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## PAPER

## Correlations and energy in mediated dynamics

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E-mail: [tanjungkrisnanda@gmail.com](mailto:tanjungkrisnanda@gmail.com) and [tomasz@paterek.info](mailto:tomasz@paterek.info)**Keywords:** quantum speed limit, quantum entanglement, mediated dynamics, correlations and energy

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## Abstract

The minimum time required for a quantum system to evolve to a distinguishable state is set by the quantum speed limit, and consequently influences the change of quantum correlations and other physical properties. Here we study the time required to maximally entangle two principal systems interacting either directly or via a mediating ancillary system, under the same energy constraints. The direct interactions are proved to provide the fastest way to entangle the principal systems, but it turns out that there exist mediated dynamics that are just as fast. We show that this can only happen if the mediator is initially correlated with the principal systems. These correlations can be fully classical and can remain classical during the entangling process. The final message is that correlations save energy: one has to supply extra energy if maximal entanglement across the principal systems is to be obtained as fast as with an initially correlated mediator.

An evolution of a quantum state into a distinguishable one requires finite time. The shortest time to achieve this task is governed by the quantum speed limit (QSL). The first lower bound on the shortest time was derived in a pioneering work by Mandelstam and Tamm [1]. Thereafter, important advancements and extensions of the QSL were reported, for example, for pure states [2–4] as well as mixed states [5–8]. The applications of these fundamental findings have been valuable in many areas, e.g. in the analysis for the rate of change of entropy [9], coherence and correlations in bipartite settings [10, 11], the limitations in quantum metrology [12] and quantum computation [13, 14], and the limit on charging capability of quantum batteries [15–17]. See also [18, 19] for studies showing the application of QSL in the classical regime.

The widely accepted time bound for an evolution of a quantum state  $\rho$  (in general, mixed) to another state  $\sigma$  is known as the *unified* QSL [20, 21], which reads

$$\tau(\rho, \sigma) \geq \hbar \frac{\Theta(\rho, \sigma)}{\min\{\langle H \rangle, \Delta H\}}, \quad (1)$$

where  $\Theta(\rho, \sigma) = \arccos(\mathcal{F}(\rho, \sigma))$  denotes a distance measure known as the Bures angle,  $\mathcal{F}(\rho, \sigma) = \text{tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})$  the Uhlmann root fidelity [22, 23],  $\langle H \rangle = \text{tr}(H\rho) - E_g$  the mean energy taken relative to the ground level of the Hamiltonian,  $E_g$ , and  $\Delta H = \sqrt{\text{tr}[H^2\rho] - \text{tr}[H\rho]^2}$  the standard deviation of energy (SDE). Note also that other distances have been employed [21]. In essence, equation (1) is often described as a version of time-energy uncertainty relation as the evolution time is lower bounded by the amount of energy (mean or variance) initially accessible to the system.

Here we investigate the evolution speed of two principal objects  $A$  and  $B$ , which interact either directly or via an ancillary system  $C$ . While direct interactions place no restrictions on the joint Hamiltonian  $H_{AB}$ , the mediated dynamics is mathematically encoded in the assumption that the tripartite Hamiltonian is a sum  $H_{AC} + H_{BC}$  that excludes the terms coupling  $A$  and  $B$  directly. Note that local Hamiltonians, i.e.  $H_A$ ,  $H_B$ , and  $H_C$ , are already included in these general forms. These scenarios are quite generic and applicable to ample situations. We are interested in contrasting them and in identifying resources different than energy that play a role in speeding up the evolution. We therefore impose the same energy constraint (the denominator in equation (1)) in both bipartite and tripartite settings. Under this condition we show achievable minimal time required to maximally entangle principal systems starting from disentangled states. It turns out that the mediated dynamics cannot be faster than the optimal direct dynamics, but it can be just as fast provided that the mediator is initially correlated with the principal systems. We show additionally, with an explicit example, that although entanglement gain between  $A$  and  $B$  is the desired quantity, the correlations to the mediator can remain classical at all times, see also [24, 25]. These results can be interpreted in terms of trading correlations for energy. If one starts with an uncorrelated mediator and aims at entangling the principal systems as fast as with a correlated mediator, additional energy has to be supplied initially. On the other hand, due to energy conservation, the same energy must be invested in order to prepare the correlated mediator, see [26–28] for a discussion from a thermodynamic perspective.

## 1. Preliminaries

Figure 1 summarises different considered generic scenarios. We shall refer to the case of direct interactions as  $\mathcal{DI}$  and split the mediated interactions into two cases where mediator  $C$  either interacts with the principal systems at all times ( $\mathcal{CMI}$  for continuously mediated interactions) or where it first interacts with  $A$  and then with  $B$  ( $\mathcal{SMI}$  for sequentially mediated interactions). Note that  $\mathcal{SMI}$  in particular covers the case of commuting Hamiltonians  $H_{AC}$  and  $H_{BC}$ . We begin by explaining the energy constraints imposed on these scenarios.

Consider, for the moment, a unitary evolution of a quantum state  $\rho(0)$  to  $\rho_{\text{tar}}$  generated by a Hamiltonian  $H$ . One can see from the unified QSL in equation (1) that there are two relevant quantities: one being the fidelity  $\mathcal{F}(\rho(0), \rho_{\text{tar}})$  between the initial and target state and the other  $\min\{\langle H \rangle, \Delta H\}$ , which is the minimum of the non-negative mean energy or SDE. It is straightforward to check that scaling of the Hamiltonian,  $H \rightarrow kH$ , where  $k$  is a constant, leads to the rescaled energy factors  $\langle H \rangle \rightarrow k\langle H \rangle$  and  $\Delta H \rightarrow k\Delta H$ . A trivial option to speed up the evolution of the quantum state is therefore to supply more energy, e.g. by having stronger coupling. We wish to focus on other quantities playing a role in the speed of evolution and therefore, in what follows, we put the strength of all interactions on equal ground by setting  $\min\{\langle H \rangle, \Delta H\} = \hbar\Omega$ , where  $\Omega$  is a frequency unit. This allows us to write the unified QSL in equation (1) as

$$\Gamma(\rho(0), \rho_{\text{tar}}) \geq \frac{\Theta(\rho(0), \rho_{\text{tar}})}{\min\{\langle M \rangle, \Delta M\}}, \quad (2)$$

where  $\Gamma = \Omega\tau$  stands for the dimensionless minimal time, whereas  $\langle M \rangle = \langle H \rangle / \hbar\Omega$  and  $\Delta M = \Delta H / \hbar\Omega$  respectively denote the non-negative mean energy and SDE, normalised with respect to  $\hbar\Omega$ . Hereafter, we assume the condition

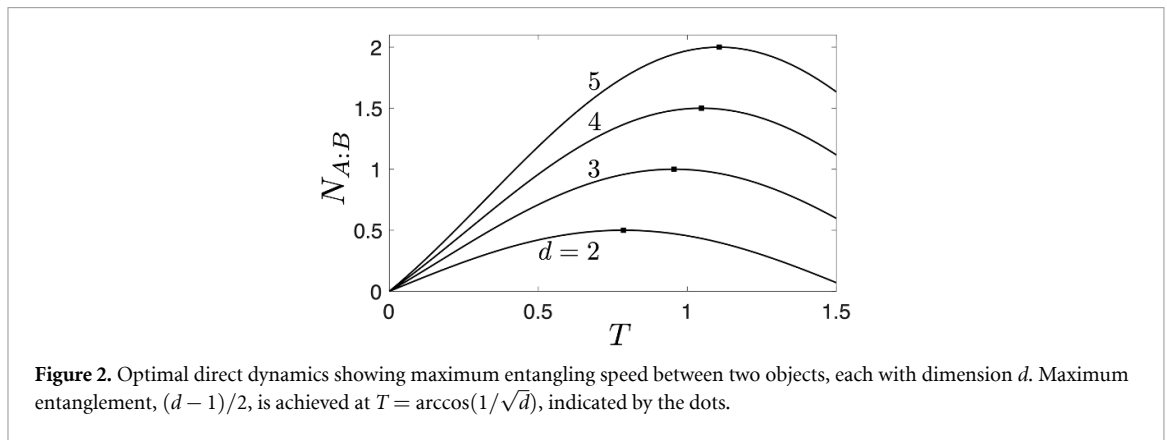
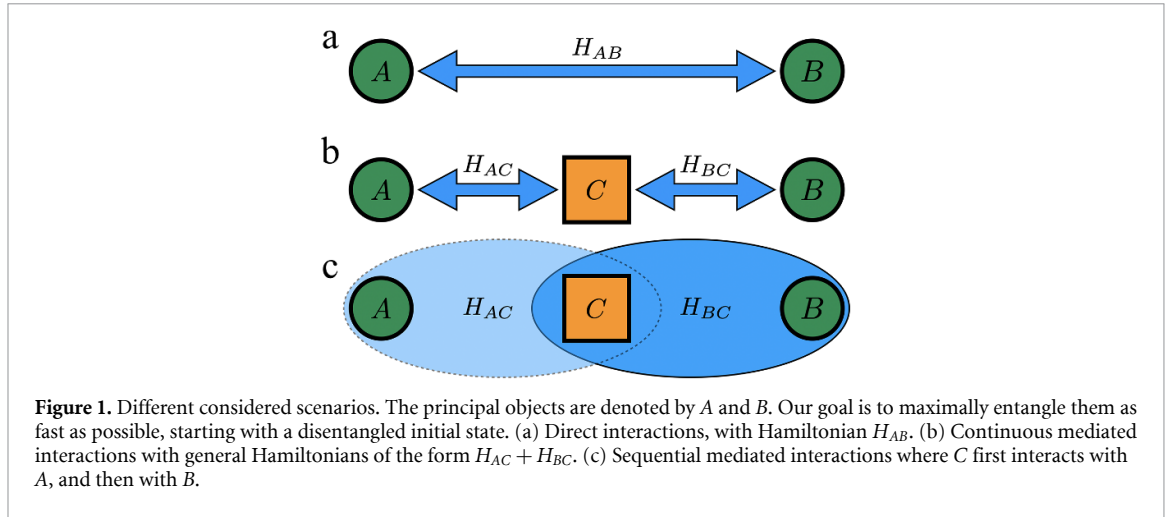
$$\min\{\langle M \rangle, \Delta M\} = 1, \quad (3)$$

which can always be ensured with appropriate scaling  $k$ . We refer to this condition as *resource equality*.

To quantify the amount of entanglement in concrete situations we use negativity, which is a well known computable entanglement monotone [29–33]. We stress, however, that the conclusions of the paper hold for any entanglement monotone. Negativity is defined as the sum of negative eigenvalues after the state of a bipartite system is partially transposed. The bipartite entanglement between objects  $X$  and  $Y$  is denoted by  $N_{X:Y}$  and admits maximum value  $(d-1)/2$ , where  $d = \min\{d_X, d_Y\}$  and  $d_X$  ( $d_Y$ ) is the dimension of object  $X$  ( $Y$ ). For simplicity, we shall assume that the principal objects have the same dimension. Maximally entangled states, for any entanglement monotone [34], are given by pure states of the form

$$|\Psi_{XY}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |x_j\rangle |y_j\rangle, \quad (4)$$

where  $\{|x_j\rangle\}$  and  $\{|y_j\rangle\}$  are orthonormal bases for object  $X$  and  $Y$ , respectively.



## 2. Direct interactions

Let us begin with optimal entangling dynamics for any dimension  $d$ , with direct interactions. Since the initial state we take is disentangled, it has to be a pure product state as the dynamics is purity preserving and the final maximally entangled state is pure, see equation (4). One easily verifies with the help of Cauchy–Schwarz inequality that the fidelity between a product state and maximally entangled state is bounded as  $\mathcal{F} = \langle \alpha\beta | \Psi_{AB} \rangle \leq 1/\sqrt{d}$ . From the resource equality, the time to maximally entangle two systems via direct interactions follows

$$\Gamma_{\mathcal{DI}} \geq \arccos(\mathcal{F}) \geq \arccos(1/\sqrt{d}). \quad (5)$$

This bound is tight and can be achieved with the following exemplary dynamics. Under an initial state of  $|00\rangle$ , we take an optimal (to be shown below) Hamiltonian

$$H_{AB} = \frac{\hbar\Omega}{2\sqrt{d-1}} \sum_{j=1}^{d-1} (X_A^j + Y_A^j) \otimes (X_B^j + Y_B^j), \quad (6)$$

where the subscripts indicate the corresponding system and we have defined  $X^j \equiv |0\rangle\langle j| + |j\rangle\langle 0|$  and  $Y^j \equiv -i|0\rangle\langle j| + i|j\rangle\langle 0|$ . Note that the constant factor ensures the resource equality. One can show that the state at time  $t$  takes the form  $|\psi_{AB}(t)\rangle = \cos(\Omega t)|00\rangle + \sin(\Omega t)(\sum_{j=1}^{d-1}|jj\rangle)/\sqrt{d-1}$ , and therefore it oscillates between the disentangled state  $|00\rangle$  and a maximally entangled state  $|\Psi_{AB}\rangle$ . The latter is achieved earliest at time  $T \equiv \Omega t = \arccos(1/\sqrt{d})$ , see figure 2.

Alternatively, the optimality of this dynamics can be understood from the triangle inequality of the Bures angle [35]:  $\Theta(0, T) + \Theta(T, \arccos(1/\sqrt{d})) \geq \Theta(0, \arccos(1/\sqrt{d}))$ , where we have used a short notation  $\Theta(T_1, T_2) \equiv \Theta(\rho(T_1), \rho(T_2))$ . Under the resource equality, the optimal time should be equal to the Bures angle. Indeed this is the case for the above dynamics as  $\Theta(T_1, T_2) = T_2 - T_1$ , saturating the triangle inequality. Therefore, not only the maximally entangled state is reached in the shortest time, but the

evolution between any intermediate states, i.e. from  $T_1$  to  $T_2$ , with  $T_1, T_2 \in (0, \arccos(1/\sqrt{d}))$ , is the fastest possible.

The described fastest entangling dynamics has the following special features. (a) The Bures angle between the initial state,  $|\psi_{AB}(0)\rangle$ , and the state at any time  $t$  before reaching maximal entanglement,  $|\psi_{AB}(t)\rangle$ , is monotonic with entanglement gain, so that QSL directly translates to the limits on entanglement generation. (b) This generation has its origin in components  $(\sum_{j=1}^{d-1} |jj\rangle)/\sqrt{d-1}$  and the high entangling speed comes from the fact that already the linear term in the expansion of the evolution operator  $\exp(-i\Delta t H_{AB}/\hbar)$  introduces these components. That is, the rate of change of entanglement is strictly positive  $\dot{N}_{A:B}(t) > 0$ , for all times up to maximally entangling time.

### 3. Can mediator speed up entangling process?

At first sight, one might wonder whether the use of quantum mechanical mediator could speed up the evolution by utilising non-commuting Hamiltonians, as revealed through the Baker–Campbell–Hausdorff (BCH) formula. Namely, the dynamics generated by direct coupling  $H_{AB} = A \otimes B$  could be reconstructed through the mediator system  $C$  interacting via  $H_{AC} + H_{BC} = A \otimes p_C + x_C \otimes B$ , where  $x_C$  and  $p_C$  are the position and momentum operators acting on the mediator. Due to the canonical commutation relation the BCH equation reduces to:

$$e^{-it(A \otimes p_C + x_C \otimes B)/\hbar} = e^{-itA \otimes p_C/\hbar} e^{-itx_C \otimes B/\hbar} e^{-it^2 A \otimes B/2\hbar}. \quad (7)$$

Effective direct coupling is now identified in the last term on the right-hand side. Since the corresponding exponent is proportional to squared time, it is legitimate and interesting to enquire about the speeding up possibility.

On the other hand, the special features described at the end of the previous section make it unlikely that any other dynamics is faster than the fastest direct one. Indeed, this is shown in Theorem 1 presented in appendix A. Any dynamics (direct or mediated) that starts with disentangled principal systems can maximally entangle them in time lower bounded as

$$\Gamma_{\text{any}} \geq \arccos(1/\sqrt{d}), \quad (8)$$

where the resource equality is assumed. One then wonders whether mediated dynamics can achieve the same speed as the direct one. At this stage initial correlations with the mediator enter the picture.

We shall show that if the mediator is initially completely uncorrelated from the principal systems, the time required to reach the maximally entangled state is *strictly* longer than  $\arccos(1/\sqrt{d})$ . Then we provide explicit examples of mediated dynamics, with initially correlated mediators, that achieve the shortest possible entangling time.

Consider the initial tripartite state of the form  $\rho(0) = \rho_{AB} \otimes \rho_C$  (with separable  $\rho_{AB}$ ) and, to give a vivid illustration first, take a Hamiltonian  $H_{AC} + H_{BC} = (H_A + H_B) \otimes H_C$ , or any commuting Hamiltonians for which one can identify common eigenbasis  $\{|c\rangle\}$ . Let us take a specific product state  $|\alpha\beta\gamma\rangle$  in the decomposition of the initial state  $\rho(0)$ , and write  $|\gamma\rangle = \sum_c \lambda_c |c\rangle$ . Since  $[H_{AC}, H_{BC}] = 0$  the evolution is mathematically equivalent to  $U_{BC} U_{AC} = \exp(-itH_{BC}/\hbar) \exp(-itH_{AC}/\hbar)$  and the initial product state evolves to  $|\psi(t)\rangle = \sum_c \lambda_c |\alpha_c(t)\rangle |\beta_c(t)\rangle |c\rangle$ , where  $|\alpha_c(t)\rangle = \exp(-itE_c H_A/\hbar) |\alpha\rangle$  and  $|\beta_c(t)\rangle = \exp(-itE_c H_B/\hbar) |\beta\rangle$  with the corresponding eigenvalue  $E_c$  of the Hamiltonian  $H_C$ . By tracing out system  $C$  we note that the state of  $AB$  is a mixture of product states and hence not entangled. Application of this argument to all the product states in the decomposition of  $\rho(0)$  shows that this evolution cannot generate any entanglement between the principal systems whatsoever, i.e.  $\Gamma_{CMI} = \infty$  in this case. This stark contrast with the QSL comes from the fact that the Bures angle is no longer related to the amount of entanglement in the subsystem  $AB$ .

Consider now a general Hamiltonian  $H_{AC} + H_{BC}$ . In Theorem 2 presented in appendix B we show that starting with  $\rho(0) = \rho_{AB} \otimes \rho_C$  the mediated dynamics has non-positive entanglement rate at time  $t = 0$ , i.e.  $\dot{N}_{A:B}(0) \leq 0$  if the three systems are open to their local environments and  $\dot{N}_{A:B}(0) = 0$  for any closed mediated tripartite system. This delay is causing a departure from the optimal entangling path and cannot be compensated in the future. We show rigorously in Theorem 3 presented in appendix C that starting with an uncorrelated mediator, i.e.  $\rho(0) = \rho_{AB} \otimes \rho_C$  the time required to maximally entangle  $A$  and  $B$  via  $CMI$  satisfies a strict bound

$$\Gamma_{CMI} > \arccos(1/\sqrt{d}). \quad (9)$$

Furthermore, we have performed numerical checks with random initial states and Hamiltonians (see appendix D for details) and conjecture that the actual time to maximally entangle the principal systems with initially uncorrelated mediator is  $\Gamma_{\text{conj}} \geq 2 \arccos(1/\sqrt{d})$ . The following two examples with three quantum bits shed light on the origin of this hypothetical lower bound. As initial state, consider  $|000\rangle$ , in the order  $ABC$ , and first take a Hamiltonian  $H = \hbar\Omega(X_A Y_C + Y_B X_C)/\sqrt{2}$ , where  $X$  and  $Y$  denote Pauli operators for the respective qubits. One verifies that the resource equality holds and the state at time  $t$  reads  $|\psi(t)\rangle = \cos(\Omega t)|000\rangle + \sin(\Omega t)|\psi^+\rangle|1\rangle$ , where  $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$  is the Bell state. The maximally entangled state is obtained at time  $\Omega t = \pi/2$  because one has to wait until the dynamics completely erases the  $|000\rangle$  component. In contradistinction, the direct dynamics introduces  $|11\rangle$  component (already in linear time  $\Delta t$ ) and hence the evolution can stop at  $\Omega t = \pi/4$ . Another natural way to entangle two systems via mediator is to entangle the mediator with one of the systems first and then swap this entanglement. Each of these processes takes time at least  $\arccos(1/\sqrt{d})$  and hence again we arrive at the bound anticipated above (the swapping step actually takes a bit longer, see appendix E). A rigorous proof of this bound is left as an open problem.

We finally give examples of mediated dynamics, starting with a correlated mediator, that entangles as fast as the fastest direct dynamics. One may think of utilising an extreme option where the dynamics is initialised with a maximally entangled mediator. This is indeed possible but it is also possible to utilise purely classical correlations with the mediator. Let us start with the entangled mediator first. Consider three qubits with an initial state and the Hamiltonian written as

$$\begin{aligned} |\psi(0)\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \\ H &= \frac{\hbar\Omega}{2\sqrt{2}}(Z_A \otimes H_{C_1} + Z_B \otimes H_{C_2}), \end{aligned} \quad (10)$$

where  $H_{C_1} = -(\mathbb{1} + X_C + Y_C + Z_C)$  and  $H_{C_2} = \mathbb{1} - X_C - Y_C + Z_C$ . The principal system is initially disentangled but the mediator is maximally entangled with the rest of the systems,  $N_{AB:C}(0) = 1/2$ . One verifies that  $N_{A:B}$  follows the curve for  $d = 2$  in figure 2.

As mentioned, quantum correlations to the mediator are not necessary. Consider the following example:

$$\begin{aligned} \rho(0) &= \frac{1}{2}|\psi_m\rangle\langle\psi_m| \otimes |0\rangle\langle 0| + \frac{1}{2}|\tilde{\psi}_m\rangle\langle\tilde{\psi}_m| \otimes |1\rangle\langle 1|, \\ H &= \frac{\hbar\Omega}{2}(Z_A \otimes Z_C + Z_B \otimes Z_C), \end{aligned} \quad (11)$$

where  $|\psi_m\rangle = (|+-\rangle + |-+\rangle)/\sqrt{2}$  and  $|\tilde{\psi}_m\rangle = (|--\rangle + |++\rangle)/\sqrt{2}$  are two Bell-like states of  $AB$  with  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ . This example is similar to those in [24, 25] used to demonstrate entanglement localisation via classical mediators and to indicate that controlled quantum teleportation can be realised without genuine multipartite entanglement [36]. Note that initially the principal system is disentangled (an even mixture of Bell states) and this time the mediator is only classically correlated—its states flag in which maximally entangled state is the principal system [37]. Furthermore, Hamiltonians  $H_{AC}$  and  $H_{BC}$  in equation (11) commute, with the common  $Z$  eigenbasis, and hence in the absence of initial correlations with the mediator entanglement in the principal system would be impossible. One can now verify via standard computations that the dynamics of  $N_{A:B}$  resulting from equation (11) is the same as in figure 2 for  $d = 2$ . Note that the states of the mediator are the eigenstates of  $H$  and hence they are stationary. Accordingly, only the Bell-like states evolve in time. It has been shown recently in a general case of  $\mathcal{CM}\mathcal{I}$  where the state contains only classical correlations in the partition  $AB : C$  at all times, that the entanglement gain, quantified by the relative entropy of entanglement [38], is bounded by the initial mutual information, i.e.  $E_{A:B}(t) - E_{A:B}(0) \leq I_{AB:C}(0)$  [25, 39]. Here,  $E_{A:B}(t) = \min_{\{\sigma_{AB}\}} -\text{tr}(\rho_{AB}(t) \log(\sigma_{AB})) - S(\rho_{AB}(t))$  denotes the relative entropy of entanglement and  $I_{AB:C}(0) = S(\rho_{AB}(0)) + S(\rho_C(0)) - S(\rho(0))$  stands for the initial mutual information. The subscripts indicate the corresponding subsystems,  $\sigma_{AB}$  is any separable state in the partition  $A : B$ , and  $S(\rho) = -\text{tr}(\rho \log(\rho))$  is the von Neumann entropy. In the particular example of equation (11) this bound is achieved as we initially have  $I_{AB:C}(0) = 1$  and  $E_{A:B}(0) = 0$  which get converted to maximal entanglement  $E_{A:B}(T) = 1$ . More generally, for the task discussed here an immediate strategy is to start with  $I_{AB:C}(0)$  at least equal to the entanglement  $E_{A:B}$  of the target state  $|\Psi_{AB}\rangle$ .

#### 4. Sequential mediated interactions

At last we move to the  $\mathcal{SM}\mathcal{I}$  scenario, where system  $C$  first interacts only with  $A$  and then only with  $B$ . This setting was studied to some degree in [40] where, in the present context, it was found that in order to prepare

a maximally entangled state between the principal systems the dimension of  $C$  has to be at least  $d$ . We therefore set it to  $d$  and take the initial state as  $\rho(0) = \rho_{AB} \otimes \rho_C$ . Under these conditions Theorem 4 in appendix E shows the following lower bound on the entangling time:

$$\Gamma_{\mathcal{SMI}} \geq \arccos(1/\sqrt{d}) + \arccos(1/d). \quad (12)$$

Our numerical simulations indicate that this bound is tight. Note that this is even longer than  $2\arccos(1/\sqrt{d})$  already demonstrated to be achievable with  $\mathcal{CMI}$ .

## 5. Discussion

We wish to conclude with a few comments on the obtained results. Since a maximally entangled state  $|\Psi_{AB}\rangle$  is pure and the direct closed dynamics preserves the purity, the maximal entanglement cannot be achieved via direct coupling if one starts with a mixed state. After introducing an ancillary system, the reduced  $AB$  dynamics is, in general, not unitary and hence the purity of  $\rho_{AB}$  may change. For a concrete example see below equation (11), where the initial purity of  $1/2$  is increased to 1 while the disentangled initial state becomes maximally entangled. Therefore, for states of  $AB$  that are initially mixed, the only way to deterministically achieve maximum entanglement and saturate the time bound of  $\mathcal{DI}$  is to make use of a correlated mediator. If the deterministic condition is lifted, a pure state can also be obtained by incorporating a projective measurement.

Having said this, a possibility emerges to maximally entangle initially mixed principal systems by opening just one of them to a correlated local environment. This is reasonable because the incoherent evolution may increase the purity of  $\rho_{AB}$  and previously established entanglement with the environment can flow to the principal systems. A simple example is as follows. Suppose  $A$  and  $B$  are qubits and only qubit  $A$  interacts with its single-qubit environment  $C$ . As the initial state, we take the one in equation (11) and consider a Hamiltonian  $H = \hbar\Omega Z_A \otimes Z_C$  for the local interaction with environment. One verifies that the resulting dynamics gives the same entanglement  $N_{A:B}$  as in figure 2 for  $d = 2$ . It is therefore the fastest possible entangling process.

The last example is interesting from the point of view of open quantum systems. Note that the mutual information in the principal system grows from the initial value  $I_{A:B}(0) = 1$  to the final value  $I_{A:B}(\pi/4) = 2$ . Yet, subsystem  $B$  has not been operated on—only system  $A$  interacts with its local environment. One therefore asks what happens to the data processing inequality stating that information can only decay under local operations [35]. The answer is that the inequality is derived for local maps which are completely positive and trace preserving. Accordingly, the example just given is likely one of the simplest of non-completely-positive dynamics. Violation of data processing inequality has already been discussed as a witness of such forms of evolution [41]. In the present example, the high entangling speed comes from the lack of complete positivity. The initial mutual information of a separable state satisfies  $I_{A:B}(0) \leq \min\{S(\rho_A), S(\rho_B)\}$  and it cannot be improved via any evolution respecting data processing inequality. In contradistinction, entangled states admit mutual information as high as  $I_{A:B} = 2 \min\{S(\rho_A), S(\rho_B)\}$ . Such gain via local operations is possible only with non-completely-positive dynamics.

Our main result shows that correlations play a similar role to energy in speeding up dynamics. In tripartite mediated system  $A$ - $C$ - $B$ , where principal systems  $A$  and  $B$  are coupled via mediator  $C$ , it takes strictly longer to maximally entangle  $AB$  when the evolution is initialised with uncorrelated mediator than when it begins with a correlated mediator. We conjecture that the required minimal time for the case of uncorrelated mediator is twice as long. In other words, if one would like to start with an uncorrelated mediator and reach a maximally entangled state at the same time as with a correlated mediator, one has to supply twice as much energy.

## Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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## Appendix A. No speeding up with mediators

**Theorem 1.** Consider dynamics described by a Hamiltonian  $H$ , involving three objects  $A$ ,  $B$ , and  $C$  (direct or mediated). For initial states  $\rho(0) = \rho_{ABC}$ , having disentangled  $\rho_{AB}$ , the lower bound on the time required to maximally entangle  $AB$  satisfies

$$\Gamma_{\text{any}} \geq \arccos\left(1/\sqrt{d}\right), \quad (\text{A1})$$

where the resource equality is assumed.

**Proof.** In the target state the principal systems are maximally entangled, which implies that their state is pure and uncorrelated with the mediator  $C$ , i.e.  $\rho_{\text{tar}} = |\Psi_{AB}\rangle\langle\Psi_{AB}| \otimes \rho_C$ . We evaluate the fidelity of the initial and target states:

$$\begin{aligned} \mathcal{F}(\rho(0), \rho_{\text{tar}}) &= \mathcal{F}(\rho_{ABC}, |\Psi_{AB}\rangle\langle\Psi_{AB}| \otimes \rho_C) \\ &\leq \mathcal{F}(\rho_{AB}, |\Psi_{AB}\rangle\langle\Psi_{AB}|) \\ &\leq \max_{p_j, |a_j b_j\rangle} \sqrt{\sum_j p_j |\langle a_j b_j | \Psi_{AB} \rangle|^2} \\ &\leq \max_{|a_j b_j\rangle} |\langle a_j b_j | \Psi_{AB} \rangle| = \frac{1}{\sqrt{d}}, \end{aligned} \quad (\text{A2})$$

where the steps are justified as follows. The first inequality is due to monotonicity of fidelity under trace-preserving completely positive maps [42] (here, tracing out  $C$ ). Then we expressed the disentangled state as  $\rho_{AB} = \sum_j p_j |a_j b_j\rangle\langle a_j b_j|$  and used its convexity properties. The final equation follows from the form of maximally entangled state. Finally, by having the resource equality, one gets  $\Gamma_{\text{any}} \geq \arccos(\mathcal{F}(\rho(0), \rho_{\text{tar}})) \geq \arccos(1/\sqrt{d})$ .  $\square$

## Appendix B. No initial entanglement gain with uncorrelated mediator

**Theorem 2.** Consider the case of  $\mathcal{CM}\mathcal{I}$ , where all objects can be open to their own local environments (for generality). For initial states where the mediator is uncorrelated, i.e.  $\rho(0) = \rho_{AB} \otimes \rho_C$ , the rate of any entanglement monotone follows  $\dot{E}_{A:B}(0) \leq 0$ .

**Proof.** We take the evolution of the whole tripartite system following the Lindblad master equation to include the contribution from interactions with local environments:

$$\begin{aligned} \frac{\rho(\Delta t) - \rho(0)}{\Delta t} &= -i[H, \rho(0)] + \sum_{X=A,B,C} L_X \rho(0), \quad (\text{B1}) \\ L_X \rho(0) &\equiv \sum_k Q_k^X \rho(0) Q_k^{X\dagger} - \frac{1}{2} \{Q_k^{X\dagger} Q_k^X, \rho(0)\}. \end{aligned}$$

We set  $\hbar$  to unity in this proof for simplicity. Note that the first term in the RHS of equation (B1) corresponds to the coherent part of the dynamics, while the second constitutes incoherent processes from interactions with local environments, that is, the operator  $Q_k^X$  only acts on system  $X$ . We take the total Hamiltonian as  $H = H_A \otimes H_C + H_B \otimes H_C$ , without loss of generality, and note that the proof easily follows for a general Hamiltonian  $H = \sum_\mu H_A^\mu \otimes H_C^\mu + \sum_\nu H_B^\nu \otimes H_C^\nu$ .

Following equation (B1), the state of the principal objects at  $\Delta t$  reads

$$\begin{aligned} \rho_{AB}(\Delta t) &= \text{tr}_C(\rho(\Delta t)) \\ &= \text{tr}_C(\rho(0) - i\Delta t[H, \rho(0)] + \Delta t \sum_X L_X \rho(0)) \end{aligned}$$



$$= \rho_{AB} - i\Delta t[H_A E_C + H_B E_{C'}, \rho_{AB}] + \Delta t(L_A + L_B)\rho_{AB}, \quad (\text{B2})$$

where  $E_C = \text{tr}(H_C \rho_C)$  and  $E_{C'} = \text{tr}(H_{C'} \rho_C)$  denote the initial mean energies, and we have used  $\rho(0) = \rho_{AB} \otimes \rho_C$ . Also,  $\text{tr}_C(Q_k^C \rho_C Q_k^{C\dagger} - \frac{1}{2}\{Q_k^{C\dagger} Q_k^C, \rho_C\}) = 0$  follows from the cyclic property of trace.

Effectively, the evolution of the principal objects leading to  $\rho_{AB}(\Delta t)$ , as written in equation (B2), consists of local Hamiltonians weighted by the corresponding mean energies  $H_A E_C + H_B E_{C'}$ , and interactions with respective local environments. Therefore, for any entanglement monotone, a measure that is non-increasing under local operations and classical communication, one concludes that  $E_{A:B}(\Delta t) \leq E_{A:B}(0)$ , and hence,  $\dot{E}_{A:B}(0) \leq 0$ .  $\square$

Unitary dynamics is a special case of Theorem 2 without incoherent interactions with local environments. Since entanglement monotones are invariant under local unitary operations  $E_{A:B}(\Delta t) = E_{A:B}(0)$  or  $\dot{E}_{A:B}(0) = 0$ . As a consequence, changes in entanglement between the principal objects (positive or negative) are only possible if the mediator  $C$  is correlated with them.

By applying this argument to the final state  $|\Psi_{AB}\rangle\langle\Psi_{AB}| \otimes \rho_C$  and backwards in time, we conclude that any dynamics (direct or mediated) approaches the final state at a rate  $\dot{E}_{A:B}(T) = 0$ , clearly seen in figure 2.

### Appendix C. Strict bound for uncorrelated mediator

We revisit the condition where  $C$  is initially uncorrelated, i.e.  $\rho(0) = \rho_{AB} \otimes \rho_C$ , which is a special case of Theorem 1. In this case, we have

$$\mathcal{F}(\rho(0), \rho_{\text{tar}}) = \mathcal{F}(\rho_{AB}, |\Psi_{AB}\rangle\langle\Psi_{AB}|) \mathcal{F}(\rho_C, \rho'_C), \quad (\text{C1})$$

where  $\rho'_C$  is the state of  $C$  in the target  $\rho_{\text{tar}}$ . The only way to saturate the optimal bound of direct dynamics is to set  $\mathcal{F}(\rho_C, \rho'_C) = 1$ , i.e.  $\rho_C = \rho'_C$ . Accordingly, the initial state of  $AB$  has to be in a pure product form. Having this in mind, the theorem below shows that the time bound is still strict.

**Theorem 3.** *For the initial state of the form  $\rho(0) = |\alpha\beta\rangle\langle\alpha\beta| \otimes \rho_C$ , the time required to maximally entangle the principal systems via  $\mathcal{CMI}$  follows a strict bound*

$$\Gamma_{\mathcal{CMI}} > \arccos(1/\sqrt{d}). \quad (\text{C2})$$

**Proof.** Recall that the dynamics identified in the  $\mathcal{DI}$  case saturates the triangle inequality and is characterised by a straight line in Bures angles. Any other optimal dynamics (e.g. generated by other Hamiltonians) has to follow the same straight line. Along the line the states of  $AB$  remain pure at all times. However, Theorem 2 shows that entanglement gain between  $A$  and  $B$  is possible only when the mediating system is correlated with the principal systems at some time  $t$  during the dynamics. In the present case, this means that at  $t$ , the state of  $AB$  is not pure, in particular, the mediator is not in a decoupled form  $|\psi_{AB}(t)\rangle\langle\psi_{AB}(t)| \otimes \rho_C$ , where  $|\psi_{AB}(t)\rangle$  is the state from the optimum  $\mathcal{DI}$ . Since  $\mathcal{F}(\rho_{AB}(t), |\psi_{AB}(t)\rangle\langle\psi_{AB}(t)|) < 1$ , we use the triangle inequality of the Bures angle to conclude the strict bound:

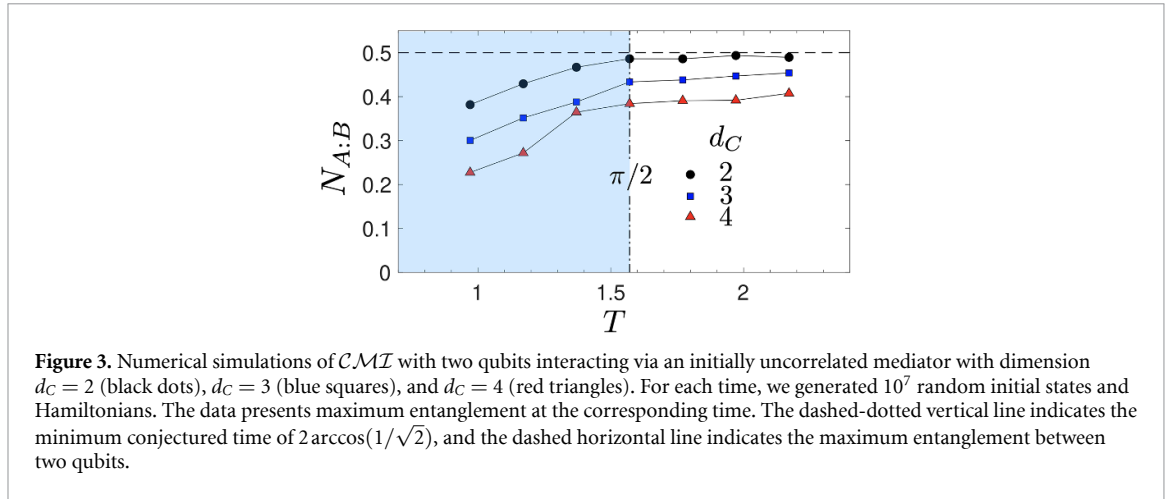
$$\begin{aligned} \Gamma_{\mathcal{CMI}} &= \Gamma_1 + \Gamma_2 \\ &\geq \Theta(0, t) + \Theta(t, \arccos(1/\sqrt{d})) \\ &> \Theta(0, \arccos(1/\sqrt{d})) = \arccos(1/\sqrt{d}), \end{aligned} \quad (\text{C3})$$

where  $\Gamma_1$  and  $\Gamma_2$  respectively denote the minimum time for evolution  $0 \rightarrow t$  and  $t \rightarrow \arccos(1/\sqrt{d})$ . In other words, the dynamics strictly does not follow the optimum (straight line) path, where at time  $t$  the state is uniquely  $|\psi_{AB}(t)\rangle\langle\psi_{AB}(t)| \otimes \rho_C$ .  $\square$

### Appendix D. Numerical simulations for uncorrelated mediator

Here we present results of numerical simulations behind the conjectured minimal time of  $2 \arccos(1/\sqrt{d})$  to maximally entangle the principal systems with initially uncorrelated mediator (recall that  $d_A = d_B = d$ ). Based on the discussion prior to Theorem 3, we consider initial states of the form  $\rho(0) = |\alpha\beta\rangle\langle\alpha\beta| \otimes \rho_C$  and Hamiltonians  $H = H_{AC} + H_{BC}$  scaled to satisfy the resource equality condition.

Let us first describe the case of two qubits interacting via a mediating qubit, i.e.  $d_A = d_B = d_C = 2$ . We randomise the initial state, i.e.  $|\alpha\rangle$ ,  $|\beta\rangle$ , and  $\rho_C$  as well as the Hamiltonians  $H_{AC}$  and  $H_{BC}$  using the quantinf package by Toby Cubitt. For a particular evolution time, we sample  $10^7$  random instances and compute the corresponding entanglement. We present the maximum entanglement at each time in figure 3 (black dots).



As seen, our simulations suggest that the fastest time to reach maximum entanglement of 0.5 is indeed  $2\arccos(1/\sqrt{2})$ . We have also performed simulations where the dimension of the mediator is  $d_C = 3$  as well as  $d_C = 4$ . In this case, we present the corresponding maximum entanglement at each time in figure 3 as blue squares ( $d_C = 3$ ) and red triangles ( $d_C = 4$ ). One can see from figure 3 that maximum entanglement is increasing with time for  $T < 2\arccos(1/\sqrt{2})$  and fluctuates below the maximum value (0.5) for  $T > 2\arccos(1/\sqrt{2})$ , further supporting the conjecture. We also performed simulations for the case of two qutrits interacting via a mediating qutrit, i.e.  $d_A = d_B = d_C = 3$ . In this case, entanglement does not even come close to the maximum possible value at time  $2\arccos(1/\sqrt{3})$ , indicating that this is a correct lower bound on the entangling time.

## Appendix E. Sequential mediated dynamics

**Theorem 4.** Starting with  $\rho(0) = \rho_{AB} \otimes \rho_C$ , maximal entanglement in  $AB$  is achieved via  $SMI$  in time

$$\Gamma_{SMI} \geq \arccos(1/\sqrt{d}) + \arccos(1/d). \quad (E1)$$

**Proof.** The final state has the form  $\rho_f = |\Psi_{AB}\rangle\langle\Psi_{AB}| \otimes \rho_C$ . In this scenario it is to be obtained by the sequence of operations  $\rho_f = U_{BC}U_{AC}\rho(0)U_{AC}^\dagger U_{BC}^\dagger$ . We start with the following argument

$$E_{A:B}(\rho_f) \leq E_{A:BC}(\rho_f) = E_{A:BC}(U_{AC}\rho(0)U_{AC}^\dagger) \quad (E2)$$

where the inequality is due to the monotonicity of entanglement under local operations (here, tracing out  $C$ ) and the equality is due to the fact that the second unitary,  $U_{BC}$ , is local in the considered bipartition. Thus the only way to establish maximal final entanglement between the principal systems is to already prepare it with operation  $U_{AC}$ . This consumes time  $\arccos(1/\sqrt{d})$  and requires initial state of  $A$  and  $C$  to be pure, i.e.  $|\alpha\gamma\rangle$  because  $C$  is not correlated with  $AB$  initially (note that it does not pay off to start with partial entanglement in  $\rho_{AB}$ ). Furthermore, since the final state is pure and we are left with application of  $U_{BC}$  only, the state of particle  $B$  also has to be pure. Summing up, after the first step the tripartite state reads  $|\Psi_{AC}\rangle|\beta\rangle$ . In the remaining step we need to swap this maximal entanglement into the principal systems. To estimate the time required by the swapping we compute the fidelity:

$$\begin{aligned} \mathcal{F} &= |\langle\Psi_{AC}|\langle\beta|\Psi_{AB}\rangle|\gamma\rangle| \\ &= \frac{1}{d} \left| \sum_{j=1}^d \sum_{k=1}^d \langle a_j|a'_k\rangle \langle\beta|b'_k\rangle \langle c_j|\gamma\rangle \right| \\ &\leq \frac{1}{d} \sqrt{\sum_j \left| \sum_k \langle a_j|a'_k\rangle \langle\beta|b'_k\rangle \right|^2} \sqrt{\sum_j |\langle\gamma|c_j\rangle|^2} \\ &= \frac{1}{d} \sqrt{\sum_{j,k,l} \langle a'_l|a_j\rangle \langle a_j|a'_k\rangle \langle\beta|b'_k\rangle \langle b'_l|\beta\rangle} = \frac{1}{d}, \end{aligned} \quad (E3)$$

where we have written  $|\Psi_{AC}\rangle = \sum_j |a_j c_j\rangle/\sqrt{d}$  and  $|\Psi_{AB}\rangle = \sum_k |a'_k b'_k\rangle/\sqrt{d}$  as the maximally entangled states (note possibly different Schmidt bases). Then we used the Cauchy-Schwarz inequality to obtain the third

line. Since  $\{|c_j\rangle\}$  form a complete basis the last square root in the third line equals 1 (sum of probabilities). Rewriting the remaining mod-squared and using the completeness of the bases  $\{|a_j\rangle\}$  and  $\{|b'_k\rangle\}$  we arrive at the final result. The total time required by both steps is therefore at least  $\Gamma_{SM\mathcal{I}} = \Gamma_1 + \Gamma_2 \geq \arccos(1/\sqrt{d}) + \arccos(1/d)$ .  $\square$

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