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Optimally preserving quantum correlations and coherence with eternally non-Markovian dynamics

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Abstract

We demonstrate, both analytically and experimentally, the usefulness of non-Markovianity for preserving correlations and coherence in quantum systems. For this, we consider a broad class of qubit evolutions, having a decoherence matrix separated from zero for large times. While any such Markovian evolution leads to an exponential loss of correlations, non-Markovianity can help to preserve correlations even in the limit $t \to \infty$. In fact, under general assumptions, eternally non-Markovian evolution naturally emerges as the one that allows for optimal preservation of quantum correlations. For covariant qubit evolutions, we also show that non-Markovianity can be used to preserve quantum coherence at all times, which is an important resource for quantum metrology. We explicitly demonstrate this effect experimentally with linear optics, by implementing the optimal non-Markovian quantum evolution.

1. Introduction

In quantum resource theories [1] correlations, such as entanglement [2], are seen as expendable resource to perform certain tasks, e.g. quantum teleportation [3]. On the other hand, every quantum setup we try to control is subject to noise, as it interacts with the environment [4]. Historically, the way to treat these interactions and solve the equations of motion was with the aim of the Born–Markov approximation which assumes that the characteristic time evolution of the environment is very short [5]. In other words, the environment immediately loses memory of its contact with the system and is restored to its initial condition instantaneously. Over time, several different mathematical descriptions of this feature have been proposed [6–11].

The description that we will adopt in this work is based on the notion of divisibility of a dynamical map. An evolution is termed *Markovian*, or CP-divisible, if it can be decomposed into [6–8]:

$$\Lambda_t = V_{t,s} \circ \Lambda_s, \tag{1}$$

where $V_{t,s}$ is a valid quantum operation (a completely positive trace-preserving (CPTP) map) for all $t \geqslant s \geqslant 0$. This definition is essential, if one is to describe an open system dynamics without an explicit model of the environment. As stated in [11] this is the most general quantum version of a classical divisible Markov stochastic process, where the requirement of positivity is replaced by complete positivity to account for the possible presence of quantum correlations with some extra system. Generally, for any differentiable Markovian evolution, the state evolves according to the (time-dependent) Gorini–Kossakowski–Sudarshan–Lindblad equation [12–16]:

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \mathcal{L}_t(\rho) = -\mathrm{i}\,[H(t), \rho(t)] + \sum_{i,j} \gamma_{ij}(t) \left(A_i \rho(t) A_j^\dagger - \frac{1}{2} \left\{ A_j^\dagger A_i, \rho(t) \right\} \right), \tag{2}$$

where $H(t)$ is a time-dependent Hermitian operator, and $\gamma_{ij}(t)$ are elements of a positive semidefinite matrix $\gamma(t)$ we call a *decoherence matrix* [17].

Quantum dynamics which do not admit equation (1) are called *non-Markovian*. They exhibit memory effects that manifest themselves via backflow of operationally relevant quantities from the environment to the system [11]. In contrast to Markovian evolutions, the implementation of a non-Markovian dynamics requires controlling correlations between the system and an ancilla for a finite time [18, 19]. Since Markovian evolutions are easier to implement, it is reasonable to assume that they might be less useful for some tasks, when compared to non-Markovian dynamics. Examples of tasks demonstrating the usefulness of non-Markovianity are swapping the sign of entropy production rate and preserving purity in the context of thermal operations [20], as well as improving the fidelity of quantum teleportation under a noisy channel [21].

In this work, we show that the ability of a quantum evolution to preserve correlations is closely related to non-Markovianity. We consider evolutions having the property that the eigenvalues of the matrix γ are separated from zero for large times, so that the evolution does not simply become unitary at long time scales, thus avoiding trivial preservation of correlations by Markovian dynamics. To demonstrate the main features we are interested in, it is sufficient for us to concentrate on a qubit system. For a class of qubit dynamics, we show that any Markovian evolution leads to an exponential loss of correlations. These results suggest that correlations can only be preserved with the help of non-Markovianity. To make a fair comparison between Markovian and non-Markovian dynamics, we focus on covariant qubit evolutions. We show that the minimal loss of entanglement and mutual information occurs for *eternally* non-Markovian evolutions, i.e., the ones exhibiting non-Markovianity for all times $t > 0$. While entanglement vanishes in the limit $t \to \infty$, the dynamics still preserves nonzero mutual information and quantum discord.

Since covariant evolutions exhibit symmetry with respect to a given Hamiltonian [22], its eigenbasis provides a natural reference for defining quantum coherence [23, 24]. In the case of two-level systems, this corresponds to considering phase-covariant evolutions [25, 26], which cover all dynamics with rotational symmetry about an axis in the Bloch representation, e.g. the z-axis. In this case, we find the evolution which preserves quantum coherence for all finite times, including the limit $t \to \infty$. Interestingly, this dynamics converges to a map which has a two-dimensional image, having finite coherence with respect to the reference basis. As quantum coherence is a resource useful for quantum metrology [27], this dynamics allows us to estimate a parameter ω encoded in the unitary $U = \mathrm{e}^{-\mathrm{i}\omega\sigma_z}$, leading to non-zero quantum Fisher information even in the limit $t \to \infty$.

Non-Markovianity has been experimentally demonstrated based on various platforms such as linear optics [28–35], nuclear magnetic resonance [36, 37], quantum dot [38], micromechanical system [39], trapped ions [40], and superconducting qubits [41]. An attractive experimental platform for studying non-Markovian effects is offered by photonic systems, where controlled interactions between different degrees of freedom, preparation of arbitrary quantum states, and a full state tomography are highly desirable and also appealing for testing fundamental paradigms of quantum mechanics. Here, we demonstrate experimentally, using an optical system, a quantum process which is non-Markovian for all $t > 0$, and observe the optimal preservation of quantum correlations.

2. Results

2.1. Markovian qubit evolutions destroy correlations

Let us consider a two-level quantum system and its dynamical evolution given by a time-dependent Lindbladian:

$$\mathcal{L}_t \rho = \frac{1}{2} \sum_{i,j=1}^{3} \gamma_{ij}(t) \left(\sigma_i \rho \sigma_j - \frac{1}{2} \{\sigma_j \sigma_i, \rho\} \right), \tag{3}$$

where $\{\sigma_i\}_{i=1,2,3}$ are Pauli matrices and the coefficients $\gamma_{ij}(t)$ form a Hermitian matrix $\gamma(t) = (\gamma_{ij}(t))$, $\gamma(t) = \gamma(t)^\dagger$. Equation (3) specifies the evolution of the system as an initial value problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho(t) = \mathcal{L}_t \rho(t), \quad \rho(0) = \rho_0. \tag{4}$$

We will make use of the standard notation: $\dot{\rho}_t = \frac{\mathrm{d}}{\mathrm{d}t}\rho(t)$.

We assume that the decoherence matrix $\gamma(t)$ is such that the solution to the above equation gives rise to a family Λ_t of CPTP maps: $\rho(t) = \Lambda_t \rho_0$. While it is difficult to obtain a general condition that $\gamma(t)$ must satisfy in order to generate a CPTP evolution, several special cases have been considered in the literature [26, 42]. Nevertheless, it is known that the condition $\gamma(t) \geqslant 0$ for all $t \geqslant 0$ is necessary and sufficient for Λ_t to be CP-divisible [16, 43]. The reader is referred to the appendix for more detailed discussion of possible solutions of equation (4).

We now consider Markovian qubit dynamics, having the property that all eigenvalues of γ are separated from zero, i.e., $\gamma(t) \geqslant c\mathbb{1}$ for some $c > 0$. We show that such evolutions lead to the exponential decay of any kind of correlations.

Proposition 1 *Let $\mathcal{L}_t$ be a Lindbladian giving rise to the qubit dynamics Λ_t. If there is a constant $c > 0$ and time $T \geqslant 0$, such that $\gamma(t) \geqslant c\mathbb{1}$ for all $t \geqslant T$, the corresponding qubit dynamics Λ_t fulfills*

$$\min_{\sigma^A \otimes \sigma^B} \left\| \Lambda_t \otimes \mathbb{1}(\rho^{AB}) - \sigma^A \otimes \sigma^B \right\|_1 \leqslant 2\,\mathrm{e}^{-2ct} \tag{5}$$

for all two-qubit states ρ^{AB} and the trace norm $\|M\|_1 = \mathrm{Tr}\sqrt{M^\dagger M}$.

For the proof, we refer the reader to the appendix.

Proposition 1 shows that certain Markovian qubit dynamics destroy all correlations in bipartite quantum states. Moreover, the decay of correlations is exponentially fast. As we will see in the following, finely tuned non-Markovian systems can preserve certain correlations for all times, including in the limit $t \to \infty$.

2.2. Eternally non-Markovian covariant evolutions preserve correlations and coherence in an optimal way

We now focus on non-Markovian quantum evolutions that could potentially exhibit slower rates of decay of entanglement and other quantum correlations. We will show that, apart from entanglement, non-Markovianity is useful for preserving coherence. As coherence is a basis-dependent quantity, we consider evolutions commuting with the unitary encoding the phase, which we assume to be in the z-direction. Hence, we restrict our discussion to *covariant* evolutions [25, 26], with the following decoherence matrix in the Pauli basis:

$$\gamma(t) = \begin{pmatrix} a(t) & \mathrm{i}x(t) & 0 \\ -\mathrm{i}x(t) & a(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix}. \tag{6}$$

The eigenvalues of $\gamma(t)$ are given by $a(t) \pm x(t)$, and $f(t)$.

Note that the Lindbladian, expressed in the basis $\{\sigma_+, \sigma_-, \sigma_3\}$, where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$, has diagonal decoherence matrix $\gamma(t)$, satisfying the requirement for a general covariant quantum evolution [22].

For any covariant qubit dynamics, we can solve the equations of motion (4), using the parametrisation: $\rho(t) = \frac{1}{2}\left(\mathbb{1} + \sum_k r_k(t)\sigma_k\right)$, (see also equation (29)):

$$\dot{r}_1(t) = -[a(t) + f(t)]r_1(t), \tag{7a}$$

$$\dot{r}_2(t) = -[a(t) + f(t)]r_2(t), \tag{7b}$$

$$\dot{r}_3(t) = -2a(t)r_3(t) - 2x(t). \tag{7c}$$

The solution to the above equations gives rise to a CPTP dynamics if and only if

$$\mathrm{e}^{-2A(t)} + |l_z(t)| \leqslant 1, \tag{8a}$$

$$4\mathrm{e}^{-2A(t)-2F(t)} + l_z(t)^2 \leqslant (1 + \mathrm{e}^{-2A(t)})^2, \tag{8b}$$

where $A(t) = \int_0^t a(\tau)\mathrm{d}\tau$, $X(t) = \int_0^t x(\tau)\mathrm{d}\tau$, $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$, and $l_z(t) = 2\mathrm{e}^{-2A(t)}\int_0^t x(\tau)\mathrm{e}^{4A(\tau)}\mathrm{d}\tau$ (see equations (5) and (11) in reference [26]). We obtain

$$r_1(t) = \mathrm{e}^{-A(t)-F(t)}r_1(0), \tag{9a}$$

$$r_2(t) = \mathrm{e}^{-A(t)-F(t)}r_2(0), \tag{9b}$$

$$r_3(t) = \mathrm{e}^{-2A(t)}r_3(0) - l_z(t). \tag{9c}$$

We will now show that the only negative eigenvalue of $\gamma(t)$ must be $f(t)$, and any other negative eigenvalue will not result in a valid quantum dynamics. Let us assume by contradiction that one of two eigenvalues: $a(t) \pm x(t)$ is negative. Without loss of generality, we may say that there exists a constant $c > 0$ such that $x(t) > a(t) + c$ for all $t > T$. From equation (7c) we see that $\dot{r}_3(t) < -2c$ for all $t > T$, which could not lead to a valid quantum evolution, as any Bloch vector would inevitable evolve into a vector outside of the Bloch ball. Thus, $\gamma(t)$ can have only one negative eigenvalue for all $t > T$, which must be $f(t)$.

In the same spirit as in proposition 1, we assume that all eigenvalues of $\gamma(t)$ are separated from zero for all $t > T$. If all eigenvalues become eventually positive, the evolution will become Markovian and the correlations vanish, as described in proposition 1. Hence in the following, we will focus on the other case, where the decoherence matrix $\gamma(t)$ has a negative eigenvalue $f(t)$.

We will now investigate the action of the time evolution on a two-qubit quantum state ρ^{AB}. We consider a broad class of correlation quantifiers $\mathcal{C}$, making the only assumption that the amount of correlations does not increase under local noise:

$$\mathcal{C}(\Phi \otimes \mathbb{1}[\rho^{AB}]) \leqslant \mathcal{C}(\rho^{AB}), \tag{10}$$

where Φ is an arbitrary local operation. In particular, equation (10) is true for the mutual information and any measure of entanglement [2, 44, 45]. Our goal in the following is to determine the functions $f(t)$ leading to the minimal loss of correlations among all dynamics with given $a(t)$ and $x(t)$. More precisely, given a correlation quantifier $\mathcal{C}$, a two-qubit state ρ^{AB}, and time $t \geqslant 0$ we aim to maximize $\mathcal{C}(\Lambda_t \otimes \mathbb{1}[\rho^{AB}])$ over all functions $f(t)$.

It is tempting to believe that the optimal solution for $f(t)$ will in general depend on the setup, in particular on the state and the correlation quantifier. Perhaps surprisingly, we will see in the following that the optimal choice of $f(t)$ is unique, giving rise to a quantum evolution which is non-Markovian for all $t > T$.

Proposition 2 *For given functions $a(t)$ and $x(t)$ and time T such that $a(t) \geqslant |x(t)|$, for all $t > T$, the phase-covariant dynamics for which the loss of correlations is minimal at any given time $t > T$ is given by the function $f(t)$ satisfying the equality*

$$4\, e^{-2A(t)-2F(t)} + l_z(t)^2 = (1 + e^{-2A(t)})^2, \tag{11}$$

where $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$. In particular, for $x(t) = 0$, $f(t) = -a(t)\tanh A(t)$.

We refer to the appendix for the proof.

As an illustration, consider the phase-covariant dynamics Λ_t for which $a(t) = a$, $x(t) = x$ are constants such that $a \geqslant |x|$. The evolution of an initial qubit state $\rho(t) = \Lambda_t\rho_0$ is given by

$$r_{1,2}(t) = \alpha(t) r_{1,2}(0), \tag{12a}$$

$$r_3(t) = \beta(t) r_3(0) - c(t). \tag{12b}$$

with $\alpha(t) = \mathrm{e}^{-at-\int_0^t f(t)\mathrm{d}t}$, $\beta(t) = \mathrm{e}^{-2at}$, $c(t) = \frac{x}{a}(1 - \mathrm{e}^{-2at})$. We can write the Choi–Jamiołkowski state of this evolution as

$$\Omega_t = \frac{1}{4}\begin{pmatrix} 1+\beta(t) & 0 & 0 & 2\alpha(t) \\ 0 & 1-\beta(t) & 0 & 0 \\ 0 & 0 & 1-\beta(t) & 0 \\ 2\alpha(t) & 0 & 0 & 1+\beta(t) \end{pmatrix} - \frac{c(t)}{4}\,\mathrm{diag}(1, -1, 1, -1). \tag{13}$$

For the resulting evolution Λ_t to be completely positive, we require [42]: $4\alpha(t)^2 + c(t)^2 \leqslant (1 + \beta(t))^2$. This inequality is saturated for all $t \geqslant 0$ if we choose $f(t)$ as in equation (11). Note that in this case, $f(t)$ is negative for all $|x| \leqslant a$ and $t > 0$. It is straightforward to verify that the optimal choice of the function $f(t)$, as in proposition 2, is

$$f(t) = -\frac{1}{2}a\left(1 - \frac{x^2}{a^2}\right)\frac{\sinh\, 2at}{\cosh^2 at - \frac{x^2}{a^2}\sinh^2 at} \tag{14}$$

(see also proposition 4 in reference [26]).

For the choice of $f(t)$ defined in equation (14), we have

$$\alpha(t) = \sqrt{\frac{(1 + \mathrm{e}^{-2at})^2}{4} - \left(\frac{x}{a}\right)^2\frac{(1 - \mathrm{e}^{-2at})^2}{4}}. \tag{15}$$

Note that in the limit $t \to \infty$, the Bloch sphere becomes a flat disk of radius $\frac{1}{2}\sqrt{1 - \left(\frac{x}{a}\right)^2}$ with the center at $\frac{x}{a}$ along the z-axis.

In the special case when $x = 0$ and the dynamics becomes unital, we have $f(t) = -a\tanh at$. This evolution (for $a = 1$, up to a constant factor) was first proposed in [17] (see equation (14) therein). In our work, this dynamics arises naturally as a family of evolutions which is optimal for preserving correlations.

We will now consider implications of these results for concrete correlation quantifiers. We use entanglement negativity [46, 47] as a measure of entanglement

$$E(\rho) = \frac{\|\rho^{T_B}\|_1 - 1}{2}, \tag{16}$$

where T_B denotes the partial transpose. This measure was chosen for its simplicity of computation, since it only relies on the straightforward evaluation of the trace norm of the partial transpose of a mixed state [47]. We also consider the quantum mutual information $I(\rho) = S(\rho^A) + S(\rho^B) - S(\rho)$ with the von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$.

For the optimal choice of $f(t)$ as in equation (14), the negativity of the state (13) is given by

$$E(\Omega_t) = \frac{1}{2}\,\mathrm{e}^{-2at}. \tag{17}$$

We see that the evolution Λ_t preserves entanglement for all finite times, and becomes entanglement breaking in the limit $t \to \infty$ [48]. The mutual information, however, does not vanish in the limit $t \to \infty$:

$$\lim_{t\to\infty} I(\Omega_t) = \frac{h(p)}{2}, \tag{18}$$

with $p = \frac{1+\frac{x}{a}}{2}$ and the binary entropy $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$. Additionally, the Choi–Jamiołkowski state (13) exhibits a nonzero amount of quantum discord [49, 50]. Quantum discord is useful for various quantum technological tasks [51–53], an important example being distribution of entanglement between remote parties [54–60]. Following results in [61], we obtain:

$$\lim_{t\to\infty} Q(\Omega_t) = \frac{h\left(\frac{1+\frac{x}{a}}{2}\right)}{2} + h\left(\frac{1+\frac{1}{2}\sqrt{1-\left(\frac{x}{a}\right)^2}}{2}\right) - 1, \tag{19}$$

where Q is quantum discord as defined in [49, 61]. In the case of $|x| < a$, the discord remains nonzero in the limit $t \to \infty$. We refer to the appendix for more details.

Non-Markovianity is also useful for preserving quantum coherence, a fact which can be used in quantum metrology. Since we consider covariant evolutions with dephasing matrix of the form (6), coherence with respect to the eigenbasis of σ_z is a meaningful quantity in this setup. A quantifier of coherence $C(\rho)$ vanishes for all states which are diagonal in the eigenbasis of σ_z, and moreover $C(\rho)$ is monotonic under incoherent operations [23, 24, 62]. These are quantum operations $\Lambda[\rho] = \sum_i K_i\rho K_i^\dagger$ having the property that each Kraus operator does not create coherence [23, 24]. Using similar arguments as in the proof of proposition 2, we can see that a covariant qubit evolution is optimal for preserving coherence at any time $t \geqslant 0$, if $f(t)$ satisfies equation (11).

Let us consider the ℓ_1-norm of coherence, defined as $C_{\ell_1}(\rho) = \sum_{i\neq j}|\rho_{ij}|$ [23]. For a single-qubit state with Bloch vector $\boldsymbol{r} = (r_1, r_2, r_3)$, the ℓ_1-norm of coherence reduces to $C_{\ell_1} = \sqrt{r_1^2 + r_2^2}$. Using equation (12), we can evaluate C_{ℓ_1} as a function of time:

$$C_{\ell_1}(t) = \mathrm{e}^{-at-\int_0^t f(\tau)\mathrm{d}\tau} C_{\ell_1}(0), \tag{20}$$

where $C_{\ell_1}(0)$ is the initial amount of coherence at time $t = 0$. The maximal amount of coherence at any time $t \geqslant 0$ is obtained for $f(t)$ given in equation (11), leading to

$$C_{\ell_1}(t) = \frac{1}{2} C_{\ell_1}(0)\sqrt{(1+\mathrm{e}^{-2at})^2 - \frac{x^2}{a^2}(1-\mathrm{e}^{-2at})^2}. \tag{21}$$

Coherence in general does not vanish even in the limit $t \to \infty$, as long as $C_{\ell_1}(0) > 0$ and $|x| < a$.

Non-Markovianity is also useful in the context of quantum metrology [63]. Let us suppose a quantum state ρ interacts with a device through the Hamiltonian $H = \frac{\omega}{2}\sigma_z$. We would like to estimate the value of the parameter ω. We can use the fact that the evolution commutes with the Hamiltonian H and, for a suitably chosen $f(t)$, preserves coherence in the basis $\{|0\rangle, |1\rangle\}$, to facilitate the estimation of ω. The lower bound on the variance of the estimator of ω is given by the quantum Cramer–Rao bound [64]:

$$(\Delta\omega)^2 \geqslant \frac{1}{\mathcal{F}_\omega(\rho)}, \tag{22}$$

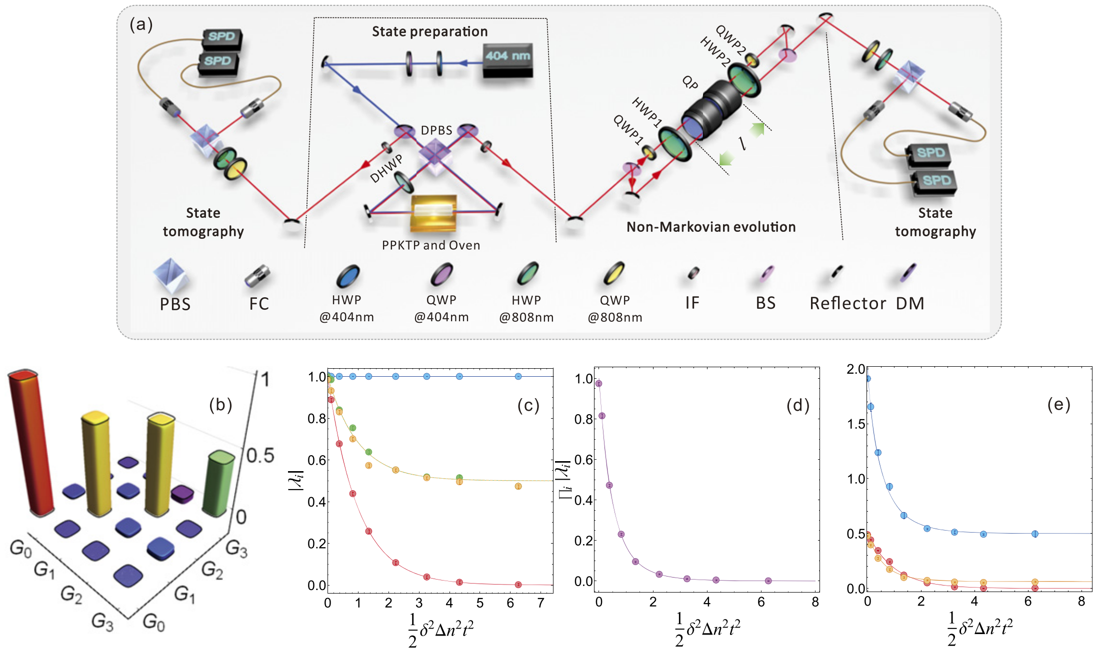

Figure 1. Experimental setup for ENM process and results. (a) The whole experimental setup includes three modules: entangled photon source, eternally non-Markovian process, state tomography. (b) Experimentally reconstructed F matrix with black-edged transparent cubes when $\frac{1}{2}\delta^2\Delta n^2 t^2 = 0.91$. (c) The dynamical process of the absolutes of the spectral values of the process matrix, whose ideal values are given by Equation (25) and are monotonic in time, in agreement with results in [66]. The dots are experimental results, and the lines are the corresponding theoretical fits. (d) The dynamical process of the product of the absolute values of the spectral values. (e) Dynamics of mutual information (light blue disks), negativity (red disks), and geometric discord (orange disks), whose theoretical values are shown as solid lines. Key to components: PBS, polarizing beamsplitter; BS, beamsplitters; QWP, quarter-wave plate; HWP, half-wave plate; SPD, single photon source; DHWP, dichroic half wave plate; DPBS, dichroic polarizing beamsplitter; DM, dichroic mirror; FC, fiber coupler; QP, quartz plate.

where $\mathcal{F}_\omega(\rho)$ is the quantum Fisher information. The following closed formula is valid in the qubit case [65]:

$$\mathcal{F}_\omega(\rho) = |\dot{\vec{r}}|^2 + \frac{(\vec{r}\cdot\dot{\vec{r}})^2}{1-r^2} \tag{23}$$

with $\vec{r}$, the Bloch vector and $\dot{\vec{r}} = \partial\vec{r}/\partial\omega$. In case of phase-covariant dynamics considered here, the second term always vanishes and $\dot{\vec{r}} = tC_{\ell_1}(t)(\cos\omega t, -\sin\omega t, 0)$, leading to $\mathcal{F}_\omega(\rho) = t^2 C_{\ell_1}^2(t)$, with C_{ℓ_1} being the ℓ_1-norm of coherence. Hence, the non-Markovian evolution that maximizes C_{ℓ_1} in equation (21) also maximizes the quantum Fisher information (23).

2.3. Experimental implementation of an eternally non-Markovian process

We now present optical experiments, demonstrating that non-Markovianity is useful for preserving quantum coherence and correlations, as predicted in the theoretical part of this work. We achieve the goal of simulating a non-Markovian evolution by utilizing the fact that it can be obtained as a mixture of different Markovian dynamics [67]. Several attempts of simulating non-Markovian dynamics have been reported. This includes studying the transition between weak (only non CP-divisible) and strong (non P-divisible) non-Markovianity [68], experimental investigations to demonstrate the ambiguity of the extension of the definition of classical non-Markovianity to the quantum case [69, 70], using the spectrum of an evolution over time to infer non P-divisibility [66], and practical demonstration of the non-convex nature of Markovian and non-Markovian channels set [30].

Our experimental setup, illustrated in figure 1(a), relies on three stages: state preparation, implementation of the non-Markovian evolution, and state tomography. The dynamics of interest is described by four t-parameterized Kraus operators, we utilize the frequency degree and path degree of one photon as the environment and its polarization as the system of interests. The system-environment interaction is provided by the coupling between the frequency of the photons and the quartz crystal, and path-dependent operations. We implement experimentally the eternally non-Markovian evolution which is optimal for preserving quantum correlations and quantum coherence as in proposition 2. The details of our high-fidelity, all-optical implementation are explained in the appendix.

In order to verify the experimental set-up, we perform process tomography and experimentally determine the spectrum of the corresponding $n^2 \times n^2$ process matrix F:

$$F_{i,j} = \mathrm{Tr}[G_i \Lambda(G_j)], \tag{24}$$

where $G_i = \sigma_i/\sqrt{2}$, which is derived from this dynamical process at each time, following [66].

The critical experimental step in measuring the spectrum is the application of the dynamics to the basis matrices $G_{i\neq 0}$, which are not legitimate physical quantum states. Nevertheless, there always exists a finite real coefficient c and two legitimate states $\rho_{i,1}$ and $\rho_{i,2}$ satisfying $G_{i\neq 0} = (\rho_{i,1} - \rho_{i,2})/c$, which makes the F matrix and its eigenvalues $\{\lambda_i\}$ detectable in experiments.

In figures 1(b)–(d), we present the results characterizing the spectrum of the process matrix for the relevant non-Markovian evolution. In particular, we compare it against the theoretical behavior of the process eigenvalues, whose moduli read (see equation (61) and below):

$$(|\lambda_i|) = \left(1, \frac{1}{2}\left[1 + \exp\left(-\frac{1}{2}\delta^2 \Delta n^2 t^2\right)\right], \frac{1}{2}\left[1 + \exp\left(-\frac{1}{2}\delta^2 \Delta n^2 t^2\right)\right], \exp\left(-\frac{1}{2}\delta^2 \Delta n^2 t^2\right)\right). \tag{25}$$

Here, the environmental parameter δ corresponds to the variance of the frequency distribution and $\Delta n = n_H - n_V$ denotes the nonzero difference in the refraction indices of the two states, $|H\rangle$ and $|V\rangle$, of polarized photons. In particular, we verify that both the dynamics of each $|\lambda_i|$ as well as their product are in good agreement with the experimental data, which shows the high fidelity of our experimental implementation. In figure 1(e), we show explicitly the resulting dynamics of entanglement negativity, quantum discord, and mutual information—indeed, the implemented non-Markovian evolution yields these three measures of correlations to follow the optimal behavior predicted in our work.

3. Discussion

We have shown that non-Markovianity is useful for preserving correlations and coherence in quantum systems. Any Markovian qubit evolution leads to the exponential loss of correlations, if the decoherence matrix is separated from zero for large times. This complements earlier results on the role of quantum correlations in open quantum systems: while any Markovian evolution leads to monotonic decrease of entanglement [6, 71] and mutual information [72], we show that for nontrivial qubit evolutions non-Markovianity is required to preserve correlations at all times. In addition, non-Markovian qubit evolutions, with decoherence matrix separated from zero, can preserve mutual information and quantum discord even in the limit $t \to \infty$. For covariant evolutions, we have shown that non-Markovianity is also useful for preserving quantum coherence with respect to the reference basis. This effect can be used for parameter estimation: a phase encoded in a covariant unitary can be estimated with finite precision at any time, and the quantum Fisher information is nonzero also in the limit $t \to \infty$. We characterize covariant qubit evolutions that are optimal for preserving quantum coherence and correlations, and implement them experimentally using linear optics.

Our results suggest that if a certain degree of control over the noise is available, it may still be possible to distribute large amount of correlations over noisy channels. This is also demonstrated by our experiment, making our experimental methods applicable for studying fundamental problems in quantum information science. Non-Markovianity appears to be an important feature for quantum technologies, crucial to maintain and store information in the form of quantum correlations and superposition.

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Conflict of interest

The authors declare no competing interests.

Author contributions

MM and K-DW contributed equally to this work.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Markovian dynamics of qubit systems

To reduce the problem of solving equation (4) to a set of ordinary differential equations, where the quantum nature of the system is implicit in the choice of a suitable parametrisation, we use the notation:

$$\rho(t) = \frac{1}{2}\left(\mathbb{1} + \sum_k r_k(t)\sigma_k\right), \tag{26}$$

$\vec{r}(t) = (r_1(t), r_2(t), r_3(t))$, $\|\vec{r}(t)\| \leqslant 1$. Using the commutation relations of the Pauli matrices, we obtain from equation (3):

$$\mathcal{L}_t\rho(t) = \frac{1}{2}\sum_{i,k=1}^{3}\left(\frac{1}{2}(\gamma_{ik}(t) + \gamma_{ki}(t))\, r_i(t) - \gamma_{ii}(t)\, r_k(t))\sigma_k + \xi_k\sigma_k\right). \tag{27}$$

Setting $\gamma(t)^S = \frac{1}{2}(\gamma(t) + \gamma(t)^T)$ and $\vec{\xi}(t) = (\xi_k(t))$, where $\xi_k(t) = i\sum_{i,j=1}^{3} \epsilon_{ijk}\, \gamma_{ij}(t)$, we get

$$\dot{\vec{r}}(t) = (\gamma_t^S - (\mathrm{Tr}\,\gamma(t))\mathbb{1})\vec{r}(t) + \vec{\xi}(t), \quad \vec{r}(0) = \vec{r}_0. \tag{28}$$

In the following, we will make the assumption that the matrix elements $\gamma_{ij}(t)$ are such that for all $i, j = 1, 2, 3$ and $0 \leqslant t_1 < t_2 < \infty$, the integrals $\int_{t_1}^{t_2} \mathfrak{Re}\,\gamma_{ij}(t)\mathrm{d}t$, and $\int_{t_1}^{t_2} \mathfrak{Im}\,\gamma_{ij}(t)\mathrm{d}t$ are finite. We know from the general theory (see theorem 5.3 in [73], p 30), that in that case there exists a unique solution to equation (28) for $t \geqslant 0$.

A general solution to the inhomogeneous differential equation (28) is obtained in the usual way. Let X_t be the fundamental solution to the homogeneous equation: $\dot{\vec{r}}(t) = A_t\vec{r}(t)$, where $A_t = \gamma_t^S - (\mathrm{Tr}\,\gamma(t))\mathbb{1}$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}X_t = A_tX_t$ and $X_0 = \mathbb{1}$. Then the solution to equation (28) is given by

$$\vec{r}(t) = X_t\vec{r}_0 + X_t\int_0^t X_s^{-1}\vec{\xi}_s\,\mathrm{d}s. \tag{29}$$

As we can see from equation (29), the evolution of a quantum two-level system given by $\mathcal{L}_t$ splits into a sum of two evolutions: one that represents a solution to the homogeneous system of ordinary differential equations (for which $\vec{\xi}_t = 0$, or equivalently, for which $\gamma(t)$ is a real symmetric matrix that generates a unital CPTP evolution) and the other that is independent of the initial condition of the system.

Appendix B. Proof of proposition 1

At first, let us assume that $T = 0$. Because $\gamma(t) \geqslant c\mathbb{1}$, and hence $\gamma_t^S \geqslant c\mathbb{1}$, we can rewrite equation (28) as

$$\dot{\vec{r}}(t) = (A_t' - 2c\mathbb{1})\,\vec{r}(t) + \vec{\xi}(t), \tag{30}$$

where $A_t' = \gamma_t^S - \mathrm{Tr}\gamma(t)\mathbb{1} + 2c\mathbb{1} < 0$. The solution to the above equation can be written as

$$\vec{r}(t) = \mathrm{e}^{-2ct}X_t'\vec{r}_0 + \mathrm{e}^{-2ct}X_t'\int_0^t \mathrm{e}^{2cs}(X_s')^{-1}\vec{\xi}_s\,\mathrm{d}s. \tag{31}$$

Here, X'_t represents a valid CPTP dynamics: $\frac{d}{dt}X'_t = A'_t X'_t$ and $X'_0 = \mathbb{1}$. If by $\vec{\eta}(t)$ we denote the vector

$$\vec{\eta}(t) = e^{-2ct} X'_t \int_0^t e^{2cs} (X'_s)^{-1} \vec{\xi}_s \, ds, \tag{32}$$

then $|\vec{r}(t) - \vec{\eta}_t| \leqslant 2\, e^{-2ct} |\vec{r}_0|$. Hence

$$\|\Lambda_t \rho_0 - \tilde{\rho}(t)\|_1 \leqslant e^{-2ct}, \tag{33}$$

where $\tilde{\rho}(t) = \frac{1}{2}(\mathbb{1} + \vec{\eta}_t \cdot \vec{\sigma})$ for any state ρ_0. This implies

$$\|\Lambda_t - \Phi_t\| \leqslant e^{-2ct}, \tag{34}$$

where $\Phi_t \rho = (\mathrm{Tr}\, \rho)\tilde{\rho}(t)$ and the norm of a linear map is given by the infimum over all quantum states

$$\|\Lambda_t - \Phi_t\| = \inf_\rho \|\Lambda_t \rho - \Phi_t \rho\|_1. \tag{35}$$

Recall that the following inequality holds true for any pair of quantum channels Λ_1 and Λ_2 acting on a Hilbert space of dimension d (see [74], corollary 2.2.4):

$$\|\Lambda_1 \otimes \mathbb{1}_d - \Lambda_2 \otimes \mathbb{1}_d\| \leqslant d \|\Lambda_1 - \Lambda_2\|, \tag{36}$$

With equation (34), it follows that

$$\|\Lambda_1 \otimes \mathbb{1}_d - \Lambda_2 \otimes \mathbb{1}_d\| \leqslant 2\, e^{-2ct}. \tag{37}$$

The action of Φ_t on one qubit of a two-qubit state ρ^{AB} is

$$\Phi_t \otimes \mathbb{1}(\rho^{AB}) = \Phi_t(\rho^A) \otimes \rho^B. \tag{38}$$

We obtain

$$\|\Lambda_t \otimes \mathbb{1}(\rho^{AB}) - \Phi_t(\rho^A) \otimes \rho^B\|_1 \leqslant 2\, e^{-2ct}, \tag{39}$$

for any two-qubit state ρ^{AB}. Finally, if $T > 0$, we can repeat the argument above for the evolution $\Lambda'_t = \Lambda_{t+T} \Lambda_T^{-1}$, making use of the fact that Λ_t is CP-divisible. This completes the proof.

Appendix C. Proof of proposition 2

Let $\gamma(t)$ be as in equation (6) of the main text. Suppose $f_0(t)$ is the function that satisfies equation (11) of the main text. We can write the decoherence matrix $\gamma(t)$ as a sum of two matrices:

$$\gamma(t) = \begin{pmatrix} a(t) & ix(t) & 0 \\ -ix(t) & a(t) & 0 \\ 0 & 0 & f_0(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f(t) - f_0(t) \end{pmatrix}. \tag{40}$$

It is easy to see that as long as both matrices generate valid CPTP dynamics independently, the resulting evolutions commute. Since $f_0(t)$ satisfies equation (11) of the main text, the first matrix generates a valid CPTP dynamics. The dynamics generated be the second matrix is clearly CPTP for $t > T$. Indeed, because for all $t > T$, $a(t) > 0$, and hence $A(t) > 0$, then from equation (8b), we have that $0 \leqslant F(t) - F_0(t) = \int_0^t (f(\tau) - f_0(\tau))d\tau$. This is enough to satisfy the complete-positivity conditions (8a) and (8b). According to equation (10) of the main text, the amount of correlations at any given time $t > T$ cannot be larger than in the optimal case when $f(t) = f_0(t)$.

The second part of the proposition follows immediately from observing that the function $l_z(t)$ vanishes as long as we put $x(t) = 0$.

Using similar arguments, we can see that the optimal preservation of quantum coherence at any time $t \geqslant 0$ is achieved if $f(t)$ is chosen such as to satisfy equation (11) of the main text. Indeed, the map generated by the second matrix in equation (40) is a CPTP dynamics that does not create coherence (a phase-damping map) and hence the value of any quantifier of coherence cannot increase under its action.

Appendix D. Evaluation of quantum discord

We follow the results in [61] regarding the quantum discord of 4×4 X-states. The classical part of the correlations is given by equation (22) in [61] and it involves the minimization of the conditional entropy (22) (conditional respect to general von Neumann measurements B_i). The entropies $S(\rho_0)$ and $S(\rho_1)$ are defined in equations (19) and (20) in [61] and the parameters θ and θ' in equations (16) and (17) in [61]. In our case, (see equation (13) of the main text),

$$\rho_{11} = \rho_{33} = \frac{1}{4}\left(1 + \frac{x}{a}\right) \tag{41}$$

$$\rho_{22} = \rho_{44} = \frac{1}{4}\left(1 - \frac{x}{a}\right) \tag{42}$$

$$\rho_{14} = \rho_{41} = \frac{1}{4}\sqrt{1 - \left(\frac{x}{a}\right)^2} \tag{43}$$

$$\rho_{23} = \rho_{32} = 0 \tag{44}$$

$$\Theta = 4kl\left(\frac{\sqrt{1-\left(\frac{x}{a}\right)^2}}{4}\right)^2 \tag{45}$$

$$\theta = \sqrt{\frac{4kl\left(\frac{\sqrt{1-\left(\frac{x}{a}\right)^2}}{4}\right)^2}{\left[\frac{1}{2}\left(1+\frac{x}{a}\right)k + \frac{1}{2}\left(1-\frac{x}{a}\right)l\right]^2}} \tag{46}$$

$$\theta' = \sqrt{\frac{4kl\left(\frac{\sqrt{1-\left(\frac{x}{a}\right)^2}}{4}\right)^2}{\left[\frac{1}{2}\left(1+\frac{x}{a}\right)l + \frac{1}{2}\left(1-\frac{x}{a}\right)k\right]^2}}, \tag{47}$$

where k and l are the parameters of the measurements $\{B_i\}$. The minimum of the conditional entropy is attained in one of the three cases:

(a) $k = 0, l = 1$
(b) $k = 1, l = 0$
(c) $k = l = \frac{1}{2}$.

In the first two cases, $\theta = \theta' = 0$ and $S(\rho_0) = S(\rho_1) = 1$, which is not the minimal value. In the third case, $\theta = \theta' = \frac{\sqrt{1-\left(\frac{x}{a}\right)^2}}{2} = \theta_{\max}$. Because the reduced state ρ^A is maximally mixed, we obtain

$$\mathcal{C}(\rho_X) = 1 - S(\rho_0)|_{\theta_{\max}}, \tag{48}$$

where $\mathcal{C}$ is the measure of classical correlations as defined in [50, 61]. Then the quantum discord is computed as the difference between the total correlations, given by the mutual information in equations (18), and (48).

Appendix E. Methods of the experiment

The experimental set-up is shown in figure 1(a), it consists of three parts: state preparation, non-Markovian evolution, and state tomography. Note that the single qubit operations acting on the polarization state of photons induced by a half-wave plate (HWP) and a quarter wave plate (QWP) with angle θ are

$$h(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \tag{49}$$

$$q(\theta) = \begin{pmatrix} \cos^2\theta + \mathrm{i}\sin^2\theta & \frac{(1-\mathrm{i})}{2}\sin 2\theta \\ \frac{(1-\mathrm{i})}{2}\sin 2\theta & \mathrm{i}\cos^2\theta + \sin^2\theta \end{pmatrix}. \tag{50}$$

In the state preparation part, we can generate arbitrary pure qubit states

$$|\phi\rangle = \cos(\alpha)|H\rangle + \mathrm{e}^{-\mathrm{i}\beta}\sin(\alpha)|V\rangle. \tag{51}$$

In the non-Markovian evolution part, we implement the process with a success probability 1/2, as shown in figure 1(a). We now show that how our all-optical non-Markovian process acts on a single qubit. In particular, assume that we have an arbitrary initial qubit states (in basis $\{|H\rangle, |V\rangle\}$)

$$\rho_0 = \frac{1}{2}\left(I + x_0\sigma_x + y_0\sigma_y + z_0\sigma_z\right). \tag{52}$$

The first 50:50 beam splitters (BS) separate the photons into approximately two branches with equal probabilities independent of the polarization of the photons.

The upper branch is reflected by a reflector, and passes through a HWP with angle 22.5, implementing the unitary operation

$$u_1 = h(22.5) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{53}$$

The lower branch goes through a QWP with angle 0, followed by $h(22.5)$, resulting in the transformation

$$u_2 = h(22.5)q(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}. \tag{54}$$

The overall state then becomes

$$\rho_1 = \frac{1}{2} u_1 \rho_0 u_1^\dagger + \frac{1}{2} u_2 \rho_0 u_2^\dagger. \tag{55}$$

After these two wave plates, the state ρ_1 goes through a decoherence process in a quartz plate (QP). Now we will move on describing the dynamics of a photon in QP.

Unlike the previous works [28, 75], where the authors make use of a FP cavity to modify the frequency spectrum of the photons, we will not modify the spectrum of the photons, which means we will not change the spectrum of the frequency of the single photons. The dynamics of the whole system including the polarization state and frequency, $|\phi\rangle \otimes |\phi_E\rangle$, can be modeled by a unitary evolution

$$U_{\mathrm{tot}}(t) = \int \mathrm{d}\omega \left[\exp(-\mathrm{i}n_H\omega t)|\omega\rangle\langle\omega| \otimes |H\rangle\langle H| + \exp(-\mathrm{i}n_H\omega t)|\omega\rangle\langle\omega| \otimes |V\rangle\langle V| \right], \tag{56}$$

and the frequency state is modeled as

$$|\phi_E\rangle = \int \mathrm{d}\omega\, g(\omega)|\omega\rangle. \tag{57}$$

The corresponding reduced dynamical map Λ_t of the polarization degree of freedoms takes the form,

$$|H\rangle\langle H| \xrightarrow{\Lambda_t} |H\rangle\langle H|, \tag{58a}$$

$$|V\rangle\langle V| \xrightarrow{\Lambda_t} |V\rangle\langle V|, \tag{58b}$$

$$|H\rangle\langle V| \xrightarrow{\Lambda_t} \kappa(t)|H\rangle\langle V|, \tag{58c}$$

$$|V\rangle\langle H| \xrightarrow{\Lambda_t} \kappa^*(t)|V\rangle\langle H|, \tag{58d}$$

where the decoherence factor reads

$$\kappa(t) = \int \mathrm{d}\omega |g(\omega)|^2 \exp(-\mathrm{i}\Delta n\omega t), \tag{59}$$

and $\Delta n = n_H - n_V \approx 0.0089$ denotes the nonzero difference in the refraction indices of $|H\rangle$ and $|V\rangle$ polarized photons.

The spectral distribution of a single photon $|g(\omega)|^2$ in our experiments admits a Gaussian distribution, i.e.,

$$|g(\omega)|^2 = \frac{1}{\sqrt{2\pi}\delta} \exp\left[-\frac{(\omega-\omega_0)^2}{2\delta^2}\right], \tag{60}$$

where ω_0 is the central frequency and $\delta \approx 1.44 \times 10^{12}$ Hz is the variance, corresponding to the linewidth $\Delta\lambda \approx 0.5$ nm of down converted photons [76]. One can check that the normalization holds, i.e., $\int \mathrm{d}\omega |g(\omega)|^2 = 1$.

Then the decoherence factor decays exponentially with t^2, or equivalently the square of the crystal length l^2. We can calculate the decoherence factor and it can be written as

$$\kappa(l) = \exp\left(-\frac{\Delta n^2 \delta^2 l^2}{2c^2} - \frac{\mathrm{i}\Delta n \omega_0 l}{c}\right), \tag{61}$$

where l is the length of the crystal, c is the velocity of light in vacuum. We can check that $|\kappa(l)| = \exp\left(-\frac{\Delta n^2\delta^2 l^2}{2c^2}\right)$ decays exponentially according to l^2. In our experiment, we obtain the value of $\frac{1}{2}\delta^2\Delta n^2 t^2$ for each QP from the data of process tomography.

In principle, we could implement the above process based on the non-Markovian evolution part in figure 1(a) of the main text. However, there is an unpredictable phase ϕ_i between H and V polarized photons in each path, which are not equal to $\Delta n\omega_0 l/c$. To eliminate this phase, we need to tune the phase in the two paths separately. In particular we can place a wave plate in each path to remove the phase, which results in the actual setup in figure 1(a) of the main text.

After the decoherence process, photons in the upper branch are in the state

$$\rho_u = \frac{1}{2}\left[I + x_0\sigma_z + |\kappa(l)|(z_0\sigma_x - y_0\sigma_y)\right], \tag{62}$$

and the state in the lower branch is

$$\rho_l = \frac{1}{2}\left[I - y_0\sigma_z + |\kappa(l)|(z_0\sigma_x - x_0\sigma_y)\right], \tag{63}$$

then the upper branch passes through $h(22.5)$ and is converted to

$$\rho_u' = \frac{1}{2}\left[I + |\kappa(l)|z_0\sigma_z + x_0\sigma_x + |\kappa(l)|y_0\sigma_y\right], \tag{64}$$

while the photons in the lower branch are transformed to

$$\rho_l' = \frac{1}{2}\left[I + |\kappa(l)|z_0\sigma_z + |\kappa(l)|x_0\sigma_x + y_0\sigma_y\right]. \tag{65}$$

The final BS and mirror recombines the two branches and the final state is

$$\rho = \frac{1}{2}\left[I + \kappa(l)z_0\sigma_z\right] + \frac{1}{4}\left[1 + \kappa(l)\right](x_0\sigma_x + y_0\sigma_y), \tag{66}$$

thus we can realize the ENM process with the evolution time corresponding to the length of the crystal l. Since the transmissivity and reflectivity are both about 50%, we get a loss of 1/2 photons when recombining the two branches, resulting in a success probability of around 1/2.

Experimentally, the dynamical behavior of relevant physical quantities can be estimated from the reconstructed density matrix for each evolution time t. For an experimentally reconstructed state ρ_t, the negativity E, mutual information I, and geometric discord D can be evaluated directly using $E(\rho_t) = \frac{\|\rho_t^{T_B}\|_1 - 1}{2}$, $I(\rho_t) = S(\rho_t^A) + S(\rho_t^B) - S(\rho)$, and $D(\rho_t) = \frac{1}{4}(\|\mathbf{x}\|^2 + \|\mathbf{T}\|^2 - \lambda_{\max})$, where $x_i = \mathrm{Tr}(\sigma_i \otimes \mathbb{I})\rho_t$, $T_{ij} = \mathrm{Tr}(\sigma_i \otimes \sigma_j)\rho_t$, and $\lambda_{\max}$ is the largest eigenvalue of the matrix $K = \boldsymbol{x}\boldsymbol{x}^T + TT^T$ [77].

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