Is There a Finite Complete Set of Monotones in Any Quantum Resource Theory?

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Entanglement quantification aims to assess the value of quantum states for quantum information processing tasks. A closely related problem is state convertibility, asking whether two remote parties can convert a shared quantum state into another one without exchanging quantum particles. Here, we explore this connection for quantum entanglement and for general quantum resource theories. For any quantum resource theory which contains resource-free pure states, we show that there does not exist a finite set of resource monotones which completely determines all state transformations. We discuss how these limitations can be surpassed, if discontinuous or infinite sets of monotones are considered, or by using quantum catalysis. We also discuss the structure of theories which are described by a single resource monotone and show equivalence with totally ordered resource theories. These are theories where a free transformation exists for any pair of quantum states. We show that totally ordered theories allow for free transformations between all pure states. For single-qubit systems, we provide a full characterization of state transformations for any totally ordered resource theory.

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Entangled quantum systems can exhibit features which seem to contradict our intuition, based on our "classical" perception of nature [1]. Even Einstein was puzzled by some of the consequences of entanglement, concluding that quantum theory cannot be complete [2]. Today, entangled quantum systems are actively explored as an important ingredient of the emerging quantum technologies [1]. This includes applications such as quantum key distribution [3], where entangled systems are used to establish a provably secure key for communication between distant parties. Another groundbreaking application of entanglement is quantum teleportation [4], allowing us to send the state of a quantum system to a remote party by using shared entanglement and classical communication.

The development of a resource theory of entanglement [1] made it possible to study the role of entanglement for technology in a systematic way. This theory introduced the distant lab paradigm, with two remote parties (Alice and Bob) being equipped with local quantum laboratories and connected via a classical communication channel [5–7]. It has been noticed that entanglement between Alice and Bob cannot be created in this setting. Thus, entangled states become a valuable resource, allowing the remote parties to perform tasks which are not possible without it.

In recent years, it became clear that not all quantum technological tasks are based on entanglement, but can make use of other quantum features, such as quantum coherence [8,9], contextuality [10–12], or imaginarity [13–16]. This has led to the development of general quantum resource theories [17]. In analogy to entanglement, a quantum resource theory is based on the set of free states $\{\rho_f\}$ and free operations $\{\Lambda_f\}$. All states which are not free are called resource states. A free operation cannot create resource states from free states. The sets of free states and operations can be motivated by physical constraints, as is done, e.g., in the resource theory of quantum thermodynamics [18,19], where the free state is the Gibbs state, and the free operations preserve the total energy of the system and a heat bath [20]. Another motivation for a resource theory can arise from symmetries, where the free states and operations are symmetric with respect to some physical transformations. An example for such theory is the resource theory of asymmetry [21]. Also, the resource theory of coherence can be formulated in this framework, if the free states are diagonal in a reference basis and the free operations are dephasing covariant [22–26]. Similarly, the resource theory of imaginarity has free states which have only real elements in a reference basis, and the free operations are covariant with respect to transposition [27].

Two fundamental problems in any quantum resource theory are *state convertibility* and *resource quantification*. The state convertibility problem is asking whether for two quantum states there exists a free operation converting one state into the other. The goal of resource quantification is to quantify the amount of the resource in a quantum state. In general, there is no unique quantifier which captures all aspects of a resource theory, and a suitable quantifier depends on the concrete problem under study.

There are some elementary properties which are common to all resource quantifiers [17]. Recalling that resource states cannot be created from free states via free operations, it is intuitive to assume that the degree of the resource in a quantum system cannot increase under free operations, even if the initial state is not free. Thus, every meaningful resource quantifier should not increase under free operations [6,17,28,29]:

$$R(\Lambda_f[\rho]) \leq R(\rho), \tag{1}$$

for any state ρ and any free operation Λ_f. Quantifiers having this property are also called *resource monotones*.

Both problems mentioned above—state convertibility and resource quantification—are in fact closely connected. A state ρ can be converted into σ via free operations if and only if

$$R(\rho) \geq R(\sigma) \tag{2}$$

holds true for all resource monotones [30]. On the other hand, the fact that Eq. (2) holds for some resource monotone R does not guarantee that the transformation $\rho \to \sigma$ is possible via free operations. There might however exist a complete set of resource monotones $\{R_i\}$ which completely characterizes all state transformations; i.e., a transformation $\rho \to \sigma$ is possible if and only if $R_i(\rho) \geq R_i(\sigma)$ holds true for all i. The first such complete set of monotones has been presented for bipartite pure states in entanglement theory [31,32], and it was shown that there is no finite set of faithful and strongly monotonic entanglement monotones which can capture transformations between all mixed states [33]. Complete sets of monotones for concrete resource theories have been studied [34–38], and constructions for general quantum resource theories have been presented in Ref. [30]. It is worth noting that quantum resource theories that are completely governed by a majorization relation have a finite set of monotones [39,40].

Finite sets of resource monotones cannot be complete.— In this Letter we show that a finite complete set of resource monotones does not exist for a large class of quantum resource theories. Our results make only minimal assumptions on the resource monotones: additionally to Eq. (1) we require that the resource monotones are *continuous* [a resource monotone R is continuous if for all states ρ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all σ that satisfies $\|\rho - \sigma\|_1 < \delta$ we have $|R(\rho) - R(\sigma)| < \varepsilon$, where $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm] and *faithful* [a resource monotone R is faithful if $R(\rho) = 0$ if and only if ρ is a free state]. Continuity is a very natural assumption which is fulfilled for most resource monotones studied in the literature—it guarantees that the value of a resourceful state is robust to perturbations. In fact, in many cases the monotones fulfill continuity in an even stronger form; e.g., many entanglement monotones are asymptotically continuous [41,42]. Similarly, faithful monotones are often preferred since they detect some value in any nonfree state. We also use the standard assumptions that the set of free states is convex and compact, that the identity operation is free, and that any free state can be obtained from any state via free operations [this is fulfilled by resource theories that "admit a tensor product structure" [17], since for any free state σ the following measure-and-prepare channel is also free: $\Lambda(\rho) = \mathrm{Tr}(\rho)\sigma$]. The latter assumption implies that any resource monotone is minimal and constant on all free states—without loss of generality we set it to zero. We further say that a state ρ can be converted into a state σ via free operations if for any $\varepsilon > 0$ there is a free operation Λ_f such that $\|\Lambda_f(\rho) - \sigma\|_1 < \varepsilon$. Clearly, the trivial resource theory where all states and all operations are free admits a complete set of continuous monotones. Therefore, we say that a resource theory is nontrivial if there exists a free state and a nonfree state. With these assumptions, we are now ready to prove the first main result of this Letter.

Theorem 1.—For any nontrivial resource theory which contains free pure states, there does not exist a finite complete set of continuous and faithful resource monotones.

Proof.—By contradiction, let there be a complete finite set of continuous resource monotones $\{R_i\}$. Let ρ be a nonfree state. Since the set of free states is compact (and therefore closed), without loss of generality we can assume that ρ is full rank—otherwise we take a mixture with the completely mixed state. Since R_i is faithful, we have $R_i(\rho) > 0$ for all i. Moreover, we define the pure state,

$$|\psi_\varepsilon\rangle = \sqrt{1-\varepsilon}|\phi_f\rangle + \sqrt{\varepsilon}|\phi_f^\perp\rangle, \tag{3}$$

with some free pure state $|\phi_f\rangle$ and $0 < \varepsilon < 1$. Using again the fact that the set of free states is closed, the state $|\psi_\varepsilon\rangle$ can be chosen such that it is not free for all small $\varepsilon > 0$; i.e., we choose $|\phi_f\rangle$ to be on the boundary of the set of free states. Since R_i is continuous and $R_i(\psi_{\varepsilon=0}) = 0$, we can choose $\varepsilon_i > 0$ such that $R_i(\rho) \geq R_i(\psi_{\varepsilon_i})$ for each i. Take $\varepsilon = \min_i \varepsilon_i$, which must be strictly positive since there are a finite number of R_i. Using again the continuity of R_i, we have $R_i(\rho) \geq R_i(\psi_\varepsilon)$ for all i. If $\{R_i\}$ form a complete set of monotones, there must be a free operation converting ρ into $|\psi_\varepsilon\rangle$. Note that $|\psi_\varepsilon\rangle$ is a resource state and that ρ is full rank. It is however not possible to convert a full rank state into a pure resource state via free operations [43,44]; see also Supplemental Material [45]. We thus arrive at a contradiction, and the proof is complete. ■

The above theorem applies to the resource theory of entanglement, both in bipartite and multipartite setting. Moreover, the resource theories of coherence, asymmetry, and imaginarity also contain resource-free pure states, which makes our theorem applicable also to these theories. The theorem also applies to the resource theory of quantum thermodynamics in the limit $T \to 0$ if the ground state of the corresponding Hamiltonian is not degenerate, since the Gibbs state is pure in this case.

As a particular example, this means that no finite collection of continuous and faithful monotones can characterize the state transitions in positive partial transpose

(PPT) theory. This is despite the fact that for any given two states ρ, σ, checking whether there exists a PPT operation that achieves the transition $\Lambda(\rho) = \sigma$ is a semidefinite programming problem.

Surpassing the limitations: Discontinuous monotones, infinite sets, and resource catalysis.—Does the result in Theorem 1 also hold if we take discontinuous monotones into account? As we will see in the following, there exist resource theories which have a finite complete set of resource monotones in this case, at least for qubit systems. This holds for the theories of coherence and imaginarity in the single-qubit setting. For the theory of coherence, all transformations for a single qubit are described by the robustness of coherence C_R and the Δ robustness of coherence $C_{\Delta,R}$, which are given as [22–24,51–53]

$$C_R(\rho) = \min_{\tau} \left\{ s \geq 0 \colon \frac{\rho + s\tau}{1+s} \in \mathcal{I} \right\}, \tag{4}$$

$$C_{\Delta,R}(\rho) = \min_{\Delta[\sigma]=\Delta[\rho]} \left\{ s \geq 0 \colon \frac{\rho + s\sigma}{1+s} \in \mathcal{I} \right\}, \tag{5}$$

where $\mathcal{I}$ is the set of incoherent states, i.e., states which are diagonal in a reference basis. Note that in the single-qubit setting both measures can be evaluated as $C_R(\rho) = 2|\rho_{0,1}|$ and $C_{\Delta,R}(\rho) = |\rho_{0,1}|/\sqrt{\rho_{0,0}\rho_{1,1}}$ [23,24,51]. From this we see that $C_{\Delta,R}(\rho) = 1$ for all pure states which have coherence. Since $C_{\Delta,R}(\rho) = 0$ for all incoherent states, this implies that $C_{\Delta,R}$ is not continuous.

For the resource theory of imaginarity we can construct a complete set of monotones for the single-qubit setting in terms of the Bloch coordinates (r_x, r_y, r_z) of the states [14,15]:

$$I_1(\rho) = r_y^2, \tag{6}$$

$$I_2(\rho) = \frac{r_y^2}{1 - r_x^2 - r_z^2}. \tag{7}$$

As has been shown in Refs. [14,15], I_1 and I_2 do not increase under real operations, and fully describe the transformations in the single-qubit setting. Moreover, I_2 is not continuous, since $I_2(\rho) = 1$ for all pure states which have imaginarity and $I_2(\rho) = 0$ on all real states.

Note that in the resource theory of asymmetry [21,54,55] a complete set of monotones can also be constructed for single-qubit settings (see the Supplemental Material for more details [45]). While the resource theory of quantum thermodynamics [56] in general does not contain resource-free pure states, we demonstrate in the Supplemental Material that a complete set of monotones can also be found in quantum thermodynamics in the qubit setting [45].

Another way to surpass the limitations of Theorem 1 is to allow for an infinite set of resource monotones [30]. The following is a simple construction of an infinite complete set of resource monotones:

$$R_\nu(\rho) = \inf_{\Lambda_f} \|\Lambda_f[\nu] - \rho\|_1, \tag{8}$$

where ν is a quantum state which at the same time serves as a parameter of the monotone R_ν. To prove that R_ν is a resource monotone, let $\tilde{\Lambda}_f$ be a free operation such that $R_\nu(\rho) \geq \|\tilde{\Lambda}_f[\nu] - \rho\|_1 - \varepsilon$ for some $\varepsilon > 0$ (note that such $\tilde{\Lambda}_f$ exists for any $\varepsilon > 0$). Then, for any free operation Λ_f we find

$$R_\nu(\rho) \geq \|\tilde{\Lambda}_f[\nu] - \rho\|_1 - \varepsilon \geq \|\Lambda_f \circ \tilde{\Lambda}_f[\nu] - \Lambda_f[\rho]\|_1 - \varepsilon \\ \geq R_\nu(\Lambda_f[\rho]) - \varepsilon, \tag{9}$$

where we have used the fact that the trace norm does not increase under quantum operations. Since the above inequality holds true for any $\varepsilon > 0$, we conclude that $R_\nu(\rho) \geq R_\nu(\Lambda_f[\rho])$, as claimed. To prove that R_ν form a complete set, consider two states ρ and σ such that $R_\nu(\rho) \geq R_\nu(\sigma)$ for all states ν. By choosing $\nu = \rho$ and noting that $R_\rho(\rho) = 0$, it follows that $R_\rho(\sigma) = 0$. This implies that ρ can be converted into σ via free operations. The above arguments also imply that the set of all resource monotones is complete in any quantum resource theory; i.e., ρ can be converted into σ via free operations if and only if $R(\rho) \geq R(\sigma)$ for all resource monotones. We also note that different construction of a complete set of monotones for general quantum resource theories has been given in Ref. [30].

A third way to surpass the limitations of Theorem 1 is to use quantum catalysis [57]. A quantum catalyst is an additional quantum system which is not changed in the overall procedure [58]. Recently, significant progress has been achieved in the study of correlated and approximate catalysis, where a catalyst can build up correlations with the system, and the procedure is allowed to have an error which can be made negligibly small [59–64]. In this framework, a system state ρ^S can be converted into σ^S if for any $\varepsilon > 0$ there exists a catalyst state τ^C and a free operation Λ_f acting on the system S and the catalyst C such that [57,60,63,64]

$$\|\Lambda_f(\rho^S \otimes \tau^C) - \sigma^S \otimes \tau^C\|_1 \leq \varepsilon, \tag{10}$$

$$\mathrm{Tr}_S[\Lambda_f(\rho^S \otimes \tau^C)] = \tau^C. \tag{11}$$

Remarkably, in the resource theory of coherence catalytic transformations are completely described by a single quantity, known as the relative entropy of coherence $C(\rho) = S(\Delta[\rho]) - S(\rho)$ with the von Neumann entropy $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$. In particular, it is possible to transform ρ into σ via dephasing covariant operations and approximate catalysis if and only if $C(\rho) \geq C(\sigma)$ [65]; see also Supplemental Material [45]. A similar statement can be made for the resource theory of quantum thermodynamics based on Gibbs-preserving operations. In this case, catalytic transformations via Gibbs-preserving operations are fully described by the Helmholtz free energy [59]. Equivalently, a catalytic transformation $\rho \to \sigma$ is possible in this setting if and only if [59] $S(\rho||\gamma) \geq S(\sigma||\gamma)$ with the Gibbs state γ and the quantum relative entropy $S(\rho||\gamma) = \mathrm{Tr}[\rho \log_2 \rho] - \mathrm{Tr}[\rho \log_2 \gamma]$.

Single complete resource monotone and total order.—One of the early problems in entanglement theory is to find a complete set of conditions that characterizes the state transformations. While it quickly became clear that no finite set of conditions suffices [33] (except in special cases, e.g., pure states), the argument relies on specific features of entanglement. Therefore the problem remains open for other resource theories. In the last part of the Letter we will investigate the structure of "simple" resource theories which have a single complete resource monotone; i.e., a free transformation from ρ to σ is possible if and only if $R(\rho) \geq R(\sigma)$ for a single monotone R. We will show that such theories are equivalent to total ordering of the states. We will also provide some partial characterization of such theories, that shows such simple theories must have a very restricted form, which explains why state transformations in commonly considered resource theories cannot be governed by a single monotone.

In the following, we call a resource theory *totally ordered* if for any pair of states ρ and σ there exists a free transformation in (at least) one direction $\rho \to \sigma$ or $\sigma \to \rho$. We further introduce the resource monotone

$$R(\rho) = \min_{\mu \in \mathcal{F}} \|\rho - \mu\|_1, \tag{12}$$

where $\mathcal{F}$ is the set of free states. It is straightforward to see that R is a monotone in any quantum resource theory. We are now ready to prove the following theorem.

Theorem 2.—A resource theory has a single complete monotone if and only if the theory is totally ordered.

The proof idea is to use the fact that $R[(1-\varepsilon)\sigma + \varepsilon\mu]$ is a strictly nonincreasing function of ε for any free state μ. We refer to the Supplemental Material for the full proof [45]. This shows that the existence of a single monotone that is complete is equivalent to a total ordering of the set of states by the free transformations. We will now prove some additional features of totally ordered resource theories.

Theorem 3.—Any totally ordered quantum resource theory allows for free transformations between any two pure states $|\psi\rangle \to |\phi\rangle$.

See Supplemental Material for the full proof [45]. It is important to note that Theorem 3 implies

$$R(\psi) = R(\phi) \tag{13}$$

for any two pure states $|\psi\rangle$ and $|\phi\rangle$.

We will now fully characterize all totally ordered resource theories for $d = 2$. We will start by characterizing the set of free states, using again the monotone R in Eq. (12). Note for two single-qubit states ρ and σ with Bloch vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ it holds $\|\rho - \sigma\|_1 = |\boldsymbol{r} - \boldsymbol{s}|$. Since all pure states are equally far away from the set of free states due to Eq. (13), it must be that the set of free states is a ball around the maximally mixed state. Denoting the radius of this ball by t we can characterize the set of free states as follows:

$$\mathcal{F}_t = \left\{\sigma : \left\|\sigma - \frac{\mathbb{1}}{2}\right\|_1 \leq t\right\}, \tag{14}$$

with $t \in [0, 1]$. For any given t we can now evaluate the resource monotone R for any state ρ:

$$R(\rho) = \max\{|\boldsymbol{r}| - t, 0\}. \tag{15}$$

Thus, in a totally ordered resource theory for a single qubit all state transformations are determined by the length of the Bloch vector. For any two resource states ρ and σ (with Bloch vectors $\boldsymbol{r}$ and $\boldsymbol{s}$) a free transformation $\rho \to \sigma$ is possible if and only if $|\boldsymbol{r}| \geq |\boldsymbol{s}|$. Moreover, a transformation $\rho \to \sigma$ is always possible whenever $|\boldsymbol{s}| \leq t$, since σ is a free state in this case.

An example for a totally ordered resource theory in the single-qubit setting is the resource theory of purity [66,67], which corresponds to the case $t = 0$. We will now show that a totally ordered resource theory exists for any $t \in [0, 1]$. For a given t, we define the set of free operations to be all unital operations, i.e., all operations with the property $\Lambda[\mathbb{1}/2] = \mathbb{1}/2$. Additionally, all fixed-output operations such that $\Lambda[\rho] = \sigma$ with $\sigma \in \mathcal{F}_t$ are considered free. Noting that via unital operations it is possible to transform a qubit state ρ into another qubit state σ if and only if $|\boldsymbol{r}| \geq |\boldsymbol{s}|$ [67], we see that the free states and operations defined in this way give rise to a totally ordered resource theory, with $\mathcal{F}_t$ being the set of free states. Note that the key property enabling this construction is that unital operations induce a total order. However, since this property does not hold for $d \geq 3$, we cannot generalize the construction to higher dimensional systems.

Conclusions.—We have investigated the possibility to have a complete set of monotones in general quantum resource theories. Using only minimal assumptions, such as monotonicity and continuity, we have proven that a complete finite set of monotones does not exist, if a resource theory contains free pure states. This result is applicable to the theory of entanglement in bipartite and multipartite settings, and also to the theories of coherence, imaginarity, and asymmetry. It is however possible to find complete sets of monotones by either allowing discontinuity or considering infinite sets, and we gave examples for such complete sets in various resource theories.

We have further considered resource theories where the state transformations are governed by a single monotone. We proved that any such theory must be totally ordered, where any pair of states admits a free transformation in (at least) one direction. We provided a partial characterization of any such theory, any totally ordered resource theory must allow for free transformations between all pure states, and provided a full characterization of state transformations for all totally ordered resource theories for a single qubit. It remains an open question whether there exist totally ordered resource theories for $d \geq 3$. Nevertheless, this shows the severe restrictions one imposes when we assume that transformations are governed by a single monotone. Another open problem concerns the extension of our results to the resource theories of quantum channels, where—instead of states—transformations between quantum channels are considered [68]. It is not clear at this moment how the results presented in this Letter extend to these resource theories.

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