BUCKLING OF ELASTICA IN A SHEAR FLOW

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Abstract:
We analyze the three-dimensional buckling of an elastic filament in a viscous fluid under a shear flow with a low Reynolds number and a high Peclet number. We use the Euler-Bernoulli beam (elastica) model [1, 2, 3]

$$\eta\ (2\,\boldsymbol{I} - \boldsymbol{x}_s\boldsymbol{x}_s)\cdot(\boldsymbol{x}_t - \boldsymbol{U}(\boldsymbol{x})) = (T(s,t)\boldsymbol{x}_s)_s - \boldsymbol{x}_{ssss}, \qquad (1)$$

where $s \in \left[-\frac{1}{2},\frac{1}{2}\right]$ is a parameter along the filament, $\boldsymbol{x}(s,t)$ is the position of a filament segment, t is time, $T(s,t)$ is the tension, and η is the elastoviscous number. In our case, the filament is almost straight, $\boldsymbol{x} = s\hat{\boldsymbol{x}}_0(\theta,\phi) + u\hat{\boldsymbol{x}}_{\perp 1} + v\hat{\boldsymbol{x}}_{\perp 2}$, where θ is the zenith angle measured with respect to the direction of vorticity and ϕ is azimuthal angle relative to the direction of the flow. We study small perturbations u (within the shear plane) and v (out of this plane). Assuming that $u(s,t) = \Phi(s)e^{\sigma t}$ one can obtain the elastica eigenproblem

$$\left[-\frac{\partial^4}{\partial s^4} + \tilde{\chi}\left[\frac{1}{4}\left(s^2 - \frac{1}{4}\right)\frac{\partial^2}{\partial s^2} + s\frac{\partial}{\partial s}\right] - \tilde{\chi}\tilde{\sigma}\right]\Phi = 0, \quad \text{where } \tilde{\chi} = -\eta\sin 2\phi\sin^2\theta \ \text{ and } \ \tilde{\sigma} = \frac{-(2\sigma + \sin 2\phi)}{\sin 2\phi\sin^2\theta}. \qquad (2)$$

We use eigenvalues $\tilde{\sigma}$ and eigenfunctions Φ found with the Chebyshev spectral collocation method, which was obtained elsewhere [4] to show that the shape wavenumber depends linearly on $\sqrt{\tilde{\chi}}$. This scaling is the main result of this work.

Our study also shows the universal nature of the full 3D spectral problem for small perturbations u and v of the thin filament from any orientation (θ,ϕ) and explains the relation to the spectral problems solved in [3,5]. Additionally, we provide a simple analytical approximation to the eigenfunctions of the elastica equation (2).

We also perform numerical simulations for a single elastic filament of a non-negligible thickness straight at the elastic equilibrium. A filament is constructed as a chain of identical spherical beads. Its motion in a shear flow is determined with the Hydromultipole code based on the multipole expansion of the Stokes equations [6, 7]. Initially, the filament is straight with a small random perturbation. We show that at early times, the buckled shapes are similar to the eigenfunctions of the elastica equation (2) in a wide range of values of the Young modulus. This comparison is based on computation of the local curvature. Our systematic analysis of the buckled shapes extends the preliminary numerical findings from Ref. [8].

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