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Entanglement and coherence in quantum state merging

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Understanding the resource consumption in distributed scenarios is one of the main goals of quantum information theory. A prominent example for such a scenario is the task of quantum state merging where two parties aim to merge their parts of a tripartite quantum state. In standard quantum state merging, entanglement is considered as an expensive resource, while local quantum operations can be performed at no additional cost. However, recent developments show that some local operations could be more expensive than others: it is reasonable to distinguish between local incoherent operations and local operations which can create coherence. This idea leads us to the task of incoherent quantum state merging, where one of the parties has free access to local incoherent operations only. In this case the resources of the process are quantified by pairs of entanglement and coherence. Here, we develop tools for studying this process, and apply them to several relevant scenarios. While quantum state merging can lead to a gain of entanglement, our results imply that no merging procedure can gain entanglement and coherence at the same time. We also provide a general lower bound on the entanglement-coherence sum, and show that the bound is tight for all pure states. Our results also lead to an incoherent version of Schumacher compression: in this case the compression rate is equal to the von Neumann entropy of the diagonal elements of the corresponding quantum state.

Introduction. While coherence has long been known in classical physics as a fundamental waves property [1], in quantum mechanics coherent superposition is elevated to a universal principle governing all processes. Indeed, the fact that all matter exhibits wave behavior was first understood by de Broglie [2], which became the basis of the now standard formulation of quantum mechanics in Schrödinger's wave equation [3]. The universality of the superposition principle, i.e. the *tenet* that any two valid states of a system can be superposed to form a new valid state, marks a radical departure from classical physics. It is at the heart of the many counterintuitive features of quantum theory, perhaps most famously in Schrödinger's Gedankenexperiment of the cat [4]. Quantum entanglement can be considered as a particular manifestation of coherence, and both of these nonclassical phenomena have led to extensive debates in the early days of quantum mechanics [5, 6].

While the study of the resource theory of entanglement has a long tradition [7, 8], the resource theory of quantum coherence has been formulated only recently [9, 10], although other attempts in this direction have been also presented earlier [11–16]. The basis of any resource theory are free states, these are states which can be created at no cost. In entanglement theory, these are all separable states. In coherence theory these are incoherent states [9], i.e., states which are diagonal in a fixed basis $|i\rangle$. The second important ingredient of any resource theory are free operations, i.e., operations which can be performed at no additional cost. In entanglement theory this is usually the set of local operations and classical communication, although other more general sets such as separable operations [17, 18] and asymptotically nonentangling operations [19, 20] have also been considered. In coherence theory, free operations are called incoherent operations. These are precisely the quantum operations which have incoherent Kraus operators, i.e., $K_i|m\rangle \propto |n\rangle$, where $|m\rangle$ and $|n\rangle$ are elements of the incoherent basis [9].

Triggered by these recent developments, much effort is put into understanding the role of coherence as a resource in quantum theory [21–38]. Several new quantifiers of coherence have been proposed [39–52], and the dynamics of some of these quantities under noisy evolution has been investigated [53–57]. Several works also study maximally coherent states [58, 59], the role of coherence in spin models [60, 61], cohering power of quantum channels [62–64], and relations between coherence and other measures of quantumness [65–71]. Coherence also plays an important role in quantum thermodynamics [72–82], and its investigation in biological systems is an important step towards finding quantum phenomena in living objects [83–86]. Additionally, a distinction between "speakable" and "unspeakable" coherence has also been introduced recently [87]. Here we are describing coherence in a speakable sense whereas unspeakable coherence is the resource captured in resource theories of asymmetry [15].

Contrary to entanglement, which inherently implies a scenario of at least two separated parties, the resource theory of coherence has been initially introduced for one party only. Very recently, there were several approaches to extend the notion of coherence to more than one party [53, 65, 68, 70, 88–93]. Here, we build on the methods presented in [89–91], aiming to study the interplay between entanglement and coherence in the task known as quantum state merging [94, 95].

In standard quantum state merging, two parties – their names are traditionally Alice and Bob – share a mixed quantum state $\rho = \rho^{AB}$. Alice aims to send her part of the state to Bob via an additional quantum channel. The difficulty of the task arises from an extra requirement: the process has to be performed in such a way that the overall purification of the state remains intact. As was shown in [94, 95], the singlet rate required for this process is equal to the conditional entropy $S(A|B)_\rho = S(\rho^{AB}) - S(\rho^B)$, where $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy. To be precise, if the conditional entropy is positive, then merging is possible with singlets at rate $S(A|B)_\rho$, and merging is not possible if less singlets are available. Moreover, if the conditional entropy is negative, the process is possible without any entanglement. Apart from merging the state for free, Alice and Bob can additionally gain singlets at rate $-S(A|B)_\rho$.

Here, we consider the task of *incoherent quantum state merging*. This task is very similar to standard quantum state merging, up to the fact that Bob has free access to incoherent operations only, i.e., he has to pay for operations which are not incoherent. There are at least two motivations for this: On the one hand, we would like to understand better the local quantum(!) operations that Alice and in particular Bob have to perform in merging. On the other hand, coherence seems to be the resource of choice to consider here, as entanglement and coherence are both resources of superposition, one in correlation, the other locally. Thus, while the cost of standard quantum state merging is quantified by the required entanglement rate E, the cost of incoherent quantum state merging will be quantified by a pair of entanglement and coherence rate (E, C). Solving the problem of incoherent quantum state merging requires the characterization of all *optimal pairs* (E, C). These are pairs of entanglement and coherence for which merging is possible, but neither entanglement nor coherence of the pair can be reduced.

At this point we note that the term "coherence" used in this and other recent papers is, of course, also used in atomic and molecular physics, where "coherences" denote off-diagonal elements of the density matrix, typically in the basis of energy eigenstates. Note, however, that in quantum optics the term "coherence" is also used in the context on classical and quantum electrodynamics, where it describes the factorization property of certain correlation functions, ultimately related to the prominent Glauber-Sudarshan "coherent states" [96, 97]. Off-diagonal elements of the density matrix in this latter sense, are related rather to "non-classicality" of states of photos, phonons, bosons etc. (cf. [98–100] and references therein).

Incoherent quantum state merging. We consider the scenario where three parties, Alice, Bob, and a referee, share a joint quantum state $\rho = \rho^{RAB}$. In the task of incoherent quantum state merging, Alice and Bob aim to merge their parts of the total state on Bob's side by using local quantum-incoherent operations and classical communication (LQICC) [89]. Additionally, Alice and Bob have access to singlets at rate E and maximally coherent states at rate C on Bob's side.

In the following, we are interested in *achievable pairs* (E, C), these are pairs of coherence and entanglement for which the aforementioned task can be performed in the asymptotic scenario. Similar to standard quantum state merging [94, 95] we consider the most general situation, where Alice and Bob can make catalytic use of entanglement and coherence [101]. We call E_i the entanglement rate which is initially shared by Alice and Bob, and E_t will be the final amount of entanglement between them. Similarly, C_i and C_t will be the initial and the final amount of Bob's local coherence. An entanglement-coherence pair (E, C) is achievable if there exist numbers E_i, E_t, C_i, and C_t with $E = E_i - E_t$ and $C = C_i - C_t$ such that for any $\varepsilon > 0$ and any $\delta > 0$ for all sufficiently large integers $n \geq n_0$ there exists an LQICC protocol Λ between Alice and Bob such that

$$\left\|\Lambda\left[\rho_i^{\otimes n} \otimes \Phi_2^{\otimes \lfloor (E_i+\delta)n \rfloor} \otimes \Psi_2^{\otimes \lfloor (C_i+\delta)n \rfloor}\right] - \rho_t^{\otimes n} \otimes \Phi_2^{\otimes \lceil E_t n \rceil} \otimes \Psi_2^{\otimes \lceil C_t n \rceil}\right\|_1 \leq \varepsilon. \tag{1}$$

Here, $\rho_i = \rho^{RAB} \otimes |0\rangle\langle 0|^{\tilde{B}}$ is the total initial state, where $\tilde{B}$ is an additional particle in Bob's hands with dimension $d_{\tilde{B}} = d_A$. $|\Phi_2\rangle = \sqrt{\frac{1}{2}}(|00\rangle + |11\rangle)$ is a maximally entangled two-qubit state shared by Alice and Bob, and $|\Psi_2\rangle = \sqrt{\frac{1}{2}}(|0\rangle + |1\rangle)$ is a maximally coherent single-qubit state on Bob's side. The target state $\rho_t = \rho^{R\tilde{B}B} \otimes |0\rangle\langle 0|^A$ is the same as ρ_i up to relabeling the parties A and $\tilde{B}$, and $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm.

The achievable region is a closed and convex set, due to the timesharing principle [102, 103]. Namely, on block length n, and for $0 < p < 1$, we can break the n systems into two blocks of $k = \lfloor pn \rfloor$ and $\ell = \lceil (1-p)n \rceil$, and run a first protocol with asymptotic rate (E_1, C_1) on the k-block, and a second protocol with asymptotic rate (E_2, C_2) on the ℓ-block. The tensor product of these protocols is evidently an asymptotically error-free merging protocol, and achieves the rate pair $(E, C) = (pE_1 + (1-p)E_2, pC_1 + (1-p)C_2)$.

As in standard quantum state merging, the quantities E and C can be positive or negative. If E (C) is positive, it means that the merging procedure consumes entanglement (coherence) at rate E (C). If the corresponding quantity is negative, the process can be performed without the corresponding resource, and additionally singlets (maximally coherent states) are gained. Crucially, as we will see below in this paper, the latter gain is not possible for both entanglement and coherence at the same time: if entanglement is gained in the process, coherence has to be consumed, and vice versa.

Clearly, if a pair (E, C) is achievable, then any other pair (E', C') is also achievable for $E' \geq E$ and $C' \geq C$. A pair (E, C) will be called *optimal* if it is achievable and if the pairs (E, C') and (E', C) are not achievable for any $C' < C$ and $E' < E$. Since via LQICC operations a singlet can be converted into a maximally coherent state on Bob's side [89], with every achievable pair (E, C), also $(E + t, C - t)$ is achievable for $t > 0$. Thus, it is always possible to perform incoherent merging with $C = 0$, and the corresponding optimal pair will be denoted $(E_0, 0)$. Another important pair is the one with the minimal amount of entanglement $E_{\min}$ among all protocols. We denote it $(E_{\min}, C_{\max})$, since it also has the maximal amount of coherence among all optimal pairs [104].

A full solution of incoherent quantum state merging implies determining all optimal pairs for a given tripartite state. The following proposition provides a bound on the entanglement-coherence sum $E + C$.

Proposition 1. *Given a tripartite quantum state $\rho = \rho^{RAB}$, any achievable pair (E, C) fulfills the following inequality:*

$$E + C \geq S\left(\mathrm{id}^R \otimes \Delta^{AB}[\rho]\right) - S\left(\mathrm{id}^{RA} \otimes \Delta^{B}[\rho]\right), \qquad (2)$$

where $\Delta^X[\rho]$ denotes full decoherence of the state ρ in the incoherent basis of a (possibly multipartite) subsystem X: $\Delta^X[\rho] = \sum_i |i\rangle\langle i|^X \rho |i\rangle\langle i|^X$.

We refer the reader to Section I in the Supplemental Material [105] for the proof, which is based on monotonicity of QI relative entropy under LQICC operations [89].

It is instructive to compare these results to standard quantum state merging as presented in [94, 95]. In standard quantum state merging, the entanglement rate required for merging a pure state ψ^{RAB} is given by the conditional entropy of the reduced state ρ^{AB}, which can be either positive or negative. In the negative case, quantum state merging is possible without entanglement and additional singlets are produced. Since the right-hand side of Eq. (2) cannot be negative, it follows that the sum $E + C$ is also nonnegative. While each of the quantities E or C can still be negative individually, they cannot be both negative at the same time. Thus, there is no merging procedure where entanglement and coherence are gained simultaneously. This statement is true for all mixed states ρ^{RAB}.

Having presented the general framework, we will now focus on the situation where the total state is pure. Note that understanding of the pure-state scenario also gives insights for general mixed states. In particular, if a pair (E, C) is achievable for a pure state $|\psi\rangle^{RAB}$, the same pair is also achievable for any state ρ^{RAB} with the same reduction such that $\rho^{AB} = \mathrm{Tr}_R[\psi^{RAB}]$.

Incoherent merging of pure states. We will now consider incoherent quantum state merging for general pure states. By state merging [95, 110] we have $E \geq E_{\min} = S(A|B)_\rho$ with the reduced state $\rho = \rho^{AB}$. Moreover, for pure states Proposition 1 reduces to $E + C \geq S(A|B)_{\overline{\rho}}$ with the dephased state $\overline{\rho} = \Delta^{AB}[\rho^{AB}]$. As we will see in the following theorem, this bound is saturated.

Theorem 2. *Any pure state $|\psi\rangle^{RAB}$ can be merged with the optimal pair $(E_0, C = 0)$, where $E_0 = S(\overline{\rho}^{AB}) - S(\overline{\rho}^{B})$.*

We refer to Section II in the Supplemental Material [105] for the proof, which is based on an adaptation of the Slepian-Wolf distributed compression of the decohered - classical! - source. Note that $\overline{\rho}^{AB}$ is a classical state, and its conditional entropy, according to the Slepian-Wolf theorem [111], is precisely the amount of classical communication required to inform Bob about Alice's register. In fact, the proof of this theorem in Section II of the Supplemental Material [105] uses the Slepian-Wolf protocol as a building block.

The above theorem implies that for pure states ψ^{RAB} the minimal entanglement-coherence sum $E + C$ required for merging is equal to the conditional entropy of the decohered state $\overline{\rho}^{AB}$. We also mention that for pure states of the form $|\psi\rangle^{RA} \otimes |0\rangle^B$, the procedure described here can be seen as the incoherent version of Schumacher compression [112]. In particular, Theorem 2 proves that any state ρ can be faithfully compressed at rate $S(\Delta[\rho])$, under the assumption that the decompression is performed with incoherent operations only.

A final comment is in order concerning the applicability of Proposition 1 and Theorem 2 to different operational classes. Beyond the incoherent operations considered in this letter, one can consider the more general class of "maximal" incoherent operations (MIO), which consists of all non-coherence-generating maps [10, 113]. As we discuss in Section I of the Supplemental Material [105], the lower bound of Proposition 1 holds as well for MIO. On the achievability end, the rate of Theorem 2 is still achievable when Bob is limited to so-called *strictly* incoherent operations (SIO) [10, 32], and even if he is further restricted to the class of *physical* incoherent operations (PIO) [49]. Also, Alice's measurement in Theorem 2 can always be made incoherent since the protocol is one-way with her final state being incoherent. Thus our result also applies to the scenario of bipartite local incoherent operations and classical communications (LICC) [90, 91].

Coherence-entanglement tradeoff. The development so far revealed some facts about the landscape of the achievable pairs (E, C) for incoherent merging of a state ρ^{RAB}. Most importantly, there are two inaccessible regions given by the inequalities $E + C \geq S(\Delta^{AB}[\rho]) - S(\Delta^{B}[\rho])$ and $E \geq E_{\min}$. For a pure state, these simplify to $E + C \geq S(A|B)_{\overline{\rho}}$ and $E \geq S(A|B)_\rho$,

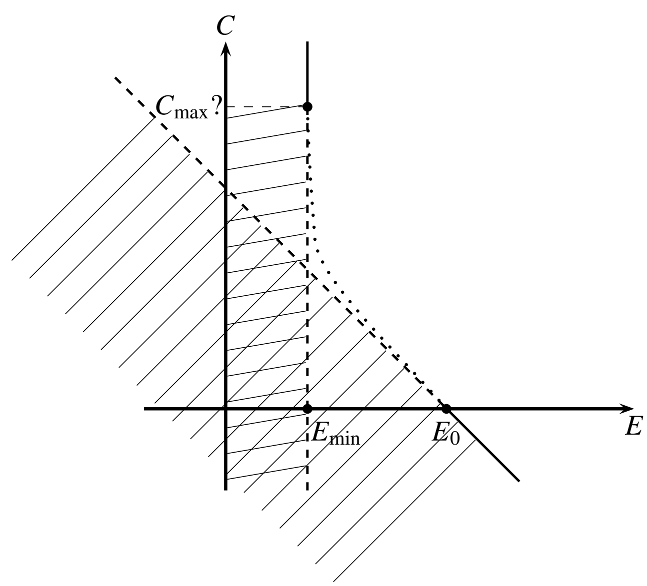

Figure 1. The achievable region and known bounds for coherence and entanglement required to merge a general pure state ψ^{RAB}. The shaded regions to the left and below the straight lines are ruled out. The solid line of slope -1 to the right downward from $(E_0, 0)$, as well as the solid vertical line upward from $(E_{\min}, C_{\max})$, are part of the boundary of the achievable region. The dotted curve connecting these two points represents $C(E)$, the general form of which is not known, however. The quantities E_0 and $E_{\min}$ are given as $E_0 = S(A|B)_{\overline{\rho}}$, $E_{\min} = S(A|B)_\rho$.

and the lower bound is tight as ($E = E_0 = S(A|B)_{\overline{\rho}}, C = 0$) is achievable. Furthermore, since with every achievable pair (E, C), also $(E + t, C - t)$ is achievable for $t > 0$, we find a boundary of the achievable region in the line of slope -1 from $(E_0, 0)$ to the right, see Fig. 1. We do not know at this point whether this boundary line continues with slope -1 also to the left of that point. The biggest open question is the characterization of $C_{\max}$, which is the coherence rate required for the minimum possible entanglement rate $E_{\min}$. Naturally, if we could show that ($E = E_{\min}, C = E_0 - E_{\min}$) is achievable, we would have characterized the entire achievable region, showing that it is delimited by the two above mentioned linear inequalities. On the other hand, it is quite conceivable that in general $C_{\max} \gg E_0 - E_{\min}$.

We are now going to present an example indicative of the second option inspired by the "flower states" [114]:

$$|\psi\rangle^{RAB} = \frac{1}{\sqrt{2d}} \sum_{i=0}^{1} \sum_{j=1}^{d} (U_i^{\top}|j\rangle)^R |i\rangle^A |j\rangle^B \tag{3}$$

where for definiteness $U_0 = \mathbb{1}$, $U_1 = $ QFT is the quantum Fourier transform. One checks that for this family of states, $E_0 = 1$ (attained by simply teleporting Alice's qubit) and $E_{\min} = 0$. Indeed, there is a simple exact merging protocol not using any entanglement, which consists of Alice measuring in the computational basis and communicating i to Bob. Bob in turn applies $U_i^{\dagger}$ after which he is left with the maximally entangled state $|\Phi_d\rangle^{RB}$ with the reference; now he creates the state $|+\rangle^{\tilde{B}} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and recovers the state $|\psi\rangle^{R\tilde{B}B}$ by the controlled unitary $|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$. Note that while U_0 is trivial, U_1 requires a large amount of coherence to be implemented, indeed, the previous procedure of Bob requires asymptotically a rate of $1 + \frac{1}{2}\log d$ of coherence. Conversely, we have the following lower bound:

Theorem 3. *Merging the state in Eq. (3) via one-way LQICC without any initial entanglement, i.e. not only $E_i = 0$ but also $\delta = 0$ in Eq. (1), requires a rate of coherence at least $C \geq 1 + \frac{1}{2}\log d \gg 1$.*

We refer to Section III of the Supplemental Material [105] for the proof. While we proved the theorem for the case where classical communication only goes in one direction, it is reasonable to believe that this result can be extended to arbitrary LQICC protocols. We also note another limitation of the result: Our proof covers only the case that entanglement is exactly zero initially. It is not clear if this result also applies when considering more general merging procedure where entanglement vanishes only in the asymptotic limit. Nevertheless, this result provides strong evidence that in the task of quantum state merging it is possible to save a large amount of local coherence by using little extra entanglement.

In Section IV of the Supplemental Material [105] we also study a family of mixed fully separable states of the form $\rho = \sum_{i,j} p_{ij} |ij\rangle\langle ij|^R \otimes |\psi_{ij}\rangle\langle\psi_{ij}|^A \otimes |i\rangle\langle i|^B$, where the states $|\psi_{ij}\rangle$ are mutually orthogonal for different j, i.e., $\langle\psi_{ij}|\psi_{ik}\rangle = \delta_{jk}$. As is shown in [105], for all these states all optimal pairs are given by $(E, C) = (aC_{\max}, [1 - a]C_{\max})$ with $a \geq 0$ and $C_{\max} = \sum_{i,j} p_{ij} S(\Delta(\psi_{ij}))$.

Conclusions. In the present paper we introduced and studied the task of incoherent quantum state merging. This task is the same as standard quantum state merging, up to the fact that one of the parties has free access to local incoherent operations only, and has to consume a coherent resource for more general operations. The amount of resources needed for merging is quantified by an entanglement-coherence pair (E, C). In general, we showed that the entanglement-coherence sum $E + C$ is nonnegative, which means that no merging procedure can gain entanglement and coherence at the same time. For pure states we gave a protocol of incoherent quantum state merging by finding the minimal entanglement-coherence sum $E + C$, which turns out to be the conditional entropy of the decohered state $\overline{\rho}^{AB}$.

Our results include an incoherent version of Schumacher compression. In particular, if we require that the decompression is performed via incoherent operations only, then the optimal compression rate is given by $S(\Delta(\rho))$. This rate is in general larger than the standard compression rate $S(\rho)$, which comes from the fact that coherence is required for the decompression in the standard case.

We have also made first steps towards an understanding of the precise tradeoff between entanglement and coherence for the task of LQICC merging. While this remains a major open problem in general, we have given strong indications that in certain situations the equivalent of one ebit can be an arbitrary amount of coherence, which we could prove in a setting of one-way LQICC and a situation where we want to reduce the available entanglement exactly (and not only asymptotically) to zero.

Another open question is the relation of LQICC merging to the results presented in [115]. In particular, the authors of [115] study the work cost for erasing a system A which is (quantum) correlated with another observer B in an environment at temperature T. As was shown in [115], this work cost is bounded above by $S(A|B)kT\ln(2)$, where k is the Boltzmann constant. At this point it is natural to ask if our results can be applied to understand the role of coherence in the erasure process. We leave these questions for future research.

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