phi(\mathfrak{B}_0;\tau) = E(\mathfrak{B}_0;\mathbf{F}(\tau)),\quad \mathbf{N}_t(\mathfrak{B}_0;\tau) = \mathbf{N}_e(\mathfrak{B}_0;\mathbf{F}(\tau)).$$

Moreover, we will assume that the function $E(\mathfrak{B}_0;\cdot)$ can be extended to a differentiable function on an open set $G\subset T_2(E^3)$ such that $M\subset G$. Then from Eqs. (3.7)–(3.12) it follows that

$$(3.13)\qquad \forall\mathbf{F}\in M,\quad \mathbf{N}_e(\mathfrak{B}_0;\mathbf{F}) = -\nabla E(\mathfrak{B}_0;\mathbf{F}),$$

where the Riemanian covariant derivative $\nabla$ is defined by the formula (2.5).

Let us note that from Eqs. (2.5), (3.9), (3.12) and (3.13) we have

$$(3.14)\qquad \begin{aligned} &\mathbf{N}(\mathfrak{B}_0;\tau) = -DE(\mathfrak{B}_0;\mathbf{F}(\tau))+\mathbf{N}_c(\mathfrak{B}_0;\tau),\\ &\mathbf{N}_c(\mathfrak{B}_0;\tau) = \mathbf{N}_n(\mathfrak{B}_0;\tau)+P^{\perp}\big(DE(\mathfrak{B}_0;\mathbf{F})\big)(\tau)\in\dot{M}\big(\mathbf{F}(\tau)\big)^{\perp},\end{aligned}$$

where $P^{\perp}(\mathbf{A})(\tau) = P^{\perp}_{\mathbf{F}(\tau)}(\mathbf{A}(\tau))$ for $\mathbf{A}:I\to T_2(E^3)$ and $P^{\perp}_{\mathbf{F}}\colon T_2(E^3)\to\dot{M}(\mathbf{F})^{\perp}$, $\mathbf{F}\in M$ are orthogonal projection operators. In the formula (3.14) the term $DE$ is independent of constraints and thereby the constraint effect is described (in contrast to the formula (3.9)) only by the generalized force $\mathbf{N}_c$ (such that $\mathbf{N}_c\cdot\dot{\mathbf{F}} = 0$ for every homogeneous $M$-process). The form (3.14) of the constraint effect description has been used up to now as the definition of this effect (e.g. [1, 5]). Both the form (3.9), (3.12) and (3.13) and the form (3.14) of the constraint effect description can be useful. For example, in the case of scalar constraints it is simpler to find the form of the generalized force $\mathbf{N}_c$ (cf. the formula (2.2) and [1], or [5]):

$$(3.15)\qquad \mathbf{N}_c(\mathfrak{B}_0;\tau) = \alpha(\mathfrak{B}_0;\tau)\cdot Dh\big(\mathbf{F}(\tau)\big).$$

The orthogonality condition (3.10) limits the form of both $\mathbf{N}_t$ and $\mathbf{N}_n$; for example, in the case (3.20) we have

$$(3.16)\qquad \begin{aligned} &\mathbf{N}(\mathfrak{B}_0;\tau) = \mathbf{N}_t(\mathfrak{B}_0;\tau)+p(\mathfrak{B}_0;\tau)\mathbf{1},\\ &\operatorname{tr}\mathbf{N}_t(\mathfrak{B}_0;\tau) = 0,\quad p(\mathfrak{B}_0;\tau) = \frac{1}{3}\operatorname{tr}\mathbf{N}(\mathfrak{B}_0;\tau).\end{aligned}$$

The "normalization" $p = 1/3\operatorname{tr}\mathbf{N}$ of the scalar $p$ is used, for example, in the theory of simple incompressible materials (cf. [5]).

The formula (3.13) describes the so-called "hyperelasticity", that is the case where the only source of elasticity is the change of the internal energy (cf. Postulate 1). Another kind of elasticity response is, for example, the so-called "rubberlike elasticity", that is the case where the only source of elasticity is the change of the entropy. In order to give a description of this kind of elasticity. let us consider "ideal elastomeric size-effect bodies" [8]. In adiabatic and isothermal processes such a body can be considered as an elastic and incompressible body such that

$$DE(\mathfrak{B}_0; \Theta_0; \mathbf{F}) = 0 \quad \text{for} \quad \det\mathbf{F} = 1 \tag{3.17}$$

where $\Theta_0$ is the temperature of both the reference configuration $\mathfrak{B}_0$ and an isothermal process, $E(\mathfrak{B}_0, \Theta_0; \mathbf{F})$ is the internal energy in this process. In this case the dissipation function reduces to the form (cf. [2])

$$\begin{aligned} \delta &= -\dot{\Phi}(\mathfrak{B}_0; \tau) + W(\mathfrak{B}_0; \tau), \\ \Phi(\mathfrak{B}_0; \tau) &= U(\mathfrak{B}_0; \mathbf{F}(\tau)), \end{aligned} \tag{3.18}$$

where $U(\mathfrak{B}_0; \mathbf{F})$, $\mathbf{F} \in M$ designates the Helmholtz free energy function for the ideal elastomeric size-effect body $\mathfrak{B}_0$ and isothermal processes, that is (cf. [2])

$$U(\mathfrak{B}_0; \mathbf{F}) = E(\mathfrak{B}_0, \Theta_0) - \Theta_0 S(\mathfrak{B}_0, \Theta_0; \mathbf{F}), \quad \mathbf{F} \in M, \tag{3.19}$$

where $S(\mathfrak{B}_0, \Theta_0; \mathbf{F})$ is the entropy of this $M$-body $\mathfrak{B}_0$ in isothermal $M$ — homogeneous processes with the constant temperature $\Theta_0$ and the constant internal energy $E(\mathfrak{B}_0, \Theta_0)$ (see Eq. (3.17)). The space $M$ of mechanical configuration has here the form

$$M = \{\mathbf{F} \in GL^+(E^3) : \det\mathbf{F} = 1\}. \tag{3.20}$$

The consideration presented in this section for the case of the hyperelasticity response can be translated to the case defined by the formulae (3.18)–(3.20) and by the assumption that the dissipation function vanishes along the $M$ — homogeneous processes ($\delta = 0$). Then the generalized force $\mathbf{N}(\mathfrak{B}_0; \tau)$ is given by the formulae (3.12) and (3.16) and the formula

$$\mathbf{N}_e(\mathfrak{B}_0; \mathbf{F}) = -\nabla U(\mathfrak{B}_0; \mathbf{F}) \tag{3.21}$$

or has the form (cf. Eqs. (3.14) and (3.16)

$$\mathbf{N}(\mathfrak{B}_0; \tau) = -DU(\mathfrak{B}_0; \mathbf{F}(\tau)) + \bar{p}(\mathfrak{B}_0; \tau)\mathbf{1}. \tag{3.22}$$

## 4. Dynamics

If $\mathbf{F} \in M$ is some fixed mechanical configur ation, then from Eqs. (3.1)–(3.3), (3.7) (3.11) and (3.12) it follows that nie zal.

$$\forall \mathbf{V} \in \dot{M}(\mathbf{F}), \quad (\dot{\mathbf{P}} - \mathbf{N}_e^T - \mathbf{M}_{ext}) \cdot \mathbf{V} = 0, \quad \mathbf{N}_n \cdot \mathbf{V} = 0. \tag{4.1}$$

$\mathbf{N}_e = \mathbf{N}_e(\mathfrak{B}_0; \mathbf{F}) \in \dot{M}(\mathbf{F})$ is defined by Eq. (3.13) (or Eq. (3.21)), $T$ denotes "transpose", $\mathbf{P} = \mathbf{P}(\mathfrak{B}_0; \tau) \in T_2(E^3)$ is the dipole moment of momentum of the $M$-body $\mathfrak{B}_0$:

$$\mathbf{P}(\mathfrak{B}_0; \tau) = \int_{\mathfrak{B}_0} \mathbf{X} \otimes d\mathbf{p}(\mathbf{X}, \tau) = \mathbf{J}(\mathfrak{B}_0)\dot{\mathbf{F}}(\tau)^T \tag{4.2}$$ — nie zal. od temp

where $d\mathbf{p}(\mathbf{X}, \tau) = \mathbf{v}(\mathbf{X}, \tau)\, dm(\mathbf{X})$, $dm(\mathbf{X}) = \varrho_0 dV(\mathbf{X})$, $\mathbf{J}(\mathfrak{B}_0)$ is the solid figure $\mathfrak{B}_0$ inertia tensor computed with respect to its mass center:

$$\mathbf{J}(\mathfrak{B}_0) = \int_{\mathfrak{B}_0} \mathbf{X} \otimes \mathbf{X}\, dm(\mathbf{X}) \tag{4.3}$$

$\varrho_0 dV(\mathbf{X})$

and $\mathbf{M}_{ext} = \mathbf{M}_{ext}(\mathfrak{B}_0; \tau) \in T_2(E^3)$ is the dipole moment of external forces:

$$\mathbf{M}_{ext}(\mathfrak{B}_0; \tau) = \int_{\mathfrak{P}_0} \mathbf{X} \otimes \mathbf{b}(\mathbf{X}, \tau) dV(\mathbf{X}) + \oint_{\mathfrak{P}_0} \mathbf{X} \otimes \mathbf{s}(\mathbf{X}, \tau) dS(\mathbf{X}). \tag{4.4}$$

Since $\dot{\mathbf{P}}$, $\mathbf{N}_e$ and $\mathbf{M}_{ext}$ are independent of $\mathbf{V} \in \dot{M}(\mathbf{F})$ and $\dot{M}(\mathbf{F}) \otimes \dot{M}(\mathbf{F})^{\perp} = T_2(E^3)$, then the formulae (3.9) and (4.1) and the Liu theorem ([9]) yield (cf. [2])

CONCLUSION. In every homogeneous $M$-process the dynamics of the elastic size-effect $M$-body $\mathfrak{B}_0$ is described by the dipole moment of the momentum balance equation:

$$\dot{\mathbf{P}}(\mathfrak{B}_0; \tau) = \mathbf{N}(\mathfrak{B}_0; \tau)^T + \mathbf{M}_{ext}(\mathfrak{B}_0; \tau). \tag{4.5}$$

Let us remark that the condition of immobility of the mass center means that the total external force acting on $\mathfrak{B}_0$ vanishes, that is we have in addition

$$\forall \tau \in I \int_{\mathfrak{P}_0} \mathbf{b}(\mathbf{X}, \tau) dV(\mathbf{X}) + \oint_{\partial \mathfrak{P}_0} \mathbf{s}(\mathbf{X}, \tau) dS(\mathbf{X}) = \mathbf{0}. \tag{4.6}$$

It follows from this conclusion that the generalized force $\mathbf{N}$ is a *dipole moment*. This dipole moment has a representation in the following form:

$$\begin{aligned} \mathbf{N}(\mathfrak{B}_0; \tau) &= -\mathbf{S}(\mathfrak{B}_0; \tau)\mathbf{F}(\tau)^*, \\ \mathbf{S}(\mathfrak{B}_0; \tau) &= \mathbf{S}_e(\mathfrak{B}_0; \mathbf{F}(\tau)) + \mathbf{S}_n(\mathfrak{B}_0; \tau) \end{aligned} \tag{4.7}$$

with the notations

$$\mathbf{S}_e(\mathfrak{B}_0; \mathbf{F}) = \nabla E(\mathfrak{B}_0; \mathbf{F})\mathbf{F}^T, \quad \mathbf{F}^* = (\mathbf{F}^T)^{-1} \tag{4.8}$$

and the condition

$$\nabla E(\mathfrak{B}_0; \mathbf{F}) \cdot \mathbf{S}_n \mathbf{F}^* = 0. \tag{4.9}$$

The internal energy $E$ must be an objective scalar function since it describes the physical properties of a material body, that is, for every tensor $\mathbf{F} \in M$ the following relations holds:

$$\forall \mathbf{Q} \in 0^+(E^3) \quad E(\mathfrak{B}_0; \mathbf{QF}) = E(\mathfrak{B}_0; \mathbf{F}). \tag{4.10}$$

It may easily be shown (on the grounds of the formula (2.5), the conditions (M1)–(M3) and basing on similar considerations as in the case of an unconstrained body — cf. [5]) that the condition (4.10) yields $\mathbf{S}_e = \mathbf{S}_e^T$. This fact leads us to assume (in order to obtain conformability with the internal constraints theory for simple bodies — cf. Final remarks and [1] or [5]) that also $\mathbf{S}_n = \mathbf{S}_n^T$.

Finally Eq. (4.5) can be rewritten as follows (cf. [2], [3]):

$$\begin{aligned} \mathbf{F}\mathbf{J}(\mathfrak{B}_0)\ddot{\mathbf{F}}^T &= -\mathbf{S} + \mathbf{F}\mathbf{M}_{ext}, \\ \mathbf{S} = \mathbf{S}^T, &\quad \mathbf{F} \in M, \end{aligned} \tag{4.11}$$

where $\mathbf{AB}$ denotes the simple contraction of the tensors $\mathbf{A}, \mathbf{B} \in T_2(E^3)$. Equation (4.11) together with Eqs. (4.3), (4.4)) and (4.6)–(4.9) or with Eqs. (3.21) and (3.22) defines the global model of dynamics for an elstic size-effect $M$-body $\mathfrak{B}_0$ (of an immovable center of mass and being homogeneously deformed).

It is interesting to make a comparison between this global model and the local model of dynamics (in the Cauchy form of the momentum balance law — e.g. [5]) for homogeneous elastic simple bodies (cf. [5] and final remarks Eqs. (5.1) and (5.3)). If body forces are absent ($\mathfrak{b} = 0$), then this local model of dynamics allows for only trivial homogeneous processes of the form $\mathbf{F}(\tau) = \mathbf{F}_0(\mathbf{1}+\tau\mathbf{F}_1)$ ([5]). Equation (4.11) does not give such a paradoxical limitation of the admissibility of homogeneous processes.

## 5. Final remarks

Let us consider an unconstrained size-effect body $\mathfrak{B}_0$ with a boundary $\partial\mathfrak{B}_0$ being a two-dimensional orientable manifold in the Euclidean vector space $E^3$. In this case $\mathbf{S} = \mathbf{S}_e$ and the dipole moment $\mathbf{S}$ has the following representation:

$$\begin{aligned} \mathbf{S} &= \oint_{\partial\mathfrak{P}_0} \mathbf{X}\otimes\mathbf{t}(\mathfrak{B}_0; \mathbf{F}, \mathbf{x})dS(\mathbf{x}) = V(\mathfrak{B})\mathbf{T}(\mathfrak{B}_0; \mathbf{F}), \\ \mathbf{t}(\mathfrak{B}_0; \mathbf{F}, \mathbf{x}) &= \mathbf{T}(\mathfrak{B}_0; \mathbf{F})\mathbf{n}(\mathbf{x}), \qquad V(\mathfrak{B}) = \mathrm{vol}\,\mathfrak{B}, \end{aligned} \tag{5.1}$$

where $\mathfrak{B} = l(\mathbf{F})(\mathfrak{B}_0)$ ($l(\mathbf{F})$ — see Eq. (2.1)) is a deformed configuration of the body $\mathfrak{B}_0$ $\mathbf{n}(\mathbf{x})$, $\mathbf{x} \in \partial\mathfrak{B}$ is the outward unit vector normal to the boundary $\partial\mathfrak{B}$ of the solid figure $\mathfrak{B}$ and $\mathbf{T}(\mathfrak{B}_0; \mathbf{F}) \in T_2(E^3)$ is a symmetric tensor (cf. the commentary to Eq. (4.10)) of the form

$$\mathbf{T}(\mathfrak{B}_0; \mathbf{F}) = \frac{1}{V(\mathfrak{B})} DE(\mathfrak{B}_0; \mathbf{F})\mathbf{F}^T. \tag{5.2}$$

If $E(\mathfrak{B}_0; \mathbf{F}) = me(\mathbf{F})$ for $m = V(\mathfrak{B}_0)\varrho_0 = V(\mathfrak{B})\varrho$ — the mass of both $\mathfrak{B}_0$ and $\mathfrak{B}$, then

$$\mathbf{T}(\mathfrak{B}_0; \mathbf{F}) = \varrho De(\mathbf{F})\mathbf{F}^T \tag{5.3}$$

this means that such a size-effect M-body is the so-called "simple body" (homogeneous and unconstrained) with the Cauchy stress tensor $\mathbf{T}(\mathfrak{B}_0; \mathbf{F})$ (cf. [5]).

The circumstance that this simple body is a particular case of the size-effect body $\mathfrak{B}_0$ enables us to take a contact interaction force (between the material solid figure $\mathfrak{B}_0$ and its exterior) at the point $\mathbf{x} \in \partial\mathfrak{B}$ in the form of the vector $\mathbf{t}(\mathfrak{B}_0; \mathbf{F}, \mathbf{x})$ (the formula (5.1)). Therefore the formula (5.2) can be considered as an extension of the stress tensor concept from simple bodies on elastic size-effect bodies. The real material, for which such a generalization can be useful, is for example brass. Namely, on the one hand brass is treated in its applications as a macroscopically homogeneous material, but on the other hand it has been confirmed for this material that, for example, the thickness of a cylinder sample influences the Young's modulus ([10]). For liquid bodies the existence of the surface tension also creates the possibility of considering this generalized stress tensor (cf. formulae (3.6), (3.4) or (3.5) and formulae (3.12), (3.13), (3.16); cf. also [7], [8]). In this case, or more generally in the case of size-effect M-bodies, a generalization of the Cauchy stress tensor can be taken in the form (cf. Eq. (5.1))

$$\mathbf{T}(\mathfrak{B}_0; \tau) = \frac{1}{V(\mathfrak{B}_\tau)}\mathbf{S}(\mathfrak{B}_0; \tau), \qquad V(\mathfrak{B}_\tau) = \mathrm{vol}\,\mathfrak{B}_\tau, \tag{5.4}$$

where $\mathbf{S}(\mathfrak{B}_0; \tau)$ is defined by Eqs. (4.7)–(4.10) and $\mathfrak{B}_\tau = l(\mathbf{F}(\tau))(\mathfrak{B}_0)$.

The dipole moment $\mathbf{S}(\mathfrak{B}_0; \tau)$ (or the generalized stress tensor $\mathbf{T}(\mathfrak{B}_0; \tau)$) represents mechanical properties of the elastic size-effect M-body $\mathfrak{B}_0$. Both the manner of description of these properties (cf. commentary after Eq. (3.3)) and the model of dynamics (cf. Eq. (4.11) and the commentary to this) have a global character and not a field character. This means that the body smallest material element, which we can endow with mechanic or dynamic properties, is the whole body; in a field theory (local or nonlocal) such an element is the infinitesimal neighbourhood of a body point.

## References


1. M. E. GURTIN, P. P. GUIDUGLI, *The thermodynamics of constrained materials*, Arch. Rational Mech Anal., **59**, 192–207, 1973.
2. A. TRZĘSOWSKI, *The size effect*, Bull. Acad. Polon. Sci., Série Sci. Techn., **27**, 255, 1979.
3. J. SŁAWIANOWSKI, *Geometry of phase space* [in Polish], PWN, Warszawa 1975.
4. R. SIKORSKI, *Introduction to differential geometry* [in Polish], PWN, Warszawa 1972.
5. C. TRUESDELL, *A first course in rational continuum mechanics*, J. Hopkins University Press, Baltimore 1972.
6. A. BLINOWSKI, A. TRZĘSOWSKI, *Surface energy in liquids and the Hadwiger integral theorem*, Arch, Mech., **33**, 133–146, 1981.
7. A. TRZĘSOWSKI, *On liquid-like response of the size-effect body*, Bull. Acad. Polon. Sci., Série Sci. Techn. **88**, 127–133, 1980.
8. A. TRZĘSOWSKI, *The ideal elastometric body with the liquid-response*, Bull. Acad. Polon. Sci., Série Sci. Techn., **28**, 135–141, 1980.
9. LIU, I-SHICH, *Method of Lagrange multipliers for exploitation of the entropy principle*, Arch. Rationa Mech., Anal., **50**, 132–148, 1972.
10. M. ZAKRZEWSKI, R. ŻUCHOWSKI, *The surface of a solid body and the internal forces equilibrium* (in Polish), Sci. Papers of Wrocław Techn. University IMSAM, 24, Studies and Research, 15, 27–37, 1975.
11. G. CAPRIZ, P. PODIO-GUIDUGLI, *Materials with spherical structure*, Arch. Rational Mech. Anal., **75**, 269–279, 1981.


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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

*Received June* 29, 1983.