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On the incremental plastic work and related aspects of invariance: Part II

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Summary. Presuming that the incremental free energy is invariant under a change of the Lagrangean finite strain measure and/or the reference configuration Hill's transformation rules for the basic quantities occurring in mechanics of elastic–plastic solids are recasted in general 3D situation. On this background the invariant incremental plastic work is defined. The basic connections between Hill–Rice theoretical framework and Eckart–Mandel approach, involving the mobile stress-free configuration, are discussed both in generalized coordinates and in the tensorial notation. To this end the selected fundamentals of solid mechanics including the work-conjugacy are recalled. The structure of the updated Lagrangean plastic increment of the total strain is exhibited accounting for the deformation and stress effects due to possible damage and pressure sensitivity of a solid. Special simple approximate relations are derived for the situations when non-dilatational elastic strains are small. The merits of using the logarithmic elastic strain as a state variable are also discussed.

1 Notations

$\boldsymbol{1} \leftrightarrow \delta_{KL}$, $(\boldsymbol{I})_{KLMN} = 0.5(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM})$

$(\overset{T}{\boldsymbol{A}})_{KL} = A_{LK}$ or $(\overset{T}{\boldsymbol{A}})_{KLMN} = A_{MNKL}$ − transpose of $\boldsymbol{A}$

$\overset{-1}{\boldsymbol{A}}$ inverse of $\boldsymbol{A}$ $(\overset{-1}{A}_{KL}A_{LM} = \delta_{KM}$ or $A_{KLMN}\overset{-1}{A}_{MNPQ} = I_{KLPQ})$, $\quad \overset{-T}{\boldsymbol{A}} = (\overset{-1}{\boldsymbol{A}})^{T} = (\overset{T}{\boldsymbol{A}})^{-1}$

$\boldsymbol{AB} \rightarrow A_{KL}B_{LM}$ or $A_{KLMN}B_{MNPQ}$, $\quad \boldsymbol{A} \otimes \boldsymbol{B} \rightarrow A_K B_L$

$\boldsymbol{A} \cdot \boldsymbol{B} \rightarrow A_K B_K$ or $A_{KL}B_{KL} = \mathrm{tr}\,(\boldsymbol{A}\overset{T}{\boldsymbol{B}})$

$(!K) \rightarrow$ no summation over index K

$\boldsymbol{A}[|\boldsymbol{B}, \boldsymbol{C}|] \rightarrow A_{KLMNPQ}\,B_{MN}\,C_{PQ}$

$\boldsymbol{A}[|., \boldsymbol{C}|] \rightarrow A_{KLMNPQ}\,C_{PQ}$

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$\boldsymbol{A}[|\boldsymbol{C},.|] \rightarrow A_{KLMNPQ}\, C_{MN}$

Two special isotropic fourth-order tensor functions $\boldsymbol{P}(\boldsymbol{A})$ and $\boldsymbol{\Theta}(\boldsymbol{X})$ of a second-order argument $\boldsymbol{A}$ and $\boldsymbol{X}$ (non-singular), respectively, are used throughout the paper:

$$2P_{KLMN}(\boldsymbol{A}) = A_{KM}\delta_{LN} + A_{KN}\delta_{LM} + A_{LM}\delta_{KN} + A_{LN}\delta_{KM}, \tag{1}$$

$$2\Theta_{KLMN}(\boldsymbol{X}) = (X_{KM}X_{LN} + X_{KN}X_{LM}). \tag{2}$$

Let $\boldsymbol{B}$ be the arbitrary symmetric tensor. The basic properties of the operators $\boldsymbol{P}$ and $\boldsymbol{\Theta}$ are

$$\boldsymbol{P}(\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}\boldsymbol{B} + \boldsymbol{B}\overset{T}{\boldsymbol{A}}, \quad \boldsymbol{\Theta}(\boldsymbol{X})\boldsymbol{B} = \boldsymbol{X}\boldsymbol{B}\boldsymbol{X}^{T}, \tag{3.1,2}$$

$$\overset{T}{\boldsymbol{P}}(\boldsymbol{A})\boldsymbol{B} = \boldsymbol{P}(\overset{T}{\boldsymbol{A}})\boldsymbol{B} = \overset{T}{\boldsymbol{A}}\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}, \quad \boldsymbol{P}(\boldsymbol{1}) = 2\boldsymbol{I}, \tag{3.3,4}$$

$$\overset{T}{\boldsymbol{\Theta}}(\boldsymbol{X}) = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{X}}), \quad \overset{-1}{\boldsymbol{\Theta}}(\boldsymbol{X}) = \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{X}}), \quad \boldsymbol{\Theta}(\boldsymbol{1}) = \boldsymbol{I}, \tag{3.5-7}$$

$$\boldsymbol{\Theta}(\boldsymbol{X}\boldsymbol{Y}) = \boldsymbol{\Theta}(\boldsymbol{X})\boldsymbol{\Theta}(\boldsymbol{Y}), \tag{3.8}$$

$$\boldsymbol{\Theta}(\boldsymbol{X})\boldsymbol{P}(\boldsymbol{X}) = \boldsymbol{P}(\boldsymbol{X})\boldsymbol{\Theta}(\boldsymbol{X}), \quad \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{X}})\boldsymbol{P}(\boldsymbol{X}) = \boldsymbol{P}(\boldsymbol{X})\boldsymbol{\Theta}(\overset{-1}{\boldsymbol{X}}) = \boldsymbol{P}(\overset{-1}{\boldsymbol{X}}), \tag{3.9,10}$$

$$\boldsymbol{\Theta}(\boldsymbol{X}) \quad \overbrace{\boldsymbol{\Theta}(\overset{-1}{\boldsymbol{X}})}^{\cdot} = \boldsymbol{P}(\overset{\cdot}{\boldsymbol{X}\overset{-1}{\boldsymbol{X}}}) = -\boldsymbol{P}(\dot{\boldsymbol{X}}\overset{-1}{\boldsymbol{X}}). \tag{3.11}$$

2 Introduction

Suppose that two observers adopt different Lagrangean strain measure and/or different reference configuration to describe the elastic–plastic behavior of the same real material. The mathematical form of the obtained constitutive equations will be in general different, and the natural question arises: what are the transformation rules linking the particular constitutive objects and what quantities and concepts are invariants. The foundations for systematic analysis of this type of invariance in Lagrangean solid mechanics has been developed by Hill in [1] and later extended to thermoelasticity in [2]. The brief discussion of the invariance of thermal effects in thermoplasticity can also be found in [3] including the partially integrated thermodynamic potentials. The present paper is the companion to [4] where the basic aspects of invariance are discussed on the example of simple tension. Here a similar analysis is performed in 3D situation including the original elements: definition of invariant plastic work within Hill–Rice (H–R) theoretical framework, general transformation formulas from H–R to Eckart–Mandel (E–M) approach involving concept of stress-free configuration and accounting for deformation effects due to the possible damage and pressure-sensitivity. To exhibit the structure of updated plastic increment of the total strain in the actual configuration special attention is drawn to the most frequent situation encountered in practice – i.e., small elastic distortions.

To make the paper self-contained we present in Sect. 3.1 our own synthesis of the kinematics and work-conjugacy including a special class of confined deformations. This supports the illustrative examples of transformation rules written in the tensorial notation in Sects. 3.5 and 4.2–4.4. The concise discussion of basic elements of H–R theoretical framework and the subtle aspects of invariance, the definition of invariant incremental plastic work, as well as the transformation of the incremental equations into the mobile Lagrangean stress free configuration (E–M approach) are presented in terms of generalized coordinates in Sects. 3.2–3.4 and 4.1. The structure of the rate form of the state equations in updated Lagrangean configuration is discussed in Sect. 4.4.

3 Selected aspects of Hill–Rice theoretical framework

3.1 Preliminaries

a. Strain measures. Selected aspects of kinematics

(i) The very concept of strain involves two arbitrary factors: the strain measure and the reference configuration that can be chosen at will. We shall here restrict the attention to the popular class of strain measures defined by

$$\boldsymbol{E}(n) = \boldsymbol{f}(\boldsymbol{U}) = \frac{1}{2n}(\boldsymbol{U}^{2n} - 1) = \sum_{K=1}^{3} \bar{E}_{K}(n)\boldsymbol{N}_{K} \otimes \boldsymbol{N}_{K}, \quad \bar{E}_{K}(n) = \frac{(\lambda_{K}^{2n} - 1)}{2n}, \tag{4.1,2}$$

where $\boldsymbol{N}_{K}(K = 1, 2, 3)$ are unit vectors (Lagrangean triad) along principal directions of the right stretch tensor $\boldsymbol{U}$ occurring in the polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ of the deformation tensor $\boldsymbol{F}$ ($\boldsymbol{R}$ is the polar rotation, $\boldsymbol{R}\overset{T}{\boldsymbol{R}} = \boldsymbol{1}$). The symbols λ_{K} and $\bar{E}_{K}(n)$ $(K = 1, 2, 3)$ denote principal stretches and principal values of the strain tensor $\boldsymbol{E}(n)$. Henceforth, the over barred array symbols will denote the components of any tensor on the principal triad $\boldsymbol{N}_{K}$. The parameter n occurring in the bracket of $\boldsymbol{E}(n)$ represents the chosen strain measure $\boldsymbol{f}$ which is selected to be coaxial with $\boldsymbol{U}$ (n is a real number).

By substituting $\boldsymbol{U} = J^{1/3}\bar{\boldsymbol{U}}$ (where $J = \det\boldsymbol{F}$) into (4.1) one arrives at the following well-known dilatational-strain distortion decomposition of $\boldsymbol{E}(n)$ (cf., e.g., [5], [6]):

$$\begin{aligned} &\boldsymbol{E}(n) = \boldsymbol{E}_{v}(n) + J^{2n/3}\boldsymbol{E}_{D}(n), \\ &\boldsymbol{E}_{v}(n) = \frac{1}{2n}(J^{2n/3} - 1)\boldsymbol{1}\,, \quad \boldsymbol{E}_{D}(n) = \frac{1}{2n}(J^{-2n/3}\boldsymbol{U}^{2n} - \boldsymbol{1}), \end{aligned} \tag{5}$$

where $\boldsymbol{E}_{D}(n)$ is an appropriate measure of strain distortions. The knowledge of $\boldsymbol{E}_{D}(n)$ alone suffices to calculate the variation of angles between two material fibers in the course of straining. However, it is not sufficient to determine the dilatational changes. The standard additive decomposition of $\boldsymbol{E}$ into the spherical and deviatoric parts has a similar physical meaning only when the logarithmic strain measure is employed ($n = 0$),

$$\lim_{n\to 0}\boldsymbol{E}_{v}(n) \equiv \boldsymbol{E}_{v}(0) = (1/3)(\ln J)\boldsymbol{1}, \quad \lim_{n\to 0} J^{2n/3}\boldsymbol{E}_{D}(n) \equiv \boldsymbol{E}^{d}(0) = \mathrm{dev}\boldsymbol{E}(0). \tag{6}$$

The ratios λ_{L}/λ_{K} are expressible in the elementary manner in terms of principal components $\bar{E}_{K}^{d}(0)$ of $\boldsymbol{E}^{d}(0)$, $\lambda_{L}/\lambda_{K} = \exp[E_{L}^{d}(0) - E_{K}^{d}(0)]$.

(ii) The incremental variation of strain is $d\boldsymbol{E}(n) = \dot{\boldsymbol{E}}(n)dt$, where $\dot{\boldsymbol{E}}(n)$ is the time derivative of Lagrangean strain $\boldsymbol{E}(n)$ (direct strain flux [1]). Denote by $\boldsymbol{D}^{R}$ the Lagrangean strain-rate induced from the Eulerian strain-rate $\boldsymbol{D}$ (stretching) by the polar rotation $\boldsymbol{R}$,

$$\boldsymbol{D}^{R} \equiv \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}})\boldsymbol{D}, \quad 2\boldsymbol{D} = \boldsymbol{L} + \overset{T}{\boldsymbol{L}}, \quad \boldsymbol{L} = \dot{\boldsymbol{F}}\overset{-1}{\boldsymbol{F}}, \tag{7}$$

where $\boldsymbol{\Theta}$ is defined in (2), and suppose that at the generic instant t of a process of deformation the current shape of the material element (m.e.) is known (i.e., known is $\boldsymbol{U}$ or $\boldsymbol{E}(n)$). The important question arises: what is the *kinematic relation* between two different incremental variations of strains $\dot{\boldsymbol{E}}(n)dt$ and $\boldsymbol{D}^{R}dt$?

The answer may be found in Hill [1] (see also [7, Chap. 2]):

$$\dot{\bar{E}}_{KL}(n) = \bar{\boldsymbol{\mathcal{E}}}^{(n)}_{KLMN}\bar{D}_{MN} \equiv \begin{cases} \lambda_{K}^{2n}\bar{D}_{KK} & \text{for } K = L, \quad (!K) \\ \dfrac{\lambda_{K}\lambda_{L}(\lambda_{K}^{2n} - \lambda_{L}^{2n})}{n(\lambda_{K}^{2} - \lambda_{L}^{2})}\bar{D}_{KL} & \text{for } K \neq L, \quad (!K, L), \end{cases} \tag{8}$$

where $\bar{\dot{E}}_{KL}(n)$ and $\bar{D}_{KL}$ are the components[1] of $\dot{\boldsymbol{E}}(n)$ and $\boldsymbol{D}^R$ on the Lagrangean triad $\boldsymbol{N}_K$. The array $\bar{\boldsymbol{\mathcal{E}}}^{(n)}_{KLMN}$ represents the components (on the principal triad) of the family (parameter n) of fourth-order pairwise symmetric tensor functions $\boldsymbol{\mathcal{E}}^{(n)}(\boldsymbol{U})$ defined by (8)

$$\dot{\boldsymbol{E}}(n) = \boldsymbol{\mathcal{E}}^{(n)}(\boldsymbol{U})\boldsymbol{D}^R, \quad \boldsymbol{D}^R = \overset{-1}{\boldsymbol{\mathcal{E}}}{}^{(n)}\dot{\boldsymbol{E}}(n). \tag{9}$$

Equation (8) is valid also in the limit when two different principal stretches become equal. If $\lambda_K \to \lambda_L$ the limiting relation is

$$\bar{\dot{E}}_{KL}(n) = \lambda_K^{2n}\bar{D}_{KL} \quad (!K).$$

It can readily be verified that $\overset{-1}{\boldsymbol{\mathcal{E}}}{}^{(1/2)} = 0.5\boldsymbol{P}(\overset{-1}{\boldsymbol{U}}), \boldsymbol{\mathcal{E}}^{(1)} = \boldsymbol{\Theta}(\boldsymbol{U})$ [cf. (1)–(3)] and $\boldsymbol{\mathcal{E}}^{(-1)} = \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{U}})$, $\overset{-1}{\boldsymbol{\mathcal{E}}}{}^{(-1)} = \boldsymbol{\mathcal{E}}^{(1)}$. Hence, Eq. (8) includes the familiar index-free relations between $\boldsymbol{D}^R$ and $\dot{\boldsymbol{E}}(n)$ for $n = 1$ (Green measure), $n = -1$ (Almansi measure) and for the stretch strain measures $n = \pm 1/2$,

$$\boldsymbol{D}^R = \boldsymbol{\Theta}(\overset{\mp 1}{\boldsymbol{U}})\dot{\boldsymbol{E}}(\pm 1) \Rightarrow \boldsymbol{D} = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\dot{\boldsymbol{E}}(1) = \boldsymbol{\Theta}(\boldsymbol{F})\dot{\boldsymbol{E}}(-1), \tag{10.1}$$

$$2\boldsymbol{D}^R = \boldsymbol{P}(\overset{\mp 1}{\boldsymbol{U}})\dot{\boldsymbol{E}}(\pm 1/2). \tag{10.2}$$

To find the index-free form of the inverse of (10.2) denote by $\boldsymbol{\omega} = 0.5(\boldsymbol{L} - \overset{T}{\boldsymbol{L}})$ the usual Eulerian body spin and by $\Delta\boldsymbol{\omega} = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}})\Delta\boldsymbol{\omega}^R$ the Lagrangean measure of the difference between the body spin and the polar spin $\boldsymbol{\omega}^R = \dot{\boldsymbol{R}}\overset{T}{\boldsymbol{R}}$, $\Delta\boldsymbol{\omega}^R = \boldsymbol{\omega} - \boldsymbol{\omega}^R$. It can be shown that $\Delta\boldsymbol{\omega}$ is the following singularity-free function of $\boldsymbol{U}$ and $\boldsymbol{D}^R$:

$$\Delta\boldsymbol{\omega} = \frac{1}{\det \boldsymbol{G}}\boldsymbol{G}(\boldsymbol{U})\big(\boldsymbol{U}\boldsymbol{D}^R_d - \boldsymbol{D}^R_d\boldsymbol{U}\big)\boldsymbol{G}(\boldsymbol{U}), \quad \boldsymbol{G}(\boldsymbol{U}) \equiv (\operatorname{tr}\boldsymbol{U})\boldsymbol{1} - \boldsymbol{U}, \tag{11}$$

where $\boldsymbol{D}^R_d = \operatorname{dev}\boldsymbol{D}^R$. It is the solution of the algebraic equation

$$\Delta\boldsymbol{\omega}\boldsymbol{U} + \boldsymbol{U}\Delta\boldsymbol{\omega} = \boldsymbol{U}\boldsymbol{D}^R_d - \boldsymbol{D}^R_d\boldsymbol{U}. \tag{12}$$

The alternative mathematical forms of (11) can be found, e.g., in [8], [9]. Once $\Delta\boldsymbol{\omega}$ is a known function of $\boldsymbol{D}^R$ the strain rates $\dot{\boldsymbol{E}}(\pm 1/2)$ may be expressed in terms of $\boldsymbol{D}^R$ by substituting (11) into the following formula:

$$\dot{\boldsymbol{E}}(\pm 1/2) = \boldsymbol{P}(\overset{\pm 1}{\boldsymbol{U}})\boldsymbol{D}^R \pm \Big[\Delta\boldsymbol{\omega}\,\overset{\pm 1}{\boldsymbol{U}} - \overset{\pm 1}{\boldsymbol{U}}\,\Delta\boldsymbol{\omega}\Big]. \tag{13}$$

When $2n$ is an integer ($n \neq 0$) the more complex index-free form of $\boldsymbol{\mathcal{E}}^{(n)}$ may be found by using the relation $\dot{\boldsymbol{E}}(1/2) \to \dot{\boldsymbol{E}}(n)$ presented in [7, Chap. 3]. We shall not pursue this matter further here. It is noted instead that the diagonal components of the tensor expressions $\boldsymbol{E}(n)\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{E}(n)$ on the principal triad $\boldsymbol{N}_K$ vanish for an arbitrary skew-symmetric tensor $\boldsymbol{\Omega}^*$. Hence the normal components $\bar{\dot{E}}_{KK}(1/2) = \bar{\dot{U}}_{KK}$ $(K!)$ on the principal triad $\boldsymbol{N}_K$ are equal to $\dot{\lambda}_K, \bar{\dot{U}}_{KK} = \dot{\lambda}_K$ $(K!)$, and (13) implies $\bar{D}_{KK} = \dot{\lambda}_K/\lambda_K$ $(K!)$ that holds "regardless of rotational history" [1], [9]. The indicated property of the above tensor expression enables also to present formally the kinematic relation (8) in the new form

$$\dot{\boldsymbol{E}}(n) + \boldsymbol{E}(n)\boldsymbol{\Omega}^{(n)} - \boldsymbol{\Omega}^{(n)}\boldsymbol{E}(n) = \boldsymbol{\Theta}(\boldsymbol{U}^n)\boldsymbol{D}^R, \quad \operatorname{tr}\dot{\boldsymbol{E}}(n) = \operatorname{tr}(\boldsymbol{U}^{2n}\boldsymbol{D}^R), \tag{14}$$

where $\boldsymbol{\Omega}^{(n)}$ is the skew-symmetric function of $\boldsymbol{U}$ and $\boldsymbol{D}^R$ (*linear in* $\boldsymbol{D}^R$). The non-zero components of $\boldsymbol{\Omega}^{(n)}$ on principal triad $\boldsymbol{N}_K$ are ($K \neq L$)

[1] The arrays $\bar{E}_{KL}$ and $\bar{D}_{KL}$ are also the components of the Eulerian tensors $\boldsymbol{e}(n) = \boldsymbol{\Theta}(\boldsymbol{R})\boldsymbol{E}(n)$ and $\boldsymbol{D}$ on the Eulerian triad $\boldsymbol{n}_K = \boldsymbol{R}\boldsymbol{N}_K$ representing the principal directions of the left stretch tensor $\boldsymbol{V}$ ($\boldsymbol{F} = \boldsymbol{V}\,\boldsymbol{R}$).

$$\bar{\Omega}^{(n)}_{LK} = \bar{\Omega}_{LK} - \frac{2n\lambda_L^n\lambda_K^n}{(\lambda_K^{2n} - \lambda_L^{2n})}\bar{D}_{KL}, \quad \bar{\Omega}_{LK} \equiv \frac{2\lambda_L\lambda_K}{(\lambda_K^2 - \lambda_L^2)}\bar{D}_{LK} \quad (!K, L) \tag{15}$$

such that $\boldsymbol{\Omega}^{(\pm 1)} = \boldsymbol{0}$ and $\lim_{\lambda_K \to \lambda_L} \bar{\Omega}^{(n)}_{LK} = 0$. The array $\bar{\Omega}_{LK}$ represents the spin $\dot{\boldsymbol{N}}_K \otimes \boldsymbol{N}_K$ of the Lagrangean triad, well determined provided that $\lambda_K \neq \lambda_L$ [1].

The Eulerian counterpart of (14) is

$$\dot{\boldsymbol{e}}(n) + \boldsymbol{e}(n)\boldsymbol{\omega}^{(n)} - \boldsymbol{\omega}^{(n)}\boldsymbol{e}(n) = \boldsymbol{\Theta}(\boldsymbol{V}^n)\boldsymbol{D}, \tag{16}$$

where the new family of *"n-strain spins"* $\boldsymbol{\omega}^{(n)}$ is defined by

$$\boldsymbol{\omega}^{(n)} = \boldsymbol{\omega} + \boldsymbol{\Theta}(\boldsymbol{R})[\boldsymbol{\Omega}^{(n)} - \Delta\boldsymbol{\omega}] \tag{17}$$

and the components on the principal triad of $\Delta\boldsymbol{\omega}$ [cf. (11)] are $\Delta\bar{\omega}_{LK} = (\lambda_L - \lambda_K)\bar{D}_{LK}/(\lambda_L + \lambda_K)$. Since the components of $\boldsymbol{\Omega}^{(n)}$ and $\boldsymbol{\omega}^{(n)}$ depend only on ratios of principal stretches the skew-symmetric tensors $\boldsymbol{\Omega}^{(n)}$ and $\boldsymbol{\omega}^{(n)}$ are unaffected by the dilatational part of the deformation. Because of $\boldsymbol{\Omega}^{(\pm 1)} = \boldsymbol{0}$ the combination of the Eulerian counterpart of (11) with (16) leads to the index-free form of the relations between $\boldsymbol{D}$ and the Zaremba–Jaumann derivatives of $\boldsymbol{e}(\pm 1)$. They will not be presented here. In the limit $n \to 0$ one arrives at the Lagrangean $\boldsymbol{\Omega}^{(0)}$ and Eulerian $\boldsymbol{\omega}^{(0)}$ "logarithmic" spins. The components of $\boldsymbol{\Omega}^{(0)}$ on $\boldsymbol{N}_K$ are ($L \neq K$)

$$\bar{\Omega}^{(0)}_{LK} = \left[\frac{2\lambda_L\lambda_K}{\lambda_K^2 - \lambda_L^2} - \frac{1}{\ln(\lambda_K/\lambda_L)}\right]\bar{D}_{KL} \quad (!K, L), \tag{18}$$

and the components of $\boldsymbol{\omega}^{(0)}$ can be found from (17),

$$\bar{\omega}^{(0)}_{LK} = \bar{\omega}_{LK} + \left[\frac{\lambda_K^2 + \lambda_L^2}{\lambda_K^2 - \lambda_L^2} - \frac{1}{\ln(\lambda_K/\lambda_L)}\right]\bar{D}_{LK} \quad (!K, L), \tag{19}$$

where $\bar{\omega}_{LK}$ are the components of the body spin $\boldsymbol{\omega}$ on the Eulerian triad $\boldsymbol{n}_K$.

The flux of any symmetric Eulerian tensor $\boldsymbol{X}$ defined by

$$\frac{\mathcal{D}^{\log}\boldsymbol{X}}{\mathcal{D}t} \equiv \dot{\boldsymbol{X}} + \boldsymbol{X}\boldsymbol{\omega}^{(0)} - \boldsymbol{\omega}^{(0)}\boldsymbol{X} \tag{20}$$

is called "logarithmic corotational rate". It is the linear function of $\boldsymbol{D}$, which in general depends on the selected reference configuration, except for $\boldsymbol{X} = \boldsymbol{e}(0)$ since $\mathcal{D}^{\log}\boldsymbol{e}(0)/\mathcal{D}t = \boldsymbol{D}$. This notion together with the notion of logarithmic spin $\boldsymbol{\omega}^{(0)}$ were introduced (by using different mathematical routes) and advanced in numerous papers by Xiao et al. [10]–[12] (see also [13]).

When the principal components $\bar{E}^d_K(0)$ of $\boldsymbol{E}^d$ (0) are small we have the following approximations of λ_L/λ_K and the function $g(x)$ occurring in (18):

$$\begin{aligned} &\frac{\lambda_L}{\lambda_K} - 1 = \bar{E}^d_L(0) - \bar{E}^d_K(0) + \cdots, \\ &g(x) = \frac{2x}{1 - x^2} + \frac{1}{\ln x} = \frac{1}{6}(x - 1) + \cdots, \end{aligned} \tag{21}$$

where $x - 1$ represents the expression $(\lambda_L/\lambda_K) - 1$. The expansion of $\bar{\Omega}^{(0)}_{LK}$ and $\bar{\omega}^{(0)}_{LK}$ of such type leads to the following approximate index-free forms of the expressions defining the logarithmic spins

$$\begin{aligned} &\boldsymbol{\Omega}^{(0)} = \frac{1}{6}\left[\boldsymbol{E}^d(0)\boldsymbol{D}^R_d - \boldsymbol{D}^R_d\boldsymbol{E}^d(0)\right] + \boldsymbol{\mathcal{O}}\Big(\left[\boldsymbol{E}^d(0)\right]^2\Big)\boldsymbol{D}^R_d, \\ &\boldsymbol{\omega} - \boldsymbol{\omega}^{(0)} = \frac{1}{3}\left[\boldsymbol{e}^d(0)\boldsymbol{D}_d - \boldsymbol{D}_d\boldsymbol{e}^d(0)\right] + \boldsymbol{\mathcal{O}}\Big(\left[\boldsymbol{e}^d(0)\right]^2\Big)\boldsymbol{D}_d, \end{aligned} \tag{22}$$

where $\boldsymbol{D}_d = \text{dev}\,\boldsymbol{D}$.

Hence

$$\dot{\boldsymbol{E}}(0) = \boldsymbol{D}^R + \boldsymbol{\mathcal{O}}([\boldsymbol{E}^d(0)]^2)\boldsymbol{D}^R_d, \tag{23.1}$$

$$\frac{\mathcal{D}\boldsymbol{e}(0)}{\mathcal{D}t} = \dot{\boldsymbol{e}}(0) + \boldsymbol{e}(0)\,\boldsymbol{\omega} - \boldsymbol{\omega}\,\boldsymbol{e}(0) = \boldsymbol{D} + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{e}^d(0)]^2\Big)\boldsymbol{D}^R \tag{23.2}$$

on account of (14) and (16) specified for $n = 0$. Here $\boldsymbol{\mathcal{O}}$ is the usual order symbol, and the Zaremba–Jaumann derivative is denoted by $\mathcal{D}/\mathcal{D}t$. The approximation of $\dot{\boldsymbol{E}}(0)$ by $\boldsymbol{D}^R$, and $\mathcal{D}\boldsymbol{e}(0)/\mathcal{D}t$ by $\boldsymbol{D}$ is close even for the moderate strain distortions [1].

(iii) Let us also recall two basic issues of the kinematics of embedded basis [14], [1]. Having known all components (contravariant, covariant and mixed) of any tensor $\boldsymbol{K}$ on the deformed embedded basis one can define four Lagrangean tensors by combining those components with the Lagrangean reference basis [1]. The two induced by deformation tensors $\boldsymbol{K}^{(I)} = \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{F}})\boldsymbol{K}$ and $\boldsymbol{K}^{(II)} = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{K}$ are symmetric[2] and their rates define the contravariant $\delta^C\boldsymbol{K}/\delta t$ and covariant $\delta_c\boldsymbol{K}/\delta t$ time derivatives of the Eulerian tensor $\boldsymbol{K}$,

$$\begin{aligned}
&\boldsymbol{\Theta}(\boldsymbol{F})\dot{\boldsymbol{K}}^{(I)} = \boldsymbol{\Theta}(\boldsymbol{F})\dot{\overbrace{\boldsymbol{\Theta}(\overset{-1}{\boldsymbol{F}})\boldsymbol{K}}} = \frac{\delta^C\boldsymbol{K}}{\delta t} = \dot{\boldsymbol{K}} - \boldsymbol{L}\boldsymbol{K} - \boldsymbol{K}(\overset{T}{\boldsymbol{L}}),\\
&\boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\dot{\boldsymbol{K}}^{(II)} = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\dot{\overbrace{\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{K}}} = \frac{\delta_c\boldsymbol{K}}{\delta t} = \dot{\boldsymbol{K}} + \overset{T}{\boldsymbol{L}}\boldsymbol{K} + \boldsymbol{K}\boldsymbol{L},
\end{aligned} \tag{24}$$

where $\boldsymbol{L} = \dot{\boldsymbol{F}}\overset{-1}{\boldsymbol{F}}$ is the usual deformation-rate tensor. The relations (24) may be verified, of course, by the direct time differentiation of $\boldsymbol{K}^{(I)}$ and $\boldsymbol{K}^{(II)}$ or by using the join property (3.9, 10) of the operators $\boldsymbol{\Theta}$ and $\boldsymbol{P}$. They hold also when $\boldsymbol{F}$ is an arbitrary orthogonal tensor, say $\boldsymbol{F} = \boldsymbol{\mathcal{R}}$ where $\overset{T}{\boldsymbol{\mathcal{R}}} = \overset{-1}{\boldsymbol{\mathcal{R}}}$. Then $\boldsymbol{L}$ becomes the associated spin $\boldsymbol{L} = \dot{\boldsymbol{\mathcal{R}}}\overset{T}{\boldsymbol{\mathcal{R}}}$. For $\boldsymbol{K} = \boldsymbol{1}$ the identity (24) reduces to (10.1).

b. Work-conjugate stress measures and their rates

(i) Let ϱ^R and ϱ be the mass densities of m.e. in a reference and the actual configuration, respectively. Denote by $\boldsymbol{\sigma}$ the Cauchy's stress and by $\boldsymbol{\tau} = \boldsymbol{\sigma}/\varrho$ the specific Kirchhoff stress. It differs from the familiar Kirchhoff stress $\varrho^R\boldsymbol{\tau}$ by a not important scalar multiplier ϱ^R. Use of $\boldsymbol{\tau}$ simplifies some transformation formulas. Moreover, the tensor $\boldsymbol{\tau}$ does not require knowledge of any reference volume of m.e. The incremental specific work dW (per unit of mass) is used to define the specific stress $\boldsymbol{T}\,(n)$ work-conjugate *to the selected strain measure* $\boldsymbol{E}\,(n)$[3],

$$dW = \boldsymbol{\tau}\cdot\boldsymbol{D}\,dt = \boldsymbol{\tau}^R\cdot\boldsymbol{D}^R\,dt = \boldsymbol{T}(n)\cdot d\boldsymbol{E}(n) = \boldsymbol{T}(n)\cdot\dot{\boldsymbol{E}}(n)\,dt. \tag{25}$$

The stress tensor $\boldsymbol{\tau}^R \equiv \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}})\,\boldsymbol{\tau}$ is induced from $\boldsymbol{\tau}$ by the polar rotation $\boldsymbol{R}$. Therefore, its representation $\overline{\tau}_{KL}$ on the Lagrangean triad $\boldsymbol{N}_K$ is the same as the representation of $\boldsymbol{\tau}$ on the Eulerian triad $\boldsymbol{n}_K$,

$$\boldsymbol{\tau}^R = \sum_{L,K=1}^{3}\overline{\tau}_{KL}\boldsymbol{N}_K\otimes\boldsymbol{N}_L, \quad \boldsymbol{\tau} = \sum_{L,K=1}^{3}\overline{\tau}_{KL}\boldsymbol{n}_K\otimes\boldsymbol{n}_L. \tag{26}$$

The general connection between $\boldsymbol{T}(n)$ and $\boldsymbol{\tau}^R$ is found by substituting (9) into (25). Since the resulting equality must hold for arbitrary $\boldsymbol{D}^R$ we have

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[2] Some authors (see, e.g., [15]) employ the terminology "pull-back" and "push-forward" to the operations $\{\boldsymbol{\Theta}(\overset{-1}{\boldsymbol{F}})$ or $\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\}$ and their inverses, respectively. The time derivatives $\delta^C/\delta t$ and $\delta_c/\delta t$ are frequently called "Lie derivatives".

[3] Hill employs the usual density-type work-conjugate $\varrho^R\boldsymbol{T}$ per unit of reference volume [1].

$$\boldsymbol{T}(n) = \overset{-1}{\boldsymbol{\mathcal{E}}}{}^{(n)}\boldsymbol{\tau}^R, \quad \boldsymbol{\tau}^R = \boldsymbol{\mathcal{E}}^{(n)}\boldsymbol{T}(n). \tag{27}$$

Hence, to get more specific relations it is sufficient to replace $\dot{\boldsymbol{E}}(n)$ with $\boldsymbol{\tau}^R$, and $\boldsymbol{D}^R$ with $\boldsymbol{T}(n)$, in all relevant equations of Sect. 3.1a. In particular, the counterparts of (10) provide the familiar definitions of the specific second Piola–Kirchhoff stress ($n = 1$), Almansi stress ($n = -1$) and Biot stress ($n = 1/2$),

$$\boldsymbol{T}(\pm 1) = \boldsymbol{\Theta}(\overset{\mp 1}{\boldsymbol{U}})\boldsymbol{\tau}^R \Rightarrow \boldsymbol{\tau} = \boldsymbol{\Theta}(\boldsymbol{F})\boldsymbol{T}(1) = \Theta(\overset{-T}{\boldsymbol{F}})\boldsymbol{T}(-1), \tag{28.1}$$

$$2\boldsymbol{T}(\pm 1/2) = \boldsymbol{P}(\overset{\mp 1}{\boldsymbol{U}})\boldsymbol{\tau}^R = \boldsymbol{P}(\overset{\pm 1}{\boldsymbol{U}})\boldsymbol{T}(\pm 1). \tag{28.2}$$

The index-free form of the inverse $\boldsymbol{T}(\pm 1/2) \to \boldsymbol{\tau}^R$ can be found in a similar manner as the inverse of (10.2). The counterpart of (23.1) is the following approximation of $\boldsymbol{T}(0)$ by $\boldsymbol{\tau}^R$ [1]:

$$\boldsymbol{T}(0) = \boldsymbol{\tau}^R + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^2\Big)\boldsymbol{\tau}^R_d \quad (\boldsymbol{\tau}^R_d = \operatorname{dev}\boldsymbol{\tau}^R). \tag{29}$$

More careful analysis of the relation (27) [cf. also (8)] shows that the principal directions of $\boldsymbol{\tau}^R$ and $\boldsymbol{\tau}$ are $\boldsymbol{N}_K$ and $\boldsymbol{n}_K$, respectively, provided that the principal directions of $\boldsymbol{T}(n)$ coincide with the Lagrangean triad ($\bar{T}_{KL} = 0$ for $K \neq L$).

This property holds perpetually during deformations of an isotropic solid. Thus, for isotropic solids the tensor $\boldsymbol{\tau}^R$ may be regarded as the stress conjugate to the logarithmic strain measure. A more detailed mathematical analysis of the work-conjugacy with $\boldsymbol{E}(0)$ can be found in [16]–[18].

(ii) The material derivative $\dot{\boldsymbol{T}}(n)$ is the direct flux of the conjugate $\boldsymbol{T}(n)$. Its connection with $\dot{\boldsymbol{\tau}}^R$ depends on strain-rate and is not simple. The time differentiation of the relation such as (27) shows that $\dot{\boldsymbol{T}}(n)$ is a linear function of $\dot{\boldsymbol{\tau}}^R$ and $\dot{\boldsymbol{E}}(n)$ which depends also on the current strain state. The general explicit index free form of this relation is also not available. The algorithm of the derivation of the component-wise connection on the principal triad can be found in [1]. Here we shall single out two known basic relations for $n = 1$ and $n = -1$. Denote by $\mathcal{D}^{(n)}\boldsymbol{\sigma}/\mathcal{D}t$ the following family (parameter n) of Eulerian stress fluxes[4] (cf. end of Sect. 3.1.c)

$$\frac{\mathcal{D}^{(n)}\boldsymbol{\sigma}}{\mathcal{D}t} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\sigma} - \frac{\dot{\varrho}}{\varrho}\boldsymbol{\sigma} - n\boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{D} = \varrho\,[\,\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\omega}\boldsymbol{\tau} - n\boldsymbol{P}(\boldsymbol{\tau})\boldsymbol{D}\,]. \tag{30.1,2}$$

The familiar connections between the rates $\dot{\boldsymbol{T}}(\pm 1)$ (specific second Piola–Kirchhoff $\boldsymbol{T}(1)$ and Almansi $\boldsymbol{T}(-1)$ stress tensors) and stress-flux $\mathcal{D}^{(\pm 1)}\boldsymbol{\sigma}/\mathcal{D}t$ can be found by substituting $\boldsymbol{K} = \boldsymbol{\tau}$ into (24) or by calculation the time derivative of (28.1)

$$\boldsymbol{\Theta}(\boldsymbol{F})\dot{\boldsymbol{T}}(1) = \frac{\delta^C\boldsymbol{\tau}}{\delta t} = \frac{1}{\varrho}\frac{\mathcal{D}^{(1)}\boldsymbol{\sigma}}{\mathcal{D}t}, \quad \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\dot{\boldsymbol{T}}(-1) = \frac{\delta_c\boldsymbol{\tau}}{\delta t} = \frac{1}{\varrho}\frac{\mathcal{D}^{(-1)}\boldsymbol{\sigma}}{\mathcal{D}t}. \tag{31}$$

c. Special cases of confined deformations

(i) The simple explicit index-free form of the basic connections $\dot{\boldsymbol{E}}(n) \Leftrightarrow \boldsymbol{D}^R$, $\boldsymbol{T}(n) \Leftrightarrow \boldsymbol{\tau}^R$, $\dot{\boldsymbol{\tau}}^R \Leftrightarrow \dot{\boldsymbol{T}}(n)$ can be found for special classes of restrictive strains. Consider first the situation when strain distortion represented by the deviatoric part $\boldsymbol{E}^d(0)$ of the logarithmic strain measure is small. In practice most of elastic–plastic materials (metal-like, rock-like and soil-like) do not sustain large elastic distortions, therefore this case is of special interest when $\boldsymbol{E}(n)$ is identified with the *elastic strain measure* in conjunction with the multiplicative decomposition of $\boldsymbol{F}$ (cf. Sect. 4.4.a).

[4] Using the local mass balance expressed in the form $-\dot{\varrho}/\varrho = \operatorname{tr}\boldsymbol{D}$ one can replace $-\dot{\varrho}/\varrho$ occurring in (30) with $\operatorname{tr}\boldsymbol{D}$.

Adopting the common reference configuration, from Eq. (4) one deduces the following relation between $\bar{E}_K(n)$ and $\bar{E}_K(0)$:

$$1 + 2n\bar{E}_K(n) = \exp[2n\bar{E}_K(0)] = J^{2n/3}\exp[2n\bar{E}_K^d(0)], \quad \ln J = \mathrm{tr}[\boldsymbol{E}(0)]. \tag{32}$$

The expansion of (32) with respect to small $\bar{E}_K^d(0)$ under fixed J leads to the following relation between $\boldsymbol{E}(n)$ and $\boldsymbol{E}(0)$:

$$\boldsymbol{E}(n) = \frac{\boldsymbol{U}^{2n} - \boldsymbol{1}}{2n} = \boldsymbol{E}_v(n) + J^{2n/3}\boldsymbol{E}^d(0)[\boldsymbol{1} + n\boldsymbol{E}^d(0)] + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^3\Big). \tag{33}$$

The second term on the right-hand side of this equation represents the approximation of $J^{2n/3}\boldsymbol{E}_D(n)$ defined in (5). Calculating the time derivative of (33) and using the observation that $\dot{\boldsymbol{E}}(0) = \boldsymbol{D}^R + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^2\Big)\boldsymbol{D}_d^R$ [cf. (23.1)] we find the approximate $\dot{\boldsymbol{E}}(n) \Leftrightarrow \boldsymbol{D}^R$ connections

$$\begin{aligned} J^{-2n/3}\dot{\boldsymbol{E}}(n) &= \boldsymbol{D}^R + n\boldsymbol{P}[\boldsymbol{E}^d(0)]\,\boldsymbol{D}^R + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^2\Big)\,\boldsymbol{D}^R, \\ J^{2n/3}\,\boldsymbol{D}^R &= \dot{\boldsymbol{E}}(n) - n\boldsymbol{P}[\boldsymbol{E}^d(0)]\,\dot{\boldsymbol{E}}(n) + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^2\Big)\,\boldsymbol{D}^R, \end{aligned} \tag{34}$$

and the work conjugacy implies the same connections between $\boldsymbol{\tau}^R$ and $\boldsymbol{T}(n)$,

$$J^{-2n/3}\boldsymbol{\tau}^R = \boldsymbol{T}(n) + n\boldsymbol{P}[\boldsymbol{E}^d(0)]\boldsymbol{T}(n) + \boldsymbol{\mathcal{O}}([\boldsymbol{E}^d(0)]^2)\boldsymbol{T}(n), \tag{35.1}$$

$$J^{2n/3}\boldsymbol{T}(n) = \boldsymbol{\tau}^R - n\boldsymbol{P}[\boldsymbol{E}^d(0)]\boldsymbol{\tau}^R + \boldsymbol{\mathcal{O}}([\boldsymbol{E}^d(0)]^2)\boldsymbol{\tau}^R. \tag{35.2}$$

Now let us calculate the time derivative of (35.1) and neglect the terms of the order $[\boldsymbol{E}^d(0)]^2$,

$$J^{2n/3}\{\boldsymbol{I} + n\boldsymbol{P}[\boldsymbol{E}^d(0)]\}\dot{\boldsymbol{T}}(n) = \dot{\boldsymbol{\tau}}^R - n\boldsymbol{P}(\boldsymbol{\tau}^R)\boldsymbol{D}^R + \boldsymbol{\mathcal{O}}[\boldsymbol{E}^d(0)]\boldsymbol{\tau}^R\boldsymbol{D}_d^R. \tag{36}$$

To eliminate $\boldsymbol{\omega}^R$ (occurring in the expression $\dot{\boldsymbol{\tau}}^R = \overset{T}{\boldsymbol{R}}[\dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{\omega}^R - \boldsymbol{\omega}^R\boldsymbol{\tau}]\boldsymbol{R}$) note that $\boldsymbol{U} = J^{1/3}[\boldsymbol{E}^d(0) + \boldsymbol{1}] + \boldsymbol{\mathcal{O}}\Big([\boldsymbol{E}^d(0)]^2\Big)$ (substitute $n = 1/2$ into (33)). If this last approximation is inserted into (12) then the result is

$$\Delta\boldsymbol{\omega}\boldsymbol{U} + \boldsymbol{U}\Delta\boldsymbol{\omega} = J^{1/3}[\boldsymbol{E}^d(0)\boldsymbol{D}_d^R - \boldsymbol{D}_d^R\boldsymbol{E}^d(0)] + \boldsymbol{\mathcal{O}}([\boldsymbol{E}^d(0)]^2)\boldsymbol{D}_d^R. \tag{37}$$

Spins $\overset{T}{\boldsymbol{R}}\boldsymbol{\omega}\boldsymbol{R}$ and $\overset{T}{\boldsymbol{R}}\boldsymbol{\omega}^R\boldsymbol{R}$ thus differ by first-order terms in $\boldsymbol{E}^d(0)$. Taking into account this observation we find the following useful version of relation (36) that holds with an error of first-order in $\boldsymbol{E}^d(0)$:

$$J^{2n/3}\dot{\boldsymbol{T}}(n) = \frac{1}{\varrho}\boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}})\frac{\mathcal{D}^{(n)}\boldsymbol{\sigma}}{\mathcal{D}t}, \tag{38}$$

where the one-parameter family of stress-fluxes $\mathcal{D}^{(n)}\boldsymbol{\sigma}/\mathcal{D}t$ is defined in (30).

(ii) We shall now consider the more restrictive situation encountered when the familiar first-order updated Lagrangean technique is applied. Let the actual configuration of m.e. at a time t be adopted as common reference configuration. Assume that the gradient of displacement $\boldsymbol{\mathcal{G}} = \boldsymbol{F} - \boldsymbol{1}$ describing the shape and orientation of a m.e. at time $t + dt$ is a small quantity. In such case

$$\boldsymbol{E}(n) = \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \quad \boldsymbol{R} = \boldsymbol{1} + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \quad J = 1 + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \quad \varrho^R = \varrho + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \tag{39}$$

and from Eqs. (34), (35) and (38) we find that

$$d\boldsymbol{E}(n) = \boldsymbol{D}dt + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \tag{40}$$

$$\boldsymbol{T}(n) = \boldsymbol{\tau} + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}), \quad d\boldsymbol{T}(n) = \frac{1}{\varrho}\frac{\mathcal{D}^{(n)}\boldsymbol{\sigma}}{\mathcal{D}t}dt + \boldsymbol{\mathcal{O}}(\boldsymbol{\mathcal{G}}). \tag{41.1,2}$$

Thus, all strain measures vanish, all strain rates coincide with $\boldsymbol{D}$, all work-conjugate stresses $\varrho\boldsymbol{T}(n)$ match with the Cauchy's stress $\boldsymbol{\sigma}$, and the family of direct fluxes $\varrho\dot{\boldsymbol{T}}(n)$ reduces to the family

(30) of stress fluxes *with an error of the first-order in* $\mathcal{G}$. The equation of the type (41.2) provides the physical interpretation of the Eulerian stress flux $\mathcal{D}^{(n)}\boldsymbol{\sigma}/\mathcal{D}t$ in terms of an infinitesimal variation of Lagrangean work-conjugate $\boldsymbol{T}(n)$. In first-order updated Lagrangean mechanics the first term occurring in the expansion (41.2) is substituted for the increment of $\boldsymbol{T}(n)$. Similar interpretations concern those Eulerian quantities (of the rate type) which occur in the expansion of the increments of some Lagrangean measures.

d. Generalized coordinates and generalized forces

In general, the knowledge of at most six variables, say q^i ($i = 1, 2,\ldots, 6$), is required to describe an actual shape of homogeneously deformed m.e.. The array q^i may be regarded as "generalized coordinates" in the abstract space of parallelepipeds.

The generalized specific p_i ($i = 1, 2,\ldots, 6$) forces are defined by the incremental total work per unit of mass,

$$dW = \sum_{i=1}^{6} p_i dq^i, \tag{42}$$

where dq^i denotes the incremental variation of the generalized coordinates. Here and in the sequel we apply usual summation convention concerning the repeated indexes. When the notion of Lagrangean strain is employed with E_{KL} -components (on some fixed in a laboratory rectangular basis) the generalized coordinates can be identified, for example, according to the following rule: $q^1 = E_{11},\ q^2 = E_{22},\ q^3 = E_{33},\ q^4 = 2\ E_{23},\ q^5 = 2\ E_{13},\ q^6 = 2\ E_{12}$. Then the corresponding elements of the array p_i are

$$p_1 = T_{11}(n),\ \ p_2 = T_{22}(n),\ \ p_3 = T_{33}(n),\ \ p_4 = T_{23}(n),\ \ p_5 = T_{13}(n),\ \ p_6 = T_{12}(n).$$

The use of variables $(q^i,\ p_i)$ is convenient in the discussion of most important aspects of the invariance and they will be frequently applied in the further part of this paper.

3.2 Free energy and incremental state equations

Similarly as in Sect. (2) of [4], the state of an elastic–plastic m.e. at fixed temperature is assumed to be described by current values of q^i (or p_i, $i = 1, 2,\ldots, 6$) and by the current pattern of internal rearrangement (PIR) symbolically denoted by H. Details concerning the physical character of H can be found in the original papers of Rice [19]–[21], Hill and Rice [22], as well as in the numerous more recent papers and books (c.f., e.g., [23]–[25, Chap. 8]) where further relevant references can be found.

We shall single out the following independent differentials of the symbolic functions $\mathcal{A}(q^i, H)$ and $\mathcal{A}(p_i, H)$

$$\begin{aligned} &\delta\mathcal{A}(q^i, H) = \mathcal{A}(q^i + \delta q^i, H) - \mathcal{A}(q^i, H),\\ &\delta\mathcal{A}(p_i, H) = \mathcal{A}(p_i + \delta p_i, H) - \mathcal{A}(p_i, H), \end{aligned} \tag{43}$$

$$\begin{aligned} &d^p\mathcal{A}(q^i, H) = \mathcal{A}(q^i, H + dH) - \mathcal{A}(q^i, H),\\ &d^p\mathcal{A}(p_i, H) = \mathcal{A}(p_i, H + dH) - \mathcal{A}(p_i, H). \end{aligned} \tag{44}$$

The differential $d^p\mathcal{A}$ is interpreted as the isothermal "plastic part" of the total incremental change in $\mathcal{A}$. For example, in the case of rate-independent plastic materials the parameters H may change only along those path segments of a process that lie on the yield surface. In general, d^p-variations of any quantity can be effected only by strain or stress plastic cyclic process that is associated with an

infinitesimal change in H. δ-Differential represents the infinitesimal change in $\mathcal{A}$ in the course of an elastic process.

When the elastic response is always of Green type then there exists the free energy function ϕ

$$\phi = \phi(q^i, H) \tag{45}$$

such that

$$p_i = \frac{\partial \phi}{\partial q^i}. \tag{46}$$

Equation (46) defines the transformation from (q^i, H)-space to the (p_i, H)-space.

Provided that the unique solution $q^i = \overset{*}{q}{}^i(H)$ of six algebraic equations $\partial\phi/\partial q^i = 0$ exists, without loss of generality the function ϕ may be rearranged into the form[5]

$$\Delta\phi = \phi - \phi_0 = \phi_e(q^i, H) + \phi_s(H) \quad \Delta\phi\Big|_{q^i = \overset{*}{q}{}^i(H)} = \phi_s(H), \tag{47}$$

where $\overset{*}{q}{}^i$ is a possible measure of permanent changes of an m.e. shape, $\phi_0 = \text{const}$ is the free energy in the thermodynamic reference state, ϕ_e (q^i, H) is the specific elastic strain energy, and $\phi_s(H)$ is the free energy stored in the m.e. in the course of prior plastic straining. By definition of ϕ_e we have

$$p_i = \frac{\partial \phi_e(q^k, H)}{\partial q^i}, \qquad \phi_e\Big|_{q^k = \overset{*}{q}{}^k} = 0, \quad p_i(\overset{*}{q}{}^k, H) = 0. \tag{48.1,3}$$

The total complementary energy $-\psi_T(p_i, H)$ and the elastic complementary energy $-\psi_e(p_i, H)$ are defined by the Legendre transformation

$$\psi_T(p_j, H) = (\phi_e - p_i q^i)\big|_{q^i = q^i(p_j, H)}, \quad \psi_e(p_j, H) \equiv \psi_T(p_j, H) + p_i \overset{*}{q}{}^i(H) \tag{49.1,2}$$

such that $\psi = \psi_T + \phi_s + \phi_0$ is the usual Gibb's potential. The inverse of (48.1) is

$$q^i = -\frac{\partial \psi_T}{\partial p_i} = -\frac{\partial \psi_e}{\partial p_i} + \overset{*}{q}{}^i(H), \quad \frac{\partial \psi_e}{\partial p_i} = \psi_e = 0 \quad \text{for } p_j = 0. \tag{50.1,2}$$

In the classical isothermal theory of plasticity it is presumed that in q^i-space (or in its dual p_i-space) there exists an elastic domain bounded by a yield surface. The elastic–plastic response of a material to incremental deformation is usually written in the form

$$dp_i = \frac{1}{\varrho^{\mathrm{R}}} l_{ij} dq^j + d^p p_i = \frac{1}{\varrho^{\mathrm{R}}} l_{ij}(dq^j - d^p q^j) \tag{51}$$

or in alternative form

$$dq^j = \varrho^{\mathrm{R}} m^{ij} dp_i + d^p q^j, \tag{52}$$

where

$$\frac{1}{\varrho^{\mathrm{R}}} l_{ij} d^p q^j = -d^p p_i. \tag{53}$$

Here $l_{ij} = l_{ji}$ and $m^{ij} = m^{ji}$ are generalized coefficients of isothermal elastic stiffness and compliances ($m^{ij} l_{jk} = \delta^i_k$). When (46) holds they are defined by

[5] The macroscopic internal variables representing PIR need not to be the same in ϕ_s and ϕ_e. The relevant examples can be found, e.g., in [24].

$$\frac{1}{\varrho^{\mathrm{R}}} l_{ij} = \frac{\partial^2 \phi}{\partial q^i \partial q^j} = \frac{\partial^2 \phi_e}{\partial q^i \partial q^j}, \quad \varrho^{\mathrm{R}} m^{ij} = \frac{\partial q^i}{\partial p_j} = -\frac{\partial^2 \psi_{_T}}{\partial p_i \partial p_j} = -\frac{\partial^2 \psi_e}{\partial p_i \partial p_j}. \tag{54}$$

The plastic parts $d^p p_i$ and $d^p q^i$ occurring in Eqs. (51) and (52) are the components of the additive decomposition of dp_i and dq^i, respectively. They are usually specified by the plastic flow rule (not necessary rate independent) and become non-zero when an appropriate condition for the plastic loading process is satisfied, otherwise $d^p q_i = 0$ and the response is purely elastic,

$$\delta p_i = \frac{1}{\varrho^{R}} l_{ij} \delta q^j, \quad \delta q^j = \varrho^{R} m^{ji} \delta p_i. \tag{55}$$

The relation (52) is sometimes regarded as the "kinetic" equation that defines operationally $d^p q^i$ in an infinitesimal loading–unloading stress cycle. However, the transformation rule for $d^p q^i$ presented in the next Section proves that such a concept is not coordinate-invariant.

By calculating the p-differential of Eqs. (46) and (50) one may furnish the following energy-interpretation of $d^p p_i$ and $d^p q^i$:

$$d^p p_i = \frac{\partial (d^p \phi)}{\partial q^i}, \quad d^p q^i = -\frac{\partial (d^p \psi_{_T})}{\partial p_i} = d\overset{*}{q}{}^i(H) - \frac{\partial [d^p \psi_e(p_j, H)]}{\partial p_i}. \tag{56.1,2}$$

Note that the incremental relations (51) can formally be derived by a calculation of the (q, H)-total differential of (46). Likewise the relation (52) is the (p, H) total differential of (50.1), whereas the p-differential of (49) results in the basic identities

$$d^p \psi = d^p \phi \Rightarrow d^p \phi_e = d^p \psi_{_T}|_{p_i = p_i(q^j, H)} = [d^p \psi_e - p_i d\overset{*}{q}{}^i]_{p_i = p_i(q^j, H)}. \tag{57}$$

Note also that the identity (53) can be obtained by calculation of the partial derivative of (57.1) with respect to q^i.

The plastic increment of the free energy ϕ, when taken with the negative sign, $-d^p \phi$, represents an incremental variation of the energy dissipation dD [3], [19]

$$dD = -d^p \phi = -d^p \phi_e - d\phi_s(H) \geq 0 \Leftrightarrow dD = -d^p \psi = -d^p \psi_{_T} - d\phi_s(H) \geq 0 \tag{58}$$

provided that p_i is regarded as an equilibrium force. The semi-positive definiteness of dD is implied by the second law of classical thermodynamics.

Denote by π the thermodynamic force energy conjugate to H, $d^p \psi = d^p \phi = -\pi dH$, and suppose that dH/dt admits the energy dissipation potential $D^*(\pi, H)$, $dH/dt = \partial D^*/\partial \pi$. Then $d^p q^i$ and $d^p p_i$ posses also the potentials

$$\frac{dH}{dt} = \frac{\partial D^*(\pi, H)}{\partial \pi} \Rightarrow d^p q^i = \frac{\partial D_p}{\partial p_i} dt, \quad d^p p_i = -\frac{\partial D_q}{\partial q^i} dt, \tag{59}$$

where

$$D_p(p_i, H) \equiv D^*[\pi(p_i, H), H], \quad D_q(q^i, H) \equiv D^*[\pi(q^i, H), H]$$

on account of (56). This is one of the crucial issues of H–R theory of inelasticity. The potential $D^*(\pi, H)$ and related constitutive equations (59.1) for dH were employed by numerous authors (cf., e.g., [15], [26]).

In what follows we shall also use the notation

$$(dW)_p \equiv p_i(q^k, H) d^p q^i = -p_i \frac{\partial (d^p \psi_{_T})}{\partial p_i} = p_i d\overset{*}{q}{}^i - \frac{\partial (d^p \psi_{_e})}{\partial p_i} p_i \tag{60}$$

to denote the "plastic" increment of the total work resulting from the additive decomposition of the infinitesimal displacement dq^i into plastic d^pq^i and elastic $dq^i - d^pq^i$ parts. The increment $(dW)_p$ should carefully be distinguished from the invariant infinitesimal work dW^p of plastic deformation which will be defined in Sect. 3.4.

3.3 Basic transformation rules under a change of generalized coordinates

(i) Consider the situation when two observers take two different strain measures, say $\boldsymbol{f}_1$ and $\boldsymbol{f}$, and two different reference configurations κ_1^R and κ^R to define the selected strain. If the constant tensor $\boldsymbol{A}$ transforms κ^R into κ_1^R then the deformation tensor transforms according to $\boldsymbol{F} = \boldsymbol{F}_1\boldsymbol{A}$. From the polar decomposition of both $\boldsymbol{F}$ and $\boldsymbol{F}_1$ one gets $\boldsymbol{U}_1^2 = (\overset{-\mathrm{T}}{\boldsymbol{A}}\boldsymbol{U}^2\overset{-1}{\boldsymbol{A}})$ and according to the definition (4) we have

$$\boldsymbol{E} = \boldsymbol{f}(\boldsymbol{U}), \quad \boldsymbol{E}_1 = \boldsymbol{f}_1(\boldsymbol{U}_1) = \boldsymbol{f}_1\left[\left(\overset{-\mathrm{T}}{\boldsymbol{A}}\boldsymbol{U}^2\overset{-1}{\boldsymbol{A}}\right)^{1/2}\right]. \tag{61}$$

Thus, there must exist the geometrical relation between $\boldsymbol{E}_1$ and $\boldsymbol{E}$ [regard $\boldsymbol{U}$ in (61) as tensorial parameter]

$$\boldsymbol{E}_1 = \boldsymbol{\Psi}(\boldsymbol{E},\boldsymbol{A}). \tag{62}$$

When $\boldsymbol{A} = \boldsymbol{1}$ (common reference configuration) the component-wise relation on the principal triad between two different strain measures $\boldsymbol{E}(n_1)$ and $\boldsymbol{E}(n)$ is [cf. (4)]

$$[1 + 2n_1\bar{E}_K(n_1)]^{2n} = [1 + 2n\bar{E}_K(n)]^{2n_1}. \tag{63}$$

The index-free form (33) of approximate relation is the another example of the connection (62).

More generally, the analogue of (62) written in terms of generalized coordinates

$$q^\alpha = q^\alpha(q^i), \quad q^i = q^i(q^\alpha) \tag{64}$$

may be interpreted as transformation of generalized coordinates in the abstract space of parallelepipeds. The generalized coordinates denoted by q^α ($\alpha = 1,\ 2,\ldots,6$) are regarded as "new" coordinates. The array q^i ($i = 1,2,\ldots,6$) thus is treated as "old" generalized coordinates. We adopt here the early convention that the symbols furnished with the Latin and the Greek indexes represent different arrays of numbers.

The infinitesimal displacements δq^i and dq^i are the prototypes of the contravariant vectors in the abstract space of shapes of m.e.. Their transformation rules are the same and follow directly from (64) ($i, j, \alpha, \beta = 1, 2,\ldots,6$)

$$\delta q^\alpha = Q_i^\alpha \delta q^i, \quad dq^\alpha = Q_i^\alpha dq^i \Rightarrow \delta q^j = Q_\alpha^j \delta q^\alpha, \quad dq^j = Q_\alpha^j dq^\alpha \tag{65.1-3}$$

where

$$Q_i^\alpha \equiv \frac{\partial q^\alpha}{\partial q^i}, \quad Q_\alpha^j \equiv \frac{\partial q^j}{\partial q^\alpha}, \quad Q_j^\beta Q_\alpha^j = \delta_\alpha^\beta, \quad Q_\alpha^j Q_i^\alpha = \delta_i^j, \quad \det \boldsymbol{Q} \neq 0. \tag{66}$$

During any infinitesimal strain cycle the net increment of a strain vanishes, no matter whether the process is associated with $dH = 0$ or not. The transformation rules (65) are such that $dq^i = 0 \Rightarrow dq^\alpha = 0$, and $\delta q^i = 0 \Rightarrow \delta q^\alpha = 0$. Hence, one concludes that *the notion of an infinitesimal strain cycle is the invariant concept*.

The incremental work (42) is the prototype of the invariant under the change of generalized coordinates. Hence,

$$p_i dq^i = p_\alpha dq^\alpha \Rightarrow p_\alpha = Q^j_\alpha p_j \tag{67.1,2}$$

i.e., p_i is an example of the covariant "vector". Since $p_i = 0$ implies $p_\alpha = 0$ we conclude that the notion of a stress cycle *that starts and terminates at stress-free state is the invariant concept*. As remarked in Part I [4] we additionally take for granted that the incremental energy dissipation (58) is also invariant,

$$-d^p\phi \geq 0 \quad d^p\phi(q^\alpha, H) = d^p\phi(q^i, H)\big|_{q^i = q^i(q^\alpha)} \geq 0 \tag{68}$$

Differentiation of (68) with respect to q^α shows that the transformation rules of p_i and $d^p p_i$ are the same,

$$d^p p_\alpha = Q^i_\alpha d^p p_i. \tag{69}$$

The above relation proves also the invariance of a plastic flow rule, provided it is specified for $d^p p_i$. This rule in combination with Eqs. (51)–(55) and (65) yields the following bilinear differential invariants:

$$\delta q^i d^p p_i = -\delta p_i d^p q^i = \delta q^i dp_i - \delta p_i dq^i, \quad dq^i d^p p_i = (d^p p_i - dp_i) d^p q^i. \tag{70}$$

The invariance property of the differential forms (70) was proved by Hill [1] in a more general situation when the potential ϕ does not exist.

(ii) The transformation rules for the increment of p_i (the counterpart of a stress rate in physical space) and for the tangent elastic moduli are more involved. Differentiation of (67.2) leads to the following formulas:

$$\begin{aligned} \delta p_\alpha &= Q^j_\alpha[\delta p_j - Q^\beta_{ij} p_\beta \delta q^i] = Q^j_\alpha[\delta p_j - \Gamma^k_{ij} p_k \delta q^i], \\ dp_\alpha &= Q^j_\alpha[dp_j - Q^\beta_{ij} p_\beta dq^i] = Q^j_\alpha[dp_j - \Gamma^k_{ij} p_k dq^i], \end{aligned} \tag{71}$$

where

$$Q^\beta_{ij} = \frac{\partial^2 q^\beta}{\partial q^i \partial q^j}, \quad \Gamma^k_{ij} = Q^k_\beta Q^\beta_{ij}. \tag{72}$$

As remarked in Part I [4], in general $Q^\beta_{ij} \neq 0$ and $dp_j = 0$ does not imply $dp_\alpha = 0$. Therefore the closed *infinitesimal* cycle in p_i does not necessary close the infinitesimal cycle in p_α until it starts and ends at stress-free state. Thus, *the notion of infinitesimal cycle in work-conjugate generalized forces (or stresses) is not an invariant concept*. Hence, the partitioning (52) of incremental strain into elastic and plastic parts is also not an invariant concept [27]. The complex transformation rule for $d^p q^i$ is presented in the formula (77).

Now we substitute $\delta p_\alpha = l_{\alpha\beta}\delta q^\beta / \varrho^R_1$ and $\delta p_j = l_{ji}\delta q^i / \varrho^R$ into (71.1) and arrive at the following transformation rule for the tangent elastic moduli [1]:

$$\frac{1}{\varrho^{\mathrm{R}}_1} l_{\alpha\beta} = Q^i_\alpha \left(\frac{1}{\varrho^{\mathrm{R}}} l_{ij} - \Gamma^k_{ij} p_k \right) Q^j_\beta, \quad \frac{1}{\varrho^{\mathrm{R}}} l_{ij} = \frac{1}{\varrho^{\mathrm{R}}_1} Q^\alpha_i l_{\alpha\beta} Q^\beta_j + Q^\beta_{ij} p_\beta, \tag{73.1,2}$$

where ϱ^R_1 and ϱ^R are mass densities of the m.e. in "new" and "old" reference configurations. From (73) it follows that *the symmetry of elastic moduli is preserved under the transformation* (64) [1]. *However, neither the elastic linearity nor the positive definiteness of the moduli are the invariant constitutive properties*.

(iii) Consider rate-independent behavior and denote by l^{e-p}_{ij} the tangent moduli measured during active elastic–plastic yielding [cf. (51)],

$$dp_i = \frac{1}{\varrho^{\mathrm{R}}} l_{ij}^{e-p} dq^j \Leftrightarrow d^p p_i = -\frac{1}{\varrho^{\mathrm{R}}}(l_{ij} - l_{ij}^{e-p}) dq^j. \tag{74.1,2}$$

The transformation rule for l_{ij}^{e-p} is the same as for l_{ij} [cf. (73)] on account of (71). The condition $\det\{l_{ij}^{e-p}\} = 0$ distinguishes the special states where infinitesimal strains can be generated under stationary work-conjugate stresses. Such states and admissible differential deformations are called by Hill [27] "eigenstates" and "eigenmodes", respectively. Suppose that $\det\{l_{\alpha\beta}^{e-p}\} = 0$ for a particular choice of work-conjugate couple (p_α, q^α). From the transformation rule for $\{l_{\alpha\beta}^{e-p}\} = 0$ [an analogy of (73)] it follows that so determined eigenstates will manifest themselves at states where

$$\det\left\{l_{ij}^{e-p} - \varrho^{R}\Gamma_{ij}^{k} p_k\right\} = 0 \tag{75}$$

for an other choice, say (p_i, q^i) of work-conjugate couple. A one-dimensional illustration of (75) is given in Part I [4].

Return to (74.2) and note that due to (69) and (65.2) the notion of "relative tangent moduli" $l_{ij} - l_{ij}^{e-p}$ is an invariant concept [27] since

$$\frac{1}{\varrho_1^{R}}(l_{\alpha\beta} - l_{\alpha\beta}^{e-p}) = \frac{1}{\varrho^{\mathrm{R}}} Q_\alpha^i (l_{ij} - l_{ij}^{e-p}) Q_\beta^j. \tag{76}$$

(iv) The transformation rule for the plastic part of the increment of q^i can be deduced from (53) written for old and new measures. Using (69) and (73) one can eventually find

$$d^p q^\alpha = \frac{\varrho_1^{R}}{\varrho^{\mathrm{R}}} Q_\beta^j l_{ij} m^{\beta\alpha} d^p q^i = Q_i^\alpha d^p q^i + \varrho_1^{R} m^{\alpha\beta} Q_\beta^j \Gamma_{ji}^k p_k d^p q^i. \tag{77}$$

To find the inverse it is sufficient to change in this formula Latin indexes into the Greek one and vice versa, and to interchange ϱ_1^{R} with ϱ^{R}.

The formula (77) is identical with Hill's [27] transformation rule for the outward normal to the yield surface in p_i-space. As remarked in Part I [4], the fact that the transformation rules for dq^i and $d^p q^i$ are different justifies the Rice statement [19] that physical dimension change of the m.e. corresponding to the plastic strain increment is dependent on the choice of the strain measure and reference configuration. Let $(dW)_p$ and $(dW)_p^1$ be the works expended on $d^p q^i$ and $d^p q^\alpha$, respectively,

$$(dW)_p = p_i d^p q^i, \quad (dW)_p^1 = p_\alpha d^p q^\alpha. \tag{78}$$

We substitute (77) into (78) and note that the so defined incremental work is not invariant, $(dW)_p^1 \neq (dW)_p$,

$$(dW)_p^1 = (dW)_p + \varrho^{R}\Gamma_{mj}^k Q_\alpha^m m^{ji} p_i p_k d^p q^\alpha. \tag{79}$$

Therefore, in general, $(dW)_p$ *cannot be regarded as rigorous measure of that increment of external work which must be irreversibly done to cause permanent changes in physical dimensions of m.e.* Both formulas (77) and (79) are independent of the physical nature of the plastic flow. The ratio $(dW)_p^1/(dW)_p$ differs from unity only by a small factor which is of the order of generalized stress divided by the typical value of elastic moduli. It can also be shown that although $d^p\psi_T + d^p\phi_S = d^p\phi$ is the example of the invariant quantity, neither the total complementary energy ψ_T itself nor its elastic counterpart ψ_e [cf. (49)] are invariants under change of the generalized coordinates q^i. Finally note that if in the physical space one adopts the common reference configuration then q^α and q^i have common origin $q^\alpha = q^i = 0$, and moreover $Q_i^\alpha|_{q^i=0} = \delta_i^\alpha$.

3.4 Invariant incremental plastic work

Let us consider a strain-cycle consisting of the following three stages (Fig. 1):

(a) the elastic unloading from a state $A(q^i, H)$ to a stress free state $A^*[\overset{*}{q}{}^i(H), H]$

(b) in elastic–plastic loading–unloading stress cycle from the stress-free state A^* through B and C to the other stress-free state $C^*[\overset{*}{q}{}^i(H+dH), H+dH]$

(c) elastic loading from the state C^* to the state $A_1(q^i, H+dH)$.

Since $p_i dq^i = d\phi_e - d^p\phi_e$ along $B-C$, $p_i\delta q^i = \delta\phi_e$ along other segments (elastic behavior) of the considered deformation path, and $\phi_e(C^*) = \phi_e(A^*) = 0$, the net work done in a transitional stress cycle $A^* - B - C - C^*$ is

$$\oint_{p_i=0} p_i dq^i = -\int_B^C d^p\phi_e, \tag{80}$$

and the difference between this work and the work expended in the whole considered strain cycle becomes

$$\oint_{p_i=0} p_i dq^i - \oint_{q^i} p_i dq^i = -d^p\phi_e(q^i, H). \tag{81}$$

It holds for all states (q^i, H) in the elastic domain and on its border (yield surface). A one-dimensional graphical illustration of (81) is presented in Part I [4]. The generalized coordinates $\overset{*}{q}{}^i(H)$ and $\overset{*}{q}{}^i(H+dH)$ shown in Fig. 1 correspond to the permanent strains in stress-free states $(p_i = 0,\ H)$ and $(p_i = 0, H+dH)$, respectively. The work-interpretation (80) constitutes the basis for the definition of the *invariant incremental plastic work* dW^p *per unit of mass* [cf. also (57)],

$$dW^p \equiv -d^p\phi_e(q^i, H) = -d^p\psi_T = p_i d\overset{*}{q}{}^i - d^p\psi_e. \tag{82}$$

The definition (82) concerns all those rate-dependent and rate-independent model materials that behave elastically in a vicinity of stress-free states, and for which it is possible to construct a strain

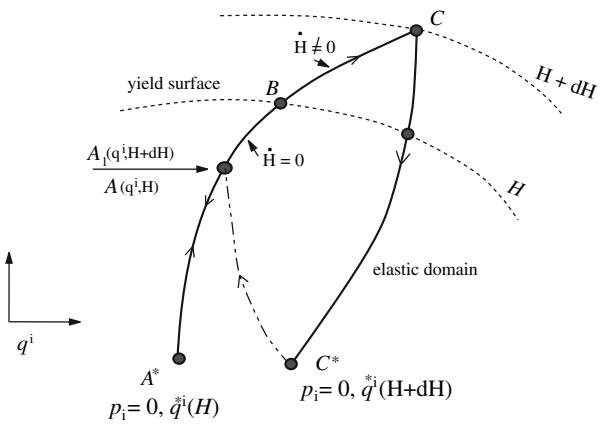

Fig. 1. Work-interpretation of basic invariants

energy function ϕ_e including materials displaying a degradation of elastic properties. *The invariant plastic work dW^p plays a role of the potential for $d^p p_i$,*

$$d^p p_i = -\frac{\partial(dW^p)}{\partial q^i}, \tag{83}$$

on account of (56.1). In the course of plastic yielding we have

$$dW = d\phi_e - d^p\phi_e \Rightarrow dW - dW^p = dW^e, \quad dW^e \equiv d\phi_e. \tag{84}$$

The total incremental work dW is additively partitioned into the invariant incremental elastic work $dW^e = d\phi_e$ and the invariant incremental plastic work dW^p. As remarked in Sect. 3.2 [cf. (58)] in classical thermodynamics of elastic–plastic materials, the incremental energy dissipation dD is equal to $-d^p\phi$, provided that the total strain is regarded as the state parameter. The d^p-differential of (47) in combination with (82) shows that the difference between dD and the invariant incremental plastic work dW^p is the total differential of the stored free energy,

$$d\phi_s(H) = dW^p - dD. \tag{85}$$

Under the common assumption that the configurational entropy is negligible, the stored free energy represents (in the course of isothermal processes) the difference between the invariant plastic work and the heat exchanged with the surrounding. The energy ϕ_s is thus a measurable quantity. The experimental results such as reported, e.g., in [28], [29] should be accounted for when proposing the specific form of ϕ_s.

In passing we note that [cf. (81)]

$$\delta \oint\limits_{p_i=0} p_i dq^i = 0 \Rightarrow \delta \oint\limits_{q^i} p_i dq^i = \delta(d^p\phi) = \delta q^i d^p p_i \tag{86}$$

because the first integral is a function of H alone. This is the work-interpretation of bilinear differential invariants (70) deduced by Hill [2] with use of the other physical arguments. It does not provide, of course, the basis for the definition of invariant incremental plastic work.

3.5 Examples of transformation rules in tensor notation

(i) In the total Lagrangean description (fixed reference configuration) the complementarities of the incremental constitutive relations (51) and (53) derived from the free energy $\phi[\boldsymbol{E}(n), H]$ are

$$\begin{aligned} d\boldsymbol{T}(n) &= \frac{1}{\varrho^{\mathrm{R}}}\boldsymbol{L}^{K_R}(n)d\boldsymbol{E}(n) + d^p\boldsymbol{T}(n) = \frac{1}{\varrho^{\mathrm{R}}}\boldsymbol{L}^{K_R}(n)[d\boldsymbol{E}(n) - d^p\boldsymbol{E}(n)], \\ d^p\boldsymbol{E}(n) &= -\varrho^R\boldsymbol{M}^{K_R}(n)d^p\boldsymbol{T}(n), \quad \boldsymbol{M}^{K_R}(n) = \overset{-1}{\boldsymbol{L}}{}^{K_R}(n), \end{aligned} \tag{87}$$

where $\boldsymbol{L}^{K_R}(n)$ is the tensor of Lagrangean tangent moduli of elasticity, defined in the reference configuration κ^R.

To find the symbolic form of transformation rules under a change of strain measure, for the quantities occurring in (87), first write Eq. (9) for two different exponents, say n_1 (new measure) and n (old measure) and eliminate $\boldsymbol{D}^R$ from the obtained equations. The results may be presented in symbolic form as

$$\dot{\boldsymbol{E}}(n_1) = \boldsymbol{Q}(n_1, n)\dot{\boldsymbol{E}}(n)\,, \; \boldsymbol{Q}(n_1, n) = \boldsymbol{\mathcal{E}}^{(n_1)}\,\overset{-1}{\boldsymbol{\mathcal{E}}}{}^{(n)} = \frac{\partial \boldsymbol{E}(n_1)}{\partial \boldsymbol{E}(n)}. \tag{88}$$

The fourth-order tensor $\boldsymbol{Q}(n_1,n)$ [the counterpart of Q_i^{α}– cf. (65)] has the following properties:

$$\overset{T}{\boldsymbol{Q}}(n_1,n)=\boldsymbol{Q}(n_1,n)\,,\ \overset{-1}{\boldsymbol{Q}}(n_1,n)=\boldsymbol{Q}(n,n_1)\,,\ \boldsymbol{Q}(n,n)=\boldsymbol{I}, \tag{89.1-3}$$

$$\boldsymbol{Q}(n_1,n_2)\boldsymbol{Q}(n_2,n_3)=\boldsymbol{Q}(n_1,n_3), \tag{89.4}$$

where (89.2) may be regarded as a "chain rule". The components of $\boldsymbol{Q}$ on the principal triad $\boldsymbol{N}_K$ may be obtained by elimination of $\bar{D}_{KL}$ between two equations of the type (8) written for n and n_1. The simple index-free form is available only for special situations, e.g., $\boldsymbol{Q}(1,1/2)=0.5\boldsymbol{P}(\boldsymbol{U})$, $\boldsymbol{Q}(-n,n)=\boldsymbol{\Theta}(\boldsymbol{U}^{-2n})$ provided that $2n$ ($n\neq 0$) is an integer (cf. [7, Chap. 3]). The operators $\boldsymbol{P}$ and $\boldsymbol{\Theta}$ are defined in (1), (2). Next calculate the derivative of (88.2) to determine the sixth-order pairwise symmetric tensor $\boldsymbol{\mathcal{Q}}(n_1,n))$ [the counterpart of Q^{α}_{ij} – cf. (72)]

$$\begin{aligned}&\boldsymbol{\mathcal{Q}}_{IJKLMN}(n_1,n)\equiv\frac{\partial^2 E_{IJ}(n_1)}{\partial E_{KL}(n)\partial E_{MN}(n)}=\frac{\partial Q_{IJKL}(n_1,n)}{\partial E_{MN}(n)},\\&\dot{\boldsymbol{Q}}(n_1,n)=\boldsymbol{\mathcal{Q}}(n_1,n)[|\,.\,,\dot{\boldsymbol{E}}(n)|]\end{aligned} \tag{90}$$

and define the sixth-order tensor $\boldsymbol{\Gamma}(n_1,n)$ [the counterpart of Γ^{k}_{ij} – cf. (72)]

$$\Gamma_{IJKLPQ}(n_1,n)=\boldsymbol{\mathcal{Q}}_{IJKLMN}(n_1,n)\,Q_{MNPQ}(n,n_1) \tag{91}$$

which has pairwise symmetry with respect to the first two pairs only. The components of $\boldsymbol{\mathcal{Q}}(n/2,1/2)$ on the principal triad $\boldsymbol{N}_K$ can be found in [7, Chap. 3]. The symbolic forms of transformation rules in the physical space are [counterparts of (67.2), (69), (71), (73.2), (77)]

$$\begin{aligned}&\boldsymbol{T}(n_1)=\boldsymbol{Q}(n,n_1)\boldsymbol{T}(n)\,,\ d^p\boldsymbol{T}(n_1)=\boldsymbol{Q}(n,n_1)\,d^p\boldsymbol{T}(n_1),\\&\boldsymbol{Q}(n,n_1)\,d\boldsymbol{T}(n_1)=\,d\boldsymbol{T}(n)-\boldsymbol{\mathcal{Q}}(n_1,n)[|d\boldsymbol{E}(n),\boldsymbol{T}(n_1)|]\Leftrightarrow\\&\boldsymbol{Q}(n_1,n)\,d\boldsymbol{T}(n)=\,d\boldsymbol{T}(n_1)-\boldsymbol{\Gamma}(n_1,n)[|d\boldsymbol{E}(n),\boldsymbol{T}(n)|],\\&(1/\varrho^R)\,\boldsymbol{L}^{K_R}(n)=(1/\varrho^R)\,\boldsymbol{Q}(n_1,n)\,\boldsymbol{L}^{K_R}(n_1)\boldsymbol{Q}(n_1,n)+\boldsymbol{\mathcal{Q}}(n_1,n)[|\boldsymbol{T}(n),\,.\,|],\\&d^p\boldsymbol{E}(n_1)=\boldsymbol{Q}(n,n_1)d^p\boldsymbol{E}(n)+\varrho^R\,\boldsymbol{M}^{K_R}(n_1)\,\boldsymbol{Q}(n,n_1)\,\boldsymbol{\Gamma}(n_1,n)[|d^p\boldsymbol{E}(n),\boldsymbol{T}(n)|].\end{aligned} \tag{92}$$

(ii) If the current configuration is chosen to be the common reference configuration then the first-order incremental relations may be written in terms of their rate counterparts

$$\begin{aligned}&\frac{\mathcal{D}^{(n)}\boldsymbol{\sigma}}{\mathcal{D}t}=\boldsymbol{\mathcal{L}}(n)\,\boldsymbol{D}+\frac{\mathcal{D}_p\,\boldsymbol{\sigma}}{\mathcal{D}t}=\boldsymbol{\mathcal{L}}(n)\,[\boldsymbol{D}-\mathcal{D}^p(n)],\\&\mathcal{D}^p(n)=-\boldsymbol{\mathcal{M}}(n)\,\frac{\mathcal{D}_p\,\boldsymbol{\sigma}}{\mathcal{D}t},\quad \boldsymbol{\mathcal{M}}(n)=\overset{-1}{\boldsymbol{\mathcal{L}}}(n)\end{aligned} \tag{93}$$

on account of (41.2). Here $\boldsymbol{\mathcal{L}}(n)$ are first-order *instantaneous tangent moduli of elasticity*. The first-order plastic increment of the specific conjugate stress is formally written in the form $(dt/\varrho)\mathcal{D}_p\boldsymbol{\sigma}/\mathcal{D}t$. The transformation rules under a change of strain measure in this case are relatively simple since all strain increments coincide with $\boldsymbol{D}dt$, and all specific conjugate stresses match with $\boldsymbol{\tau}$. Hence, all plastic increments of conjugate stresses $\mathcal{D}_p\boldsymbol{\sigma}/\mathcal{D}t$ also coincide (are independent of n), i.e., $\boldsymbol{Q}(n,n_1)=\boldsymbol{I}$ ($Q^{\alpha}_{i}=\delta^{\alpha}_{i}$). The transformation rule for stress increments (stress rates) follows directly from (30.1) and (41.2),

$$\frac{1}{\varrho}\frac{\mathcal{D}^{(n_1)}\boldsymbol{\sigma}}{\mathcal{D}t}=\frac{1}{\varrho}\frac{\mathcal{D}^{(n)}\boldsymbol{\sigma}}{\mathcal{D}t}-(n_1-n)\boldsymbol{P}(\boldsymbol{\tau})\,\boldsymbol{D}, \tag{94}$$

which implies $\boldsymbol{\mathcal{Q}}(n_1,n)[|\,.\,,\boldsymbol{\tau}|]=\boldsymbol{\Gamma}(n_1,n)[|\,.\,,\boldsymbol{\tau}|]=(n_1-n)\boldsymbol{P}(\boldsymbol{\tau})$. The counterparts of (73), (77) and (79) are

$$\begin{aligned}&\boldsymbol{\mathcal{L}}(n_1)=\boldsymbol{\mathcal{L}}(n)-(n_1-n)\,\boldsymbol{P}(\boldsymbol{\sigma})\,,\ \mathcal{D}^p(n_1)=\{\boldsymbol{I}+(n_1-n)\boldsymbol{\mathcal{M}}(n_1)\boldsymbol{P}(\boldsymbol{\sigma})\}\mathcal{D}^p(n)\\&(dW)^1_p-(dW)_p=(n_1-n)\,\boldsymbol{\tau}\cdot\boldsymbol{\mathcal{M}}(n_1)\boldsymbol{P}(\boldsymbol{\sigma})\,\mathcal{D}^p(n),\end{aligned} \tag{95}$$

where $\boldsymbol{P}$ is defined in (1). Thus, even within this approximation $(dW)_p$ is not invariant.

(iii) When the strain measure is fixed and the reference configuration κ^R is changed into κ_1^R by mapping $\boldsymbol{A}$ [set $\boldsymbol{f} = \boldsymbol{f}_1$ in (61)], the simple index-free form of transformation rules for unrestricted strains is available only for Green ($n = 1$) or Almansi ($n = -1$) strain measures. In this case the function $\boldsymbol{\Psi}$ occurring in (62) is linear. The transformation quantities $Q^i_{\alpha\beta}, \Gamma^k_{ij}$ [cf. (71)–(72)] and their counterparts in the physical space vanish.

The familiar rules for Green strain measure are

$$\begin{aligned}
&\boldsymbol{E}_1(1) = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{A}})\boldsymbol{E}(1) + 0.5\,(\overset{-T}{\boldsymbol{A}}\,\overset{-1}{\boldsymbol{A}} - \boldsymbol{1}),\\
&d\boldsymbol{E}_1(1) = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{A}})\,d\boldsymbol{E}(1), \quad \boldsymbol{T}_1(1) = \boldsymbol{\Theta}(\boldsymbol{A})\boldsymbol{T}(1),\\
&d\boldsymbol{T}_1(1) = \boldsymbol{\Theta}(\boldsymbol{A})\,d\boldsymbol{T}(1), \quad \frac{1}{\varrho_1^{\mathrm{R}}}\boldsymbol{L}_1^{K_R}(1) = \frac{1}{\varrho^{\mathrm{R}}}\boldsymbol{\Theta}(\boldsymbol{A})\boldsymbol{L}^{K_R}(1)\,\boldsymbol{\Theta}(\overset{T}{A}),\\
&d^p\boldsymbol{T}_1(1) = \boldsymbol{\Theta}(\boldsymbol{A})\,d^p\boldsymbol{T}(1), \quad d^p\boldsymbol{E}_1(1) = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{A}})\,d^p\boldsymbol{E}(1),
\end{aligned} \tag{96}$$

where the subscript "1" denotes tensorial quantities defined in the reference configuration κ_1^R.

If the current configuration $\kappa(t)$ is identified with κ_1^R then the substitution $\varrho_1^R = \varrho$ and $\boldsymbol{A} = \boldsymbol{F}(t)$ into (96) links the instantaneous quantities occurring in (93) with the one defined in the arbitrary fixed Lagrangean configuration κ^R for $n = 1$ (Green strain measure)

$$\begin{aligned}
&\boldsymbol{E}_1(1) = 0, \quad \boldsymbol{D}dt = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\,d\boldsymbol{E}(1)\,,\ \boldsymbol{\tau} = \boldsymbol{\Theta}(\boldsymbol{F})\boldsymbol{T}(1),\\
&\frac{1}{\varrho}\frac{\mathcal{D}^{(1)}\boldsymbol{\sigma}}{\mathcal{D}t}dt = \boldsymbol{\Theta}(\boldsymbol{F})\,d\boldsymbol{T}(1)\,,\ \frac{1}{\varrho}\boldsymbol{\mathcal{L}}(1) = \frac{1}{\varrho^R}\boldsymbol{\Theta}(\boldsymbol{F})\ \boldsymbol{L}^{K_R}(1)\,\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}}),\\
&\frac{1}{\varrho}\frac{\mathcal{D}_p\boldsymbol{\sigma}}{\mathcal{D}t}dt = \boldsymbol{\Theta}(\boldsymbol{F})\,d^p\boldsymbol{T}(1)\,,\ \mathcal{D}^p(1)\,dt = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}})\ d^p\boldsymbol{E}(1).
\end{aligned} \tag{97}$$

The same connections may also be obtained by application of the "push-forward" technique based on (24).

The updated Lagrangean tangent moduli $\boldsymbol{\mathcal{L}}(1)$ associated with the Green measure are related to $\boldsymbol{L}^{K_R}(1)$, defined in a fixed reference configuration κ^R, in the linear fashion. It is also worthwhile to note that in this case $(dW)_p^1 = \boldsymbol{\tau}\cdot\boldsymbol{\mathcal{D}}^p(1)\,dt = (dW)_p = \boldsymbol{T}(1)\cdot d^p\boldsymbol{E}(1)$. However, in general neither product represents the invariant (true) plastic work even in the case of materials with the elastic properties insensitive to prior plastic straining.

The similar simple transformation rules valid for unrestricted strains can also be found for the Almansi strain measure ($n = -1$).

4 Eckart–Mandel theoretical framework

4.1 Elementary connections between H–R and E–M approaches in terms of generalized coordinates

(i) Within this familiar framework (cf., e.g., [30], [31]) one employs the conceptual instantaneous unloaded configuration, say $\kappa^*(t)$, such which m.e. would have attained had the surface tractions been instantaneously reduced to zero (at the instant t) keeping all internal variables H constant. This operational definition does not impose any restriction on the orientation of the unloaded m.e. in the space (the laboratory), hence the unloaded configuration is defined with the accuracy to the rigid body rotation. The concept is associated with the multiplicative decomposition of the deformation tensor $\boldsymbol{F} = \boldsymbol{F}^e\boldsymbol{F}^p$. The total deformation tensor $\boldsymbol{F}$ maps some fixed reference configuration κ_R of m.e. into the actual one $\kappa(t)$ (mass density $\varrho(t)$), the "permanent deformation" $\boldsymbol{F}^p$ maps κ_R into $\kappa^*(t)$

(mass density $\varrho^*(t)$), whereas $\boldsymbol{F}^e$ maps $\kappa^*(t)$ into $\kappa(t)$. Both $\boldsymbol{F}^e$ and $\boldsymbol{F}^p$ are defined within the accuracy to a rigid body rotation, say $\boldsymbol{\mathcal{R}}$ ($\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^T = \boldsymbol{1}$), i.e., both $\boldsymbol{F}^e\boldsymbol{F}^p$ and $(\boldsymbol{F}^e\boldsymbol{\mathcal{R}}), (\boldsymbol{\mathcal{R}}^T\boldsymbol{F}^p)$ are admissible multiplicative decompositions of $\boldsymbol{F}$. The aspects of the solid mechanics discussed in the further part of this paper are valid for any choice of $\boldsymbol{\mathcal{R}}$, therefore $\boldsymbol{\mathcal{R}}$ will not be specified here. In passing we recall that the strain counterpart $\boldsymbol{E}_*(n)$ of the generalized measure of permanent changes in m.e. shape $\overset{*}{q}{}^i$ (Sect. 3.2) can be identified with any member of the permanent strain family

$$2n\boldsymbol{E}_*(n) = (\boldsymbol{U}^p)^{2n} - \boldsymbol{1}, \quad \boldsymbol{F}^p = \boldsymbol{R}^p\boldsymbol{U}^p \tag{98}$$

which is independent of $\boldsymbol{\mathcal{R}}$. The basic inconvenience in a direct application of $\boldsymbol{E}_*(n)$ and $\dot{\boldsymbol{E}}_*(n)$ is that their experimental determination requires the knowledge of m.e. dimensions in a certain past reference configuration.

The description of the actual shape and orientation of an m.e. in $\kappa(t)$ within the E–M theoretical framework is made relative to a moveable reference configuration $\kappa^*(t)$. Therefore, the associated description of m.e. properties may be referred to as "mobile Lagrangean description". The motion of $\kappa^*(t)$ is governed by the plastic flow rule, and $\boldsymbol{F}^p$ depends on structural changes associated plastic deformation, i.e., $\boldsymbol{F}^p = \boldsymbol{F}^p(H)$. In particular $\det(\boldsymbol{F}^p) = J^p(H)$ and the mass density ϱ^* in $\kappa^*(t)$ may also be influenced by prior plastic straining, $\varrho^*(t) = \varrho^*[H(t)]$. The state of m.e. in $\kappa^*(t)$ is called "the instantaneous natural state" [3].

The familiar kinematic quantities implied by the multiplicative decomposition of $\boldsymbol{F}$ are: the total $\boldsymbol{L}$, elastic $\boldsymbol{L}^e$ and permanent $\boldsymbol{L}^*$ deformation-rate tensors

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\overset{-1}{\boldsymbol{F}}, \quad \boldsymbol{L}^e = \dot{\boldsymbol{F}}^e\overset{-1}{\boldsymbol{F}}{}^e, \quad \boldsymbol{L}^* = \dot{\boldsymbol{F}}^p\overset{-1}{\boldsymbol{F}}{}^p, \quad \boldsymbol{L}^p \equiv \boldsymbol{F}^e\boldsymbol{L}^*\overset{-1}{\boldsymbol{F}}{}^e, \tag{99.1-4}$$

The Eulerian permanent deformation-rate tensor $\boldsymbol{L}^p$ is the component of the additive decomposition of $\boldsymbol{L}$, i.e., $\boldsymbol{L} = \boldsymbol{L}^e + \boldsymbol{L}^p$. The fundamental role plays the known *Eulerian elastic strain rate* $\boldsymbol{D}^e$, *Eulerian measure of the permanent strain-rate* $\boldsymbol{D}^p$ and Eulerian measure $\boldsymbol{\omega}^p$ of the rotation of $\kappa^*(t)$ defined by

$$2\boldsymbol{D}^e = \boldsymbol{L}^e + \overset{T}{\boldsymbol{L}}{}^e, \quad 2\boldsymbol{D}^p = \boldsymbol{L}^p + \overset{T}{\boldsymbol{L}}{}^p, \quad \boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p, \quad 2\boldsymbol{\omega}^p = \boldsymbol{L}^p - \overset{T}{\boldsymbol{L}}{}^p. \tag{100.1-4}$$

Similarly as $\boldsymbol{L}$, the elastic deformation-rate tensor $\boldsymbol{L}^e$ is independent of a reference configuration in the sense that $\boldsymbol{L}^e$ does not change when $\boldsymbol{F}^e\,\boldsymbol{A}$ is substituted for $\boldsymbol{F}^e$ ($\boldsymbol{A} = \text{const}$, $\det\boldsymbol{A} \neq 0$). In particular, on purely kinematical grounds we have $\boldsymbol{L}^e = \boldsymbol{L}$ in all those time periods when $\dot{\boldsymbol{F}}^p = \boldsymbol{0}$ ($\boldsymbol{L}^* = \boldsymbol{L}^p = \boldsymbol{0}$). It is also worthwhile to emphasize that $\text{tr}\boldsymbol{L}^p = \text{tr}\boldsymbol{D}^p = \text{tr}\boldsymbol{L}^* = \dot{J}^p/J^p$ is the proper measure of the rate of permanent dilatational changes of m.e. in the stress-free state. The increment $\boldsymbol{D}^p\,dt$ should be carefully distinguished from the plastic part of any total strain increment defined within the H–R formalism (Sect. 3.2). In particular, the increment $\boldsymbol{\mathcal{D}}^p(n)\,dt$ defined in the vicinity of the current configuration [cf. (93)] is in general not equal to $\boldsymbol{D}^p dt$. The exploration of the relation between $\boldsymbol{\mathcal{D}}^p(n)$ and $\boldsymbol{L}^p$ is therefore meaningful (see Sect. 4.4).

(ii) To find the formal connection between the total Lagrangean description used in Sect. 3 and the mobile Lagrangean description let us identify $\boldsymbol{F}_1$, $\boldsymbol{A}$ and $\boldsymbol{U}_1$ occurring in (61) and (62) with $\boldsymbol{F}^e$, $\boldsymbol{F}^p$ and $\boldsymbol{U}^e$ ($\boldsymbol{F}^e = \boldsymbol{R}^e\boldsymbol{U}^e$), respectively. Let us also identify the "new" strain $\boldsymbol{E}_1$ with the elastic strain measure $\boldsymbol{E}^e(n^*)$ ($n_1 = n^*$) belonging to the family (4). The relation (62) may then be rewritten in the symbolic form

$$\boldsymbol{E}^e(n^*) = \boldsymbol{\Psi}[\boldsymbol{E}(n),\ \boldsymbol{F}^p(H)] = \boldsymbol{\Psi}_1[\boldsymbol{E}(n), H], \tag{101}$$

where $\boldsymbol{\Psi}_1$ depends on the parameter n and n^* what is not explicitly marked in (101), and we have $\boldsymbol{\Psi}_1[\boldsymbol{E}_*(n^*), H] = 0$ [cf. (98)]. The relation between the elastic strain and total strain depends on the internal state of m.e. through the dependence of $\boldsymbol{F}^p$ on H. More generally, the counterpart of (101) written in terms of generalized coordinates

$$q^{\alpha} = q^{\alpha}(q^i, H), \quad q^i = q^i(q^{\alpha}, H), \quad q^{\alpha}(\overset{*}{q}{}^i, H) = 0, \quad q^i(0, H) = \overset{*}{q}{}^i \tag{102.1-4}$$

may be regarded as a transformation from fixed to moveable coordinates in the six-dimensional space. The array q^{α} describes the actual shape of m.e. relative to its shape in the mobile unloaded configuration κ^*. It is therefore the counterpart of the elastic strain.

During elastic infinitesimal processes $dH = 0$, whence the transformation rules for δq^i presented in (65) do not change. However, when $dH \neq 0$ the transformation rule for dq^i is different from (65),

$$dq^{\alpha} = Q_i^{\alpha} dq^i + d^p q^{\alpha}, \quad dq^i = Q_{\alpha}^i dq^{\alpha} + d^h q^i, \tag{103.1,2}$$

$$d^h q^i + Q_{\alpha}^i d^p q^{\alpha} = 0, \quad d^p q^{\alpha} + Q_i^{\alpha} d^h q^i = 0, \tag{103.3,4}$$

where the array Q_i^{α} and its inverse Q_{α}^i are defined in (66). The new differential $d^h\mathcal{A}$ of a property $\mathcal{A}$ can be applied only when $\mathcal{A}$ is expressed in terms of new coordinates q^{α} representing the elastic strain or in terms of its energy-conjugate p_{α} [cf. (106)]. It has the following meaning:

$$\begin{aligned} d^h\mathcal{A} &= \mathcal{A}(q^{\alpha}, H + dH) - \mathcal{A}(q^{\alpha}, H), \\ d^h\mathcal{A} &= \mathcal{A}(p_{\alpha}, H + dH) - \mathcal{A}(p_{\alpha}, H). \end{aligned} \tag{104}$$

The meaning of d^p-differentials remains unchanged [cf. (44)]. The identities (103) are implied by the mapping (102).

(iii) The specific free energy $\phi = \phi_0 + \phi_e(q^i, H) + \phi_s(H)$ and the elastic strain energy $\phi_e(q^i, H)$ (cf. Sect. 3.2) are invariant under the coordinate transformation (102)

$$\phi_e(q^i, H) = \phi_e(q^{\alpha}, H)\big|_{q^{\alpha}(q^i, H)}. \tag{105}$$

This is one of the basic formulae that link H–R and E–M theoretical frameworks. *In practice, the function $\phi_e(q^{\alpha}, H)$ is of primary importance*. The elastic strain can be measured in the course of unloading, and its experimental determination does not require knowledge of the physical dimensions of m.e. in any past reference configuration. The application of δ − differential to (105) yields

$$\delta\phi_e = p_i \delta q^i = p_{\alpha} \delta q^{\alpha}, \tag{106.1}$$

$$p_{\alpha} = Q_{\alpha}^i p_i = \frac{\partial \phi_e}{\partial q^{\alpha}}, \quad p_i = Q_i^{\alpha} p_{\alpha}, \tag{106.2,3}$$

where $\phi_e = \partial\phi_e/\partial q^{\alpha} = 0$ for $q^{\alpha} = 0$ on account of (48.2, 3) and (102.3, 4). The new force p_{α} is the energy-conjugate to the elastic-coordinates q^{α}. Note that the dual potential $\psi_e^{(1)}(p_{\alpha}, H)$ defined by

$$\psi_e^{(1)}(p_{\alpha}, H) = \phi_e(q^{\alpha}, H) - p_{\alpha} q^{\alpha} \Rightarrow q^{\alpha} = -\frac{\partial \psi_e^{(1)}}{\partial p_{\alpha}}, \quad \psi_e^{(1)} = \frac{\partial \psi_e^{(1)}}{\partial p_{\alpha}} = 0 \text{ for } p_{\alpha} = 0 \tag{107.1-3}$$

has a different physical meaning than ψ_e defined in (49.2) (Sect. 3.2).

Consider a non-elastic infinitesimal process $dH \neq 0$ and calculate the d − differential of (105). The total incremental work dW, the invariant incremental elastic work $dW^e = d\phi_e$, and the invariant incremental plastic work $dW^p = -d^p\phi_e(q^i, H)$ [cf. (82) and (84)] are expressed within E–M theoretical framework in terms of p_{α}, q^{α} as follows:

$$dW = p_i dq^i = dW^e + dW^p = p_{\alpha} dq^{\alpha} + p_i d^h q^i(q^{\alpha}, H), \quad p_i d^h q^i = Q_i^{\alpha} p_{\alpha} d^h q^i, \tag{108}$$

$$dW^e = p_{\alpha} dq^{\alpha} + d^h\phi_e(q^{\alpha}, H), \quad dW^p = p_i d^h q^i - d^h\phi_e \tag{109}$$

on account of (103.2) and (106.2). The expression (85) for the incremental energy dissipation $dD = -d^p\phi(q^i, H)$ becomes

$$dD = dW^p - d\phi_s(H) = p_i d^h q^i - d^h\phi_e - d\phi_s(H) \geq 0. \tag{110}$$

The relations (109) and (110) are the other examples of basic connections between H–R and E–M formalisms. It is seen that during plastic yielding ($dH \neq 0$) the incremental elastic work dW^e is not equal to $p_\alpha dq^\alpha$ unless the elastic properties of a material are not influenced by prior plastic straining. The incremental form of (106.2) and (107.2) at plastic yielding is

$$\begin{aligned} &dp_\alpha = \frac{1}{\varrho_0^*} l_{\alpha\beta} dq^\beta + d^h p_\alpha(q^\eta, H) \Leftrightarrow dq^\alpha = \varrho_0^* m^{\alpha\beta} dp_\beta + d^h q^\alpha(p_\eta, H), \\ &l_{\alpha\beta} m^{\beta\gamma} = \delta_\alpha^\gamma, \quad d^h p_\alpha(q^\alpha, H) = -\frac{1}{\varrho_0^*} l_{\alpha\beta} d^h q^\beta(p_\eta, H). \end{aligned} \tag{111}$$

Here ϱ_0^* is a *a certain conventional constant mass density*, $l_{\alpha\beta}$ are elastic tangent moduli defined in an unloaded state $\kappa^*(t)$ and associated with the *selected elastic strain measure*, and $m^{\alpha\beta}$ are corresponding elastic compliances,

$$\frac{1}{\varrho_0^*} l_{\alpha\beta} = \frac{\partial^2\phi_e}{\partial q^\alpha \partial q^\beta}, \quad \varrho_0^* m^{\alpha\beta} = -\frac{\partial^2\psi_e^{(1)}}{\partial p_\alpha \partial p_\beta}. \tag{112}$$

The increment $d^h q^\alpha(p_\eta, H)$ and its dual $d^h p_\alpha(q^\eta, H)$ defined by

$$d^h q^\alpha \equiv -\frac{\partial d^h\psi_e^{(1)}(p_\eta, H)}{\partial p_\alpha}, \quad d^h p_\alpha \equiv \frac{\partial d^h\phi_e(q^\eta, H)}{\partial q^\alpha} \tag{113}$$

represent the deformation and stress effects of the mechanisms (e.g., damaging, compacting, void-nucleation, pressure sensitivity) responsible for the sensitivity of elastic properties to prior plastic straining.

When the specific elastic strain energy is independent of H ($d^h\phi_e = d^h\psi_e^{(1)} = 0 \Rightarrow dW^e = p_\alpha dq^\alpha$, $dW^p = p_i d^h q^i$) these effects are disregarded. Such materials were classified in [4] as "materials with the elastic properties insensitive to prior plastic straining". Here we shall distinguish in addition another class of materials characterized by the following form of ϕ_e:

$$\phi_e(q^\alpha, H) = \frac{1}{\varrho^*(H)} \bar{\phi}_e(q^\alpha). \tag{114}$$

In this situation the density of the elastic strain energy $\bar{\phi}_e$ in the unloaded (natural) state $\kappa^*(t)$ (elastic strain energy per unit of volume in κ^*) is independent of prior plastic straining. The tangent elastic moduli $\bar{l}_{\alpha\beta}$ defined as

$$\frac{\varrho^*(H)}{\varrho_0^*} l_{\alpha\beta} = \bar{l}_{\alpha\beta}(q^\alpha), \quad \bar{l}_{\alpha\beta}(q^\alpha) = \frac{\partial^2\bar{\phi}_e}{\partial q^\alpha \partial q^\beta} \tag{115}$$

are the same in every stress-free state $\kappa^*(t)$, and moreover

$$d^h\phi_e = -\frac{d\varrho^*(H)}{\varrho^*(H)}\phi_e \Rightarrow dW^p = p_i d^h q^i + \frac{d\varrho^*}{\varrho^*(H)}\phi_e, \quad d^h p_\alpha = -\frac{d\varrho^*}{\varrho^*} p_\alpha. \tag{116}$$

Note also that (107.1–3) implies the identity $d^h\psi_e^{(1)}(p_\alpha, H) = d^h\phi_e(q^\alpha, H)$ that holds for every dual pair (p_α, q^α) connected by (106.2) or (107.2). Therefore, $d^h\phi_e$ occurring in (109) and (110) may be replaced by $d^h\psi_e^{(1)}(p_\alpha, H)$.

4.2 Example of the basic connections in tensor notation

(i) The connection (101) is linear when Green ($n = 1$, $n^* = 1$) or Almansi ($n = -1$, $n^* = -1$) strain measure are used to define elastic and total strain. For Green strain measure the relation (101) $q^i \leftrightarrow \boldsymbol{E}(1), q^\alpha \leftrightarrow \boldsymbol{E}^e(1)$ is

$$\boldsymbol{E}(1) = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})\boldsymbol{E}^e(1) + \boldsymbol{E}_*(1), \quad \boldsymbol{E}^e(1) = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}^p})[\boldsymbol{E}(1) - \boldsymbol{E}_*(1)] \tag{117}$$

where the special fourth-order tensor function $\boldsymbol{\Theta}$ is defined in (2). The quantity $\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})$ corresponds to Q^i_α occurring in (103) and defined in (66). According to (106.2) the transpose of $\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})$ links the usual specific Piola–Kirchhoff tensor $\boldsymbol{T}(1)$ with the specific "elastic" stress tensor $\boldsymbol{T}^e(1)$ which is the energy-conjugate to the elastic Green strain measure $\boldsymbol{E}^e(1)$ ($\boldsymbol{T}(1) = \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{F}})\boldsymbol{\tau} \leftrightarrow p_i, \boldsymbol{T}^e(1) \leftrightarrow p_\alpha$)

$$\boldsymbol{T}^e(1) = \overset{T}{\boldsymbol{\Theta}}(\overset{T}{\boldsymbol{F}^p})\boldsymbol{T}(1) = \boldsymbol{\Theta}(\boldsymbol{F}^p)\boldsymbol{T}(1) = \boldsymbol{\Theta}(\overset{-1}{\boldsymbol{F}^e})\boldsymbol{\tau} = \overset{-1}{\boldsymbol{F}^e}\boldsymbol{\tau}\overset{-T}{\boldsymbol{F}^e} \tag{118}$$

on account of the property (3.4) of $\boldsymbol{\Theta}$. Since we have $\dot{\boldsymbol{E}}(1) = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D} = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D}^e + \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D}^p$ and $\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D}^e = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^e})\boldsymbol{D}^e = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})\dot{\boldsymbol{E}}^e(1)$ the counterpart of (103.1) for the considered situation at the plastic yield point is

$$\begin{aligned} \dot{\boldsymbol{E}}(1) &= \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^p})\dot{\boldsymbol{E}}^e(1) + \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D}^p, \\ \dot{\boldsymbol{E}}^e(1) &= \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}^p})\dot{\boldsymbol{E}}(1) - \boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^e})\boldsymbol{D}^p, \end{aligned} \tag{119}$$

where $\boldsymbol{D}^p$ is the Eulerian measure of the permanent strain rate defined by (100.2).

The quantities $\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}})\boldsymbol{D}^p dt$ and $-\boldsymbol{\Theta}(\overset{T}{\boldsymbol{F}^e})\boldsymbol{D}^p dt$, associated with the Green measure in the physical space, correspond to the arrays $d^h q^i$ and $d^p q^\alpha$ occurring in (103), respectively. Similar expressions hold also for the Almansi strain tensor $\boldsymbol{E}(-1)$.

(ii) In view of (119) the invariant incremental works (108)–(109) and the incremental energy dissipation per unit of mass dD in the course of active elastic–plastic straining are

$$dW = \boldsymbol{T}(1) \cdot d\boldsymbol{E}(1) = \boldsymbol{T}^e(1) \cdot d\boldsymbol{E}^e(1) + \boldsymbol{\tau} \cdot \boldsymbol{D}^p dt, \quad \boldsymbol{T}^e(1) \cdot d\boldsymbol{E}^e(1) = \boldsymbol{\tau} \cdot \boldsymbol{D}^e dt, \tag{120}$$

$$\begin{aligned} dW^e &= \boldsymbol{T}^e(1) \cdot d\boldsymbol{E}^e(1) + d^h\phi_e[\boldsymbol{E}^e(1), H] = \boldsymbol{\tau} \cdot \boldsymbol{D}^e dt + d^h\phi_e[\boldsymbol{E}^e(1), H], \\ dW^p &= \boldsymbol{\tau} \cdot \boldsymbol{D}^p dt - d^h\phi_e[\boldsymbol{E}^e(1), H], \\ dD &= \boldsymbol{\tau} \cdot \boldsymbol{D}^p dt - d^h\phi_e[\boldsymbol{E}^e(1), H] - d\phi_s(H) \geq 0 \end{aligned} \tag{121}$$

since $p_\alpha dq^\alpha = \boldsymbol{T}^e(1) \cdot d\boldsymbol{E}^e(1)$ and $p_i d^h q^i = \boldsymbol{\tau} \cdot \boldsymbol{D}^p dt$. The latter term is sometimes (cf., e.g., [32]) presented in the form $\boldsymbol{\tau} \cdot \boldsymbol{D}^p = \boldsymbol{P}^* \cdot (\overset{T}{\boldsymbol{L}^*})$ where the non-symmetric tensor $\boldsymbol{P}^*$ defined by

$$\boldsymbol{P}^* \equiv \overset{-1}{\boldsymbol{F}^e}\boldsymbol{\tau}\boldsymbol{F}^e = \overset{-1}{\boldsymbol{U}^e}\boldsymbol{\tau}^{Re}\boldsymbol{U}^e, \quad \boldsymbol{\tau}^{Re} \equiv \overset{T}{\boldsymbol{R}^e}\boldsymbol{\tau}\boldsymbol{R}^e = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}^e})\boldsymbol{\tau} \tag{122}$$

is referred to as Mandel stress tensor and $\boldsymbol{L}^*$ is defined in (99.3). We do not pursue this concept because it increases the number of thermodynamical rates occurring in the expression for dD. As remarked in [23] nine components of $\boldsymbol{P}^*$ are not independent.

4.3 Arbitrary elastic strain measure as a state parameter

(i) When applying E–M theoretical framework (cf., e.g., [31], [25, Chap. 11]) the most frequently one uses the Green elastic strain measure $\boldsymbol{E}^e(1)$. With this measure there are associated simple and exact relations between the elastic strain-rate measures defined in $\kappa^*(t)$ and $\kappa(t)$, as well as between

corresponding rates of energy-conjugate stresses. However, as shown in Sect. 4.1 the application of any other measure belonging to the family $2n\boldsymbol{E}^e(n) = (\boldsymbol{U}^e)^{2n} - \boldsymbol{1}$ is also admissible. In particular, the application of the logarithmic elastic strain measure $\boldsymbol{E}^e(0)$ has a lot of merits [33], [34]. In the case of materials isotropic with respect to the elastic properties one can establish exact and simple connections between increments of the logarithmic elastic strains defined in $\overset{*}{\kappa}(t)$ and $\kappa(t)$, as well as between the increments of their energy conjugate stresses. Moreover, when elastic distortion $\boldsymbol{E}^{ed}(0) = \mathrm{dev}\boldsymbol{E}^e(0)$ is a small quantity the simple transition from $\overset{*}{\kappa}(t)$ to $\kappa(t)$ of all discussed properties can be made for any elastic strain measure (cf. Sect. a).

The increments $d^h\phi_e$ and dW^e *are invariant under the change of elastic strain measure*, and in the course of elastic deformation ($\dot{H} = 0 \Rightarrow \boldsymbol{L}^* = 0, \boldsymbol{D}^p = 0$) the incremental total work is equal to dW^e so that the resulting equality

$$dW|_{\dot{H}=0} = dW^e|_{\dot{H}=0} = \boldsymbol{\tau}\cdot\boldsymbol{D}^e dt = \boldsymbol{\tau}^{Re}\cdot\boldsymbol{D}^{Re}dt = \boldsymbol{T}^e(1)\cdot d\boldsymbol{E}^e(1) = \boldsymbol{T}^e(n)\cdot d\boldsymbol{E}^e(n) \tag{123}$$

defines the elastic stress measure $\boldsymbol{T}^e(n)$ energy conjugate to elastic strain $\boldsymbol{E}^e(n)$. Here $\boldsymbol{D}^{Re} = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}}{}^e)\boldsymbol{D}^e = \overset{T}{\boldsymbol{R}}{}^e\boldsymbol{D}^e\boldsymbol{R}^e$. There is full analogy between (25) and (123). All formulas presented in Sect. 3.1 can be directly applied to find the corresponding relations between elastic quantities. It is sufficient to replace $\boldsymbol{F}$ by $\boldsymbol{F}^e(\boldsymbol{F}\to\boldsymbol{F}^e)$, $\boldsymbol{N}_K \to \boldsymbol{N}^e_K$ (principal direction of $\boldsymbol{U}^e$), $\lambda_K \to \lambda^e_K$ (principal elastic stretches), $\boldsymbol{U}\to\boldsymbol{U}^e$, $\boldsymbol{n}_K \to \boldsymbol{n}^e_K$ (principal directions of $\boldsymbol{V}^e$), $\boldsymbol{V}\to\boldsymbol{V}^e$, $\boldsymbol{R}\to\boldsymbol{R}^e$, $J\to J^e = \det\boldsymbol{F}^e$, $\boldsymbol{E}(n)\to\boldsymbol{E}^e(n)$, $\boldsymbol{e}(n)\to\boldsymbol{e}^e(n)$, $\boldsymbol{D}\to\boldsymbol{D}^e$, $\boldsymbol{D}^R\to\boldsymbol{D}^{Re}$, $\boldsymbol{\omega}\to\boldsymbol{\omega}^e = 0.5(\boldsymbol{L}^e - \overset{T}{\boldsymbol{L}}{}^e) = \boldsymbol{\omega} - \boldsymbol{\omega}^p$, $\boldsymbol{\omega}^R\to\boldsymbol{\omega}^{Re} = \dot{\boldsymbol{R}}^e\overset{T}{\boldsymbol{R}}{}^e$, $\boldsymbol{\tau}^R\to\boldsymbol{\tau}^{Re}$, $\boldsymbol{T}(n)\to\boldsymbol{T}^e(n)$, $\dot{\boldsymbol{T}}(n)\to\dot{\boldsymbol{T}}^e(n)$ and $\varrho^R = \varrho^*_0$ in all formulas of Sect. 3.1. For example the analogue of (30) now becomes

$$\frac{\boldsymbol{\mathcal{D}}^{(n)}_e\boldsymbol{\sigma}}{\boldsymbol{\mathcal{D}}t} \equiv \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\omega}^e - \boldsymbol{\omega}^e\boldsymbol{\sigma} - \frac{\dot{\varrho}}{\varrho}\boldsymbol{\sigma} - n\boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{D}^e = \frac{\boldsymbol{\mathcal{D}}^{(n)}\boldsymbol{\sigma}}{\boldsymbol{\mathcal{D}}t} - \boldsymbol{\sigma}\boldsymbol{\omega}^p + \boldsymbol{\omega}^p\boldsymbol{\sigma} + n\boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{D}^p, \tag{124}$$

where $\boldsymbol{\mathcal{D}}^{(n)}\boldsymbol{\sigma}/\boldsymbol{\mathcal{D}}t$ is the family (30) of Cauchy's stress fluxes.

Likewise, such a procedure gives rise to the new notion of "elastic logarithmic spin" $\boldsymbol{\omega}^{e(0)}$ [cf. (19)]

$$\bar{\omega}^{e(0)}_{LK} = \bar{\omega}^e_{LK} + \left[\frac{(\lambda^e_K)^2 + (\lambda^e_L)^2}{(\lambda^e_K)^2 - (\lambda^e_L)^2} - \frac{1}{\ln(\lambda^e_K/\lambda^e_L)}\right]\bar{D}^e_{LK}, \tag{125}$$

where overbarred arrays are now the components of $\boldsymbol{\omega}^{e\,(0)}$, $\boldsymbol{\omega}^e$ and $\boldsymbol{D}^e$ on the principal triad $\boldsymbol{n}^e_K$ of the elastic stretch tensor. The associated "elastic logarithmic co-rotational rate" $\boldsymbol{\mathcal{D}}^{\log}(.)/\boldsymbol{\mathcal{D}}t$ may be defined in an analogous manner as (20) (substitute $\boldsymbol{\omega}^{e\,(0)}$ for $\boldsymbol{\omega}^{(0)}$) such that $\boldsymbol{\mathcal{D}}^{\log}\boldsymbol{e}^e(0)/\boldsymbol{\mathcal{D}}t = \boldsymbol{D}^e$. In general $\boldsymbol{\mathcal{D}}^{\log}\boldsymbol{X}/\boldsymbol{\mathcal{D}}t \neq \boldsymbol{\mathcal{D}}^{\log}\boldsymbol{X}/\boldsymbol{\mathcal{D}}t$. However, there exists the special kinematic relation

$$\frac{\boldsymbol{\mathcal{D}}^{\log}\boldsymbol{e}^e(0)}{\boldsymbol{\mathcal{D}}t} = \frac{\boldsymbol{\mathcal{D}}^{\log}\boldsymbol{e}(0)}{\boldsymbol{\mathcal{D}}t} = \boldsymbol{D} = \boldsymbol{D}^e \tag{126}$$

that holds in the course of elastic processes, i.e., provided that $\dot{\boldsymbol{F}}^p = \boldsymbol{0}$. Logarithmic co-rotational rates of the associated logarithmic strains are not dependent on the choice of the fixed reference configuration since stretching $\boldsymbol{D}$ has this property.

(ii) Let $\phi_e = [\boldsymbol{E}^e(n), H]$ be the elastic strain energy expressed in terms of arbitrary elastic strain measure and denote by $\psi^{(1)}_e[\boldsymbol{T}^e(n), H] \equiv \phi_e - \boldsymbol{T}^e(n)\cdot\boldsymbol{E}^e(n)$ the dual potential. Now the basic state equations in the unloaded configuration $\overset{*}{\kappa}(t)$ are

$$\boldsymbol{T}^e(n) = \frac{\partial\phi_e[\boldsymbol{E}^e(n), H]}{\partial\boldsymbol{E}^e(n)}, \quad \boldsymbol{E}^e(n) = -\frac{\partial\psi^{(1)}_e[\boldsymbol{T}^e(n), H]}{\partial\boldsymbol{T}^e(n)} \tag{127}$$

on account of (123). The equations defining the incremental elastic, true (invariant) plastic work and the energy dissipation have forms similar to (121). It suffices to replace in (121) $d^h\phi_e[\boldsymbol{E}^e(1), H]$ with

$d^h\phi_e[\boldsymbol{E}^e(n), H]$. Note that in the case of materials with the elastic properties insensitive to prior plastic straining ($d^h\phi_e = 0$) we have

$$dW^p = \boldsymbol{\tau} \cdot \boldsymbol{D}^p \, dt \quad (d^h\phi_e = 0) \tag{128}$$

where $\boldsymbol{D}^p$ is defined in (100.2).

The incremental form of the generalized state relations (127) in $\kappa^*(t)$ is

$$\begin{aligned} d\boldsymbol{T}^e(n) &= \frac{1}{\varrho_0^*} \, \boldsymbol{\mathcal{L}}^*(n) \, d\boldsymbol{E}^e(n) + d^h\boldsymbol{T}^e(n), \\ d\boldsymbol{E}^e(n) &= \varrho_0^* \, \boldsymbol{\mathcal{M}}^*(n) \, d\boldsymbol{T}^e(n) + d^h\boldsymbol{E}^e(n), \end{aligned} \tag{129}$$

where

$$\frac{1}{\varrho_0^*} \, \boldsymbol{\mathcal{L}}^*(n) = \frac{\partial^2 \phi_e}{\partial \boldsymbol{E}^e(n) \, \partial \boldsymbol{E}^e(n)}, \quad \varrho_0^* \, \boldsymbol{\mathcal{M}}^*(n) = -\frac{\partial^2 \psi_e^{(1)}}{\partial \boldsymbol{T}^e(n) \, \partial \boldsymbol{T}^e(n)} \; (\boldsymbol{\mathcal{M}}^* = \overset{-1}{\boldsymbol{\mathcal{L}}}{}^*) \tag{130}$$

are elastic tangent moduli and compliances defined in $\kappa^*(t)$ and associated with the selected elastic strain measure $\boldsymbol{E}^e(n)$. The increments

$$d^h\boldsymbol{T}^e(n) = \frac{\partial \, d^h\phi_e}{\partial \boldsymbol{E}^e(n)}, \quad d^h\boldsymbol{E}^e(n) = -\frac{\partial \, d^h\psi_e^{(1)}}{\partial \boldsymbol{T}^e(n)}, \quad d^h\boldsymbol{T}^e(n) = -\frac{1}{\varrho_0^*} \, \boldsymbol{\mathcal{L}}^*(n) \, d^h\boldsymbol{E}^e(n) \tag{131}$$

describe the stress and deformation effects associated with the sensitivity of elastic properties to prior plastic straining. Equations (129)–(131) are the counterparts of (111)–(113) in the physical space.

4.4 Rate form of state equations in updated Lagrangean configuration κ(t)

a. General case

(i) In field theories of an elastic–plastic continuum one formulates fundamental rate boundary-value problems either in a fixed or in the updated Lagrangean configuration. To this end one has first to transfer the incremental relations (129)–(131) from configuration $\kappa^*(t)$ to κ^R or to $\kappa(t)$.

We shall here restrict the attention to the latter situation. The general rate equations in the current configuration have the form (93), and the question arises what are the relations between the physical quantities occurring in (129) and (93). They are presented beneath without discussing details of their derivation. Consider the following unified transformation rule:

$$\frac{1}{\varrho} \, \boldsymbol{\mathcal{L}}(n) = \frac{1}{\varrho_0^*} \boldsymbol{\Theta}(\boldsymbol{\mathcal{F}}) \, \boldsymbol{\mathcal{L}}^*(n) \boldsymbol{\Theta}(\overset{T}{\boldsymbol{\mathcal{F}}}) \; , \; \boldsymbol{\mathcal{M}}(n) = \overset{-1}{\boldsymbol{\mathcal{L}}}(n), \tag{132.1}$$

$$\boldsymbol{\mathcal{D}}^p(n) = [\boldsymbol{I} + n \, \boldsymbol{\mathcal{M}}(n) \, \boldsymbol{P}(\boldsymbol{\sigma})] \, \boldsymbol{D}^p + \boldsymbol{D}^h(n) - \boldsymbol{\mathcal{M}}(n) \, (\boldsymbol{\sigma}\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\boldsymbol{\sigma}), \tag{132.2}$$

$$\boldsymbol{D}^h(n) \, dt = \boldsymbol{\Theta}(\overset{-T}{\boldsymbol{\mathcal{F}}}) \, d^h\boldsymbol{E}^e(n) = -\varrho \, \boldsymbol{\mathcal{M}}(n) \, \boldsymbol{\Theta}(\boldsymbol{\mathcal{F}}) \, d^h\boldsymbol{T}^e(n), \tag{132.3}$$

where $d^h\boldsymbol{T}^e(n)$ and $d^h\boldsymbol{E}^e(n)$ are defined in (131). These are exact connections for $n = 1$ provided $\boldsymbol{\mathcal{F}} = \boldsymbol{F}^e$, and for $n = -1$ when $\boldsymbol{\mathcal{F}} = \overset{-T}{\boldsymbol{F}}{}^e$ ("push forward" the physical quantities into $\kappa(t)$ using $\boldsymbol{\Theta}(\boldsymbol{F}^e)$ and $\boldsymbol{\Theta}(\overset{-T}{\boldsymbol{F}^e})$.) To justify this statement it is sufficient to substitute the elastic counterparts of (10.1) and (31) (written for $\boldsymbol{F}^e$) together with (124) into (129). For anisotropic solids and other elastic strain measures the exact index-free relations are not known. However, when *elastic distortions are small*, what is the most frequent case in practice, one may use the elastic analogues of (34)–(38) specified for $\boldsymbol{F}^e$ and arrive at Eqs. (132) with

$$\boldsymbol{\mathcal{F}} = (J^e)^{n/3}\,\boldsymbol{R}^e. \tag{133}$$

Such equations are valid for arbitrary n *with an error of first-order in the deviatoric part of the logarithmic elastic strain*. Within the same accuracy we have (cf. Sect. 3.1.c)

$$\boldsymbol{\omega}^{Re} = \dot{\boldsymbol{R}}^e \overset{T}{\boldsymbol{R}}\, e = \boldsymbol{\omega} - \boldsymbol{\omega}^p + \boldsymbol{\mathcal{O}}[\boldsymbol{e}^{ed}(0)]\,\boldsymbol{D}^e_d, \tag{134}$$

where $\boldsymbol{D}^e_d$ and $\boldsymbol{e}^{ed}(0)$ are deviators of $\boldsymbol{D}^e$ and Eulerian elastic logarithmic strain $\boldsymbol{e}^e(0)$, respectively. This equation can be used to determine $\boldsymbol{R}^e$. The relations (132) and (133) are valid for both isotropic and anisotropic materials.

b. Isotropic elastic behavior – logarithmic elastic strain as a state variable

(i) Let us assume that $\phi_e[\boldsymbol{E}^e(n), H]$ is an isotropic tensor valued function of $\boldsymbol{E}^e(n)$ (all H occurring in ϕ_e are scalars – but not necessary those present in ϕ_s) and recall some general properties of isotropic elastic behavior described by the $\boldsymbol{T}^e(n) \leftrightarrow \boldsymbol{E}^e(n)$ tensor connection (127):

(a)

$$\boldsymbol{T}^e(n) = \frac{\partial \phi_e[\boldsymbol{E}^e(n), H]}{\partial \boldsymbol{E}^e(n)} \;\Rightarrow\; \boldsymbol{t}^e(n) \equiv \boldsymbol{\Theta}(\boldsymbol{R}^e)\,\boldsymbol{T}^e(n) = \frac{\partial \phi_e[\boldsymbol{e}^e(n), H]}{\partial \boldsymbol{e}^e(n)} \tag{135.1,2}$$

where $\boldsymbol{e}^e(n) = \boldsymbol{\Theta}(\boldsymbol{R}^e)\,\boldsymbol{E}^e(n)$, $\phi_e[\boldsymbol{e}^e(n), H] = \phi_e[\boldsymbol{E}^e(n), H]$.

(b) All elastic energy conjugate couples $\{\boldsymbol{T}^e(n) \leftrightarrow \boldsymbol{E}^e(n)\}$ and the tensor $\boldsymbol{\tau}^{Re} = \boldsymbol{\Theta}(\overset{T}{\boldsymbol{R}}{}^e)\,\boldsymbol{\tau}$ are coaxial with $\boldsymbol{U}^e$ (principal directions $\boldsymbol{N}^e_K$). In particular $\boldsymbol{\tau}^{Re} = \boldsymbol{T}^e(0)$ is the energy conjugate to elastic logarithmic strain $\boldsymbol{E}^e(0)$. The justification of this property is similar to that presented in Sect. 3.1.b for the total logarithmic strain.

(c) Likewise, all Eulerian couples $\{\boldsymbol{t}^e(n), \boldsymbol{e}^e(n)\}$, Kirchhoff stress $\boldsymbol{\tau}$, Cauchy's stress $\boldsymbol{\sigma}$ are coaxial with the right elastic stretch tensor $\boldsymbol{V}^e$ (principal directions $\boldsymbol{n}^e_K$). In particular, $\boldsymbol{\tau} = \boldsymbol{t}^e(0) = (\boldsymbol{V}^e)^2\,\boldsymbol{t}^e(1) = (\boldsymbol{V}^e)^{-2}\,\boldsymbol{t}^e(-1)$.

d) In general if the connection between two symmetric tensors (Lagrangean or Eulerian), say $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{E})$, is isotropic then $\boldsymbol{TE} = \boldsymbol{ET}$ and such commutation property entails the identity

$$\frac{\partial T_{KL}}{\partial E_{MN}}\left[E_{KP}\Omega^*_{PL} - \Omega^*_{KP}E_{PL}\right] = T_{MK}\Omega^*_{KN} - \Omega^*_{MK}T_{KN} \tag{136}$$

that holds for an arbitrary skew-symmetric tensor $\boldsymbol{\Omega}^* = -\overset{T}{\boldsymbol{\Omega}}{}^*$. Due to this property one can apply any co-rotational time derivative to $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{E})$ and derive rate relations without affecting the physical content contained in the equality $\dot{\boldsymbol{T}} = (\partial \boldsymbol{T}/\partial \boldsymbol{E})\dot{\boldsymbol{E}}$ (the principle of objectivity concerns relations between physical quantities and not the individual terms occurring in such relations).

(ii) The kinematical relation between the Zaremba–Jaumann derivative of $\boldsymbol{e}^e(0)$ (co-rotational with $\boldsymbol{\omega}^e$) and $\boldsymbol{D}^e$ was derived in [33]. Using (125) it may be written in the following alternative form:

$$\frac{\boldsymbol{\mathcal{D}}_e \boldsymbol{e}^e(0)}{\boldsymbol{\mathcal{D}}t} \equiv \dot{\boldsymbol{e}}^e(0) + \boldsymbol{e}^e(0)\boldsymbol{\omega}^e - \boldsymbol{\omega}^e\boldsymbol{e}^e(0) = (\boldsymbol{I} + \Delta\boldsymbol{\mathcal{E}})\boldsymbol{D}^e \tag{137}$$

where $\Delta\boldsymbol{\mathcal{E}}$ is defined as follows.

Denote by $\Delta\bar{\boldsymbol{\mathcal{E}}}_{JKMN}$ and $\bar{X}_{MN}$ the components of $\Delta\boldsymbol{\mathcal{E}}$ and an arbitrary symmetric second-order tensor $\boldsymbol{X}$, respectively, on the Eulerian elastic triad $\boldsymbol{n}^e_K$. The fourth-order tensor-function $\Delta\boldsymbol{\mathcal{E}}$ is defined by

$$\Delta\bar{\boldsymbol{\mathcal{E}}}_{JKMN}\bar{X}_{MN} = \begin{cases} 0 & \text{for} \quad J = K \qquad (138.1) \\ \left[\dfrac{(\lambda^e_J/\lambda^e_K)^2+1}{(\lambda^e_J/\lambda^e_K)^2-1)}\ln(\lambda^e_J/\lambda^e_K)-1\right]\bar{X}_{JK} & \text{for} \quad J \neq K,\ \ (!J,K) \qquad (138.2) \end{cases}$$

where λ^e_K are the principal elastic stretches and $(\lambda^e_J/\lambda^e_K) = \exp\{e^{ed}_J(0) - e^{ed}_K(0)\}$ ($e^{ed}_K(0)$ are principal values of $\boldsymbol{e}^{ed}(0)$).

It can be shown that $\Delta\boldsymbol{\mathcal{E}}$ has the usual pairwise symmetry property. The most important other properties are

- If an arbitrary symmetric tensor $\boldsymbol{a}$ is coaxial with $\boldsymbol{V}^e$ then

$$(\boldsymbol{I}+\Delta\boldsymbol{\mathcal{E}})\boldsymbol{a} = \boldsymbol{a} \;\Rightarrow\; (\boldsymbol{I}+\Delta\boldsymbol{\mathcal{E}})^{-1}\boldsymbol{a} = \boldsymbol{a}, \quad (\Delta\boldsymbol{\mathcal{E}})\boldsymbol{a} = \boldsymbol{0}, \qquad (139)$$

$$\boldsymbol{\mathcal{L}}^{(0)}\,\Delta\boldsymbol{\mathcal{E}} = \Delta\boldsymbol{\mathcal{E}}\,\boldsymbol{\mathcal{L}}^{(0)}$$

where the moduli $\boldsymbol{\mathcal{L}}^{(0)}$ are defined below in (141).

- The Taylor expansion of the function of λ^e_J/λ^e_K occurring in (138.2)

$$\Delta\bar{\boldsymbol{\mathcal{E}}}_{JKMN}\bar{X}_{MN} = \left\{\frac{1}{3}\left[\left(\frac{\lambda^e_J}{\lambda^e_K}\right)-1\right]^2+\cdots\right\}\bar{X}_{JK} \qquad (140)$$

shows that the leading term is second-order and $\Delta\bar{\boldsymbol{\mathcal{E}}}_{JKMN}$ is small, e.g., when $5/6 \le \lambda^e_J/\lambda^e_K \le 7/6$ the term of curl bracket is less than $1/96$. Thus, $\Delta\boldsymbol{\mathcal{E}}$ is negligible and can be set to be zero in practice.

Now apply the Zaremba–Jaumann derivative $\boldsymbol{\mathcal{D}}^e(.)/\boldsymbol{\mathcal{D}}t$ to the relation $\boldsymbol{\tau} = \partial\phi_e/\partial\boldsymbol{e}^e(0)$ [substitute $n = 0$ and $\boldsymbol{t}^e = \boldsymbol{\tau}$ into (135.2)], and transform the obtained equality to the form that matches with (93). The eventual structure of the terms present in (93) for $n = 0$ is (note the difference between moduli $\boldsymbol{\mathcal{L}}(0)$ and $\boldsymbol{\mathcal{L}}^{(0)}$) :

$$\boldsymbol{\mathcal{L}}(0) = \boldsymbol{\mathcal{L}}^{(0)}\{\boldsymbol{I}+\Delta\boldsymbol{\mathcal{E}}\}, \quad \frac{1}{\varrho}\,\boldsymbol{\mathcal{L}}^{(0)} = \frac{\partial^2\phi_e}{\partial\boldsymbol{e}^e(0)\partial\boldsymbol{e}^e(0)}, \quad \boldsymbol{\mathcal{M}}^{(0)} = \overset{-1}{\boldsymbol{\mathcal{L}}}{}^{(0)}, \qquad (141)$$

$$\begin{aligned}\boldsymbol{\mathcal{D}}^p(0) &= \boldsymbol{D}^p + \boldsymbol{D}^h(0) - \overset{-1}{\boldsymbol{\mathcal{L}}}(0)\,(\boldsymbol{\sigma}\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\boldsymbol{\sigma}), \\ &= \boldsymbol{D}^p + \boldsymbol{D}^h(0) - (\boldsymbol{I}+\Delta\boldsymbol{\mathcal{E}})^{-1}[\boldsymbol{e}^{ed}(0)\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\boldsymbol{e}^{ed}(0)],\end{aligned} \qquad (142)$$

$$\boldsymbol{D}^h(0)\,dt = -\varrho\,\boldsymbol{\mathcal{M}}^{(0)}\,d^h\boldsymbol{\tau} = -\frac{\partial[d^h\psi^{(1)}_e(\boldsymbol{\tau},H)]}{\partial\boldsymbol{\tau}}, \quad d^h\boldsymbol{\tau} = \frac{\partial(d^h\phi_e)}{\partial\boldsymbol{e}^e(0)}. \qquad (143)$$

Here $\psi^{(1)}_e = \phi_e - \boldsymbol{\tau}\cdot\boldsymbol{e}^e(0)$ is the elastic complementary energy which should not be confused with ψ_e present in (48). The same identification can be found by applying the operation $\mathcal{D}^{\mathrm{elog}}/\mathcal{D}t$ [cf. (125)] to Eq. (135.2) specified for $n = 0$, and by subsequent use of the property (136). Note that the set of Eqs. (141)–(143) coincides with the Eqs. (132)–(133) specialized for $n = 0$ provided that $\Delta\boldsymbol{\mathcal{E}} = \boldsymbol{0}$. This situation was discussed in detail in [33] where the hydrodynamical theory of metals was combined with the usual theory of thermoplasticity.

Remarks

(a) It can be readily verified using (132) that in general $dW^p = \boldsymbol{\tau}\cdot\boldsymbol{D}^p dt - d^h\phi_e \neq \boldsymbol{\tau}\cdot\boldsymbol{\mathcal{D}}^p(n)dt$, even if $d^h\phi_e = 0$.

In the case of materials isotropic with respect to the elastic properties we have $\mathrm{tr}\{\boldsymbol{\tau}[\boldsymbol{e}^{ed}(0)\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\boldsymbol{e}^{ed}(0)]\} = 0$, and the spin $\boldsymbol{\omega}^p$ influences neither the product $\boldsymbol{\tau}\cdot\boldsymbol{\mathcal{D}}^p(0)$ nor

$\boldsymbol{\tau}\cdot\boldsymbol{\mathcal{D}}^p(n)$ on account of (95.2) and (142). If additionally the elastic properties are not influenced by prior plastic straining ($d^h\phi_e=0$, $\boldsymbol{D}^h\ (n)=\boldsymbol{0}$) then in a vicinity of the current configuration the plastic increment of the total logarithmic strain $\boldsymbol{\mathcal{D}}^p(0)$ may be given the invariant plastic work interpretation, i.e.,

$$dW^p=\boldsymbol{\tau}\cdot\boldsymbol{D}^p\,dt=\boldsymbol{\tau}\cdot\boldsymbol{\mathcal{D}}^p(0)\,dt \tag{144}$$

From the transformation rule (95.2) it then follows that such property cannot be attributed to the plastic increment $\boldsymbol{\mathcal{D}}^p(n)$ of any other strain measure ($n\neq 0$).

(b) The situation $d^h\psi_e^{(1)}=0$ ($\boldsymbol{D}^h(0)=\boldsymbol{0}$) is considered in the paper [35] where instead of the elastic the total logarithmic rate $\mathcal{D}^{\log}\boldsymbol{\tau}/\mathcal{D}t$ of the Kirchhoff stress $\boldsymbol{\tau}$ is applied to get the rate form of the state equations, notwithstanding the fact that such procedure leads to the implicit dependence of the instantaneous elastic moduli $\boldsymbol{\mathcal{L}}(0)$ [cf. (93)] on prior plastic straining [past plastic deformation can make current principal stretches occurring in (19) to be arbitrarily large/small]. Likewise, the plastic part of the total stretching defined in [35] is also influenced by the past plastic strain. This approach is inconsistent with (141)–(143) and with the approach remarked beneath.

(c) In computational mechanics it has become popular (cf. [26, Chap. 9]) to use $\boldsymbol{b}^e=\boldsymbol{F}^e\overset{T}{\boldsymbol{F}^e}=(\boldsymbol{V}^e)^2=2\boldsymbol{e}^e(1)+\boldsymbol{1}$ as a state variable, $\phi_e=\phi_e(\boldsymbol{b}^e)$, and to postulate the plastic flow rule for the strain-rate tensor $\boldsymbol{D}'$ defined as [cf. (98)]

$$\boldsymbol{D}'\equiv-\frac{1}{2}\frac{\delta^c\boldsymbol{b}^e}{\delta t}=-\frac{1}{2}\boldsymbol{\Theta}(\boldsymbol{F})\dot{\overbrace{(\boldsymbol{U}^p)^{-2}}}=\boldsymbol{\Theta}(\boldsymbol{F})\dot{\boldsymbol{E}}_*(-1)=\boldsymbol{\Theta}(\boldsymbol{F}^e)\boldsymbol{D}^*, \tag{145}$$

where $2\boldsymbol{D}^*=\boldsymbol{L}^*+\overset{T}{\boldsymbol{L}^*}$ [cf. (99)] is the "permanent stretching" tensor, and the contravariant time derivative $\delta^c(.)/\delta t$ is defined in terms of the total deformation-rate tensor $\boldsymbol{L}$ as in (24). For elastically isotropic materials insensitive to prior plastic straining we have $\phi_e(\boldsymbol{b}^e)=\phi_e[\overset{-1}{\boldsymbol{U}^p}\boldsymbol{C}\overset{-1}{\boldsymbol{U}^p}](\boldsymbol{C}=\overset{T}{\boldsymbol{F}}\boldsymbol{F})$ and this proposal enables to avoid the need to postulate the constitutive equation for the plastic spin $\boldsymbol{\omega}^p$ [cf. (100)]. It is incompatible, however, with the two general theoretical frameworks discussed in the present paper. The following kinematic connection between $\boldsymbol{D}'$ and $\boldsymbol{D}^p$ defined by (100.2) holds:

$$\boldsymbol{D}'=\boldsymbol{D}^p+\boldsymbol{P}[\boldsymbol{e}^e(1)]\boldsymbol{D}^p+\boldsymbol{\omega}^p\,\boldsymbol{e}^e(1)-\boldsymbol{e}^e(1)\,\boldsymbol{\omega}^p \tag{146}$$

showing that the knowledge of $\boldsymbol{D}'$ is insufficient to separate $\boldsymbol{D}^p$ from $\boldsymbol{\omega}^p$.

We noted that in the discussed situation $[\boldsymbol{\tau}\boldsymbol{e}^e(1)=\boldsymbol{e}^e(1)\boldsymbol{\tau}\,,\ d^h\phi_e=0]$ the tensor $\boldsymbol{D}'$ may be furnished with the special plastic work interpretation

$$dW^p=\boldsymbol{\tau}\cdot\boldsymbol{D}\,dt-\dot{\phi}_e\,dt=\boldsymbol{\tau}\cdot\boldsymbol{D}^p\,dt=\boldsymbol{t}^e(1)\cdot\boldsymbol{D}'\,dt,\quad \boldsymbol{t}^e(1)=\boldsymbol{\Theta}(\overset{-1}{\boldsymbol{V}}e)\,\boldsymbol{\tau}=\overset{-1}{b}{}^e\boldsymbol{\tau}. \tag{147}$$

The similar approach, valid for elastically isotropic solids and based either on multiplicative decomposition of the metric tensors of the embedded basis or on the additive decompositions of their rates, was applied earlier (e.g., in [36]–[40]).

c. Relation between nominal rate of stress and deformation rate

In the updated Lagrangean field theory one applies the rate equilibrium in the form [41]–[44]:

$$\frac{\partial\pi_{ij}^{\diamond}}{\partial x_j}+b_j^{\diamond}=0, \tag{148}$$

where $\boldsymbol{x}$ is the position of a particle in the current configuration of a body (at the same time the position $\boldsymbol{x}$ identifies the particle of a body), $\boldsymbol{b}^{\diamond}$ is the nominal-rate of the body forces and $\boldsymbol{\pi}^{\diamond}$ is the nominal-rate of stress defined by

$$\boldsymbol{\pi}^{\diamond} \equiv \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathrm{tr}\boldsymbol{D}. \tag{149}$$

To formulate the rate boundary value problem for the increment $\boldsymbol{v}\, dt$ ($\boldsymbol{v}$-velocity field in the current configuration) one needs the $\boldsymbol{\pi}^{\diamond} \leftrightarrow \boldsymbol{L}$ relation. Under the assumed known current state of the body such relation can be found by elimination of $\dot{\boldsymbol{\sigma}}$ between (149) and (30) and by subsequent use of (93),

$$\boldsymbol{\pi}^{\diamond}(\boldsymbol{L}) = \boldsymbol{\mathcal{L}}(n)[\boldsymbol{D} - \boldsymbol{\mathcal{D}}^{p}(n)] + (n-1)\,\boldsymbol{P}(\boldsymbol{\sigma})\boldsymbol{D} + \boldsymbol{\sigma}\overset{T}{\boldsymbol{L}}. \tag{150}$$

A more specific structure of the first term one obtains by eliminating $\boldsymbol{\mathcal{D}}^{p}(n)$ with the aid of (132.2) or (142). In order for the set of differential equations to be closed the extra constitutive equations (plastic flow rules) should be postulated for $\boldsymbol{D}^{p}$, $\dot{H}$ and $\boldsymbol{\omega}^{p}$. The latter spin does not occur in the expression for the incremental energy dissipation. Therefore, the equation for $\boldsymbol{\omega}^{p}$ can be specified at will, until the definite experimental observations concerning the rearrangement of the internal structure (crystal plasticity [45], evolution of texture [46]) are intended to be included into the theoretical framework. Mandel [47], [48] has introduced the concept of "isoclinic" unloaded configuration. One of the basic element of this concept is that $\boldsymbol{D}^{p} = \boldsymbol{0} \Rightarrow \boldsymbol{\omega}^{p} = \boldsymbol{0}$ and this is one of the rational restrictions that may be imposed on the choice of $\boldsymbol{\omega}^{p}$, in particular it can be set to be zero even in the case of anisotropic materials. Other representative proposals are presented, e.g., in [49] (cf. also, e.g., [25, Chap. 11]). The uniqueness theorems concerning the rate boundary-value problems, the foundation of the related bifurcation theories as well as stability criteria can be found in [42]–[44], [50]–[52]. The adequate computational methods were developed, e.g., in [53], [54, Chap. 6].

5 Brief discussion

Most of the invariance aspects of the finite-strain theory of plasticity illustrated in Part I [4] on the background of the simple tension have been shown to be valid also in 3D situation. Hill's transformation rules of the basic constitutive quantities were recasted for materials with the Green type of elasticity in a manner slightly different, by presuming that the incremental free energy is the basic invariant. They help to distinguish the universal concepts (e.g., notion of incremental strain cycle, plastic increment of work-conjugate stresses) from others that are not invariant (e.g., notion of incremental stress cycle, positive semi-definiteness of the tangent moduli, plastic increment of the total strain). The special cyclic process has been analyzed to define the invariant incremental plastic work within H–R formalism.

The E–M theoretical framework has naturally resulted from H–R approach as a transition from a fixed generalized coordinates system to a moving one the changes of which correspond to the changes in the internal state of m.e. The structure of the updated Lagrangean plastic increment of total strain has been shown in terms of thermodynamic properties described by the free-energy potential including those representing stress and deformation effects due to damage, pressure sensitivity etc. Special attention has been paid to the situations when *elastic distortions* remain small. These are of special engineering interest since the most not rubber-like materials undergoing permanent deformations do not experience large elastic distortions. The derived approximate relations (132)–(133), connecting the updated Lagrangean quantities with those defined in an unloaded configuration are valid for general anisotropic materials. In our view they constitute the appropriate compromise between simplicity, generality and usefulness. Notwithstanding the merits they were not discussed in the literature. The generalized Hooke's law can be recasted by assuming that either the moduli $\boldsymbol{\mathcal{L}}^{*}(n)$ or $\varrho^{*}(H)\,\boldsymbol{\mathcal{L}}^{*}(n)/\varrho_{0}^{*}$ presented in (132) are at most functions of H. Their image $\boldsymbol{\mathcal{L}}(n)$ in the actual configuration $\kappa(t)$ depends on the extra constant n. This illustrates the

concept "let the experiments judge what is the most optimal strain measure for the investigated material". Such concept in more general context was advanced in the nonlinear elasticity of rubber-like materials by Hill [1] and Ogden [55]–[57].

References

[1] Hill, R.: Aspects of invariances in solids mechanics. In: Chia-Shien, Y. (ed.) Adv. Appl. Mech., vol. 18, pp. 1–75. Academic Press, New York (1978)

[2] Hill, R.: Invariance relations in thermoelasticity with generalized variables. Proc. Camb. Philos. Soc. **90**, 373–384 (1981)

[3] Kleiber, M., Raniecki, B.: Elastic–plastic materials at finite strain. In: Sawczuk, A. (ed.) Plasticity Today, pp. 3–46. Elsevier, Amsterdam (1985)

[4] Nguyen, H.V., Raniecki, B., Ziółkowski, A.: On the incremental plastic work and related aspects of invariance, Part I. Acta Mech. **189**, 1–22 (2007)

[5] Flory, R.J.: Thermodynamic relations for highly elastic materials. Trans. Faraday Soc. **57**, 829–838 (1961)

[6] Bazant, Z.P.: Finite strain generalization of small-strain constitutive relations for any finite strain tensor and additive volumetric–deviatoric split. Int. J. Solids Struct. **33**, 2887–2897 (1996)

[7] Ogden, R.W.: Nonlinear elastic deformations. Dover, New York (corrected republication) (1997)

[8] Mehrabadi, M.M., Nemat-Nasser, S.: Some basic kinematical relations for finite deformations of continua. Mech. Materials **6**, 127–138 (1987)

[9] Dienes, J.K.: On the analysis of rotation and stress rate in deforming bodies. Acta Mech. **32**, 217–232 (1979)

[10] Xiao, H., Bruhns, O.T., Meyers, A.: Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mech. **124**, 89–105 (1997)

[11] Xiao, H., Bruhns, O.T., Meyers, A.: The choice of objective rates in finite elastoplasticity: general results on the uniqueness of the logarithmic rate. Proc. R. Soc. Lond. **A456**, 1865–1882 (2000)

[12] Xiao, H., Bruhns, O.T., Meyers, A.: Existence and uniqueness of the integrable-exactly hypoelastic equation $\boldsymbol{\tau} = \lambda(\mathrm{tr}\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D}$ and its significance to finite inelasticity. Acta Mech. **138**, 31–50 (2000)

[13] Lehmann, Th., Guo, Z.H., Liang, H.Y.: The conjugacy between Cauchy stress and logarithm of the left stretch tensor. Eur. J. Mech. A/Solids **10**, 395–404 (1991)

[14] Sedov, L.I.: Different definitions of the rate of change of a tensor. Prikl. Mat. i Mekh. **24**, 393–398 (1960)

[15] Simo, J.C.: A framework for finite strain elastoplasticity based on maximum plastic dissipation and multiplicative decomposition: Part I. Continuum formulation. Comput. Math. Appl. Mech. Engng. **66**, 199–220 (1988)

[16] Gurtin, M.E., Spear, K.: On the relationship between the logarithmic strain rate and the stretching tensor. Int. J. Solids Struct. **19**, 437–444 (1983)

[17] Hoger, A.: The material time derivative of logarithmic strain. Int. J. Solids Struct. **22**, 1019–1032 (1986)

[18] Hoger, A.: The stress conjugate to logarithmic strain. Int. J. Solids Struct. **23**, 1645–1656 (1987)

[19] Rice, J.R.: Inelastic constitutive relations for solids: an internal variable theory and its application to metal plasticity. J. Mech. Phys. Solids **19**, 433–455 (1971)

[20] Rice, J.R.: Continuum plasticity in relation to microscale deformation mechanisms. In: Rohde, R.W., Butcher, B.M., Holland, J.R., Karners, C.H. (eds.) Metallurgical Effects at High Strain Rates, pp. 93–106. Plenum Press, New York (1973)

[21] Rice, J.R.: Continuum mechanics and thermodynamics of plasticity in relation to misroscale deformation mechanisms. In: Argon, A.S. (ed.) Constitutive Equations in Plasticity, pp. 23–79. The MIT Press, Cambridge (1975)

[22] Hill, R., Rice, J.R.: Elastic potentials and the structure of inelastic constitutive laws. SIAM J. Appl. Math. **25**, 448–461 (1973)

[23] Svendson, B.: A thermodynamics formulation of finite-deformation elastoplasticity with harderning based on the concept of material isomorphism. Int. J. Plast. **14**, 473–488 (1998)

[24] Svendson, B.: On the modelling of anisotropic elastic and inelastic material behaviour at large deformation. Int. J. Solids Struct. **38**, 9579–9599 (2001)

[25] Lubarda, V.A.: Elastoplasticity theory. CRC Press LLC, West Palm Beach (2002)
[26] Simo, J.C., Hughes, T.J.R.: Computational inelasticity. In: Interdisciplinary Appl. Math., vol. 7. Springer, Heidelberg (2000)
[27] Hill, R.: On intrinsic eigenstates in plasticity with generalized variables. Math. Proc. Camb. Philos. Soc. **93**, 177–189 (1983)
[28] Oliferuk, W., Raniecki, B.: Thermodynamic analysis of energy storage rate during uniaxial tensilc deformation of polycrystalline metal. Arch. Metal. **47**, 261–273 (2002)
[29] Oliferuk, W., Maj, M., Raniecki, B.: Experimental analysis of energy storage rate components during tensil deformation of polycrystals. Mater. Sci. Engng. **A374**, 77–81 (2004)
[30] Eckart, G.: Theory of elasticity and inelasticity. Phys. Rev. **73**, 373–380 (1948)
[31] Mandel, J.: Plasticité et viscoplasticité. In: CISM Lectures Notes, vol. 97, Udine. Springer, Wien (1971)
[32] Halphen, B.: Sur les champ des vitesses en thermoplasticité finie. Int. J. Solids Struct. **11**, 947–960 (1975)
[33] Raniecki, B., Nguyen, H.V.: Isotropic elastic–plastic solids at finite strain and arbitrary pressure. Arch. Mech. **36**, 687–704 (1984)
[34] Becker, R.: Effects of crystal plasticity on materials loaded at high pressure and strain rates. Int. J. Plast. **20**, 1983–2006 (2004)
[35] Meyers, A., Xiao, H., Bruhns, O.T.: Thermodynamics law and consistent Eulerian formulation for finite elastoplasticity with thermal effects. J. Mech. Phys. Solids **55**, 338–365 (2007)
[36] Lehmann, Th.: Some thermodynamic considerations of phenomenological theory of non-isothermal elastic–plastic deformations. Arch. Mech. **24**, 975–989 (1972)
[37] Lehmann, Th.: Einige Betrachtungen zur Thermodynamik großer elasto-plastischer Formänderungen. Acta Mech. **20**, 187–207 (1974)
[38] Lehmann, Th.: On the theory of large, non-isothermic, elastic–plastic and elastic–viscoplastic deformations. Arch. Mech. **29**, 393–409 (1977)
[39] Lehmann, Th. (ed.) The constitutive laws in thermoplasticity. In: CISM Courses and Lecture, vol. 281. Springer, Wien (1984)
[40] Holsapple, K.A.: A finite elastic–plastic theory and invariance requirements. Acta Mech. **17**, 277–290 (1973)
[41] Hill, R.: Some basic principles in the mechanics of solids without a natural time. J. Mech. Phys. Solids **7**, 209–225 (1959)
[42] Hill, R.: Uniqueness and extremum principles in self-adjoint boundary-value problems in continuum mechanics. J. Mech. Phys. Solids **10**, 185–194 (1962)
[43] Hill, R.: Uniqueness in general boundary-value problems for elastic and inelastic solids. J. Mech. Phys. Solids **7**, 209–225 (1962)
[44] Raniecki, B., Bruhns, O.T.: Bounds to bifurcation stresses in solids with non-associated plastic flow at finite strain. J. Mech. Phys. Solids **29**, 153–172 (1981)
[45] Bassani John, L.: Plastic flow of crystals. In: Adv. Appl. Mech., vol. 30, pp. 191–257. Academic Press, New York (1994)
[46] Raniecki, B., Mróz, Z.: On the strain-induced anisotropy and texture in rigid-plastic solids. In: Kleiber, M., König, A. (eds.) Inelastic Solids and Structure, A. Sawczuk Memorial Volume, pp. 13–32. Pineridge Press, Swansea (1989)
[47] Mandel, J.: Director vectors and constitutive equations for plastic and viscoplastic media. In: Sawczuk, A., (ed.) Problems of Plasticity, pp. 135–143. Nordhoff, Gronigen (1974)
[48] Mandel, J.: Définition d'un repère privilégié pour l'étude des transformations anélastiques du polycristal. J. de Méc. Thé. et App. **1**, 1–23 (1982)
[49] Dafalias, Y.F.: The plastic spin. J. Appl. Mech. **52**, 865–871 (1985)
[50] Petryk, H.: Material instability and strain-rate discontinuities in incrementally nonlinear continua. J. Mech. Phys. Solids **40**, 1227–1250 (1992)
[51] Petryk, H.: Thermodynamic stability of equilibrium in plasticity. J. Non-Equilib. Thermodyn. **20**, 132–149 (1995)
[52] Petryk, H.: Instability of plastic processes. In: Proc. 19th International Congress of Theoretical and Applied Mechanics, Kyoto 1996 (Tatsumi, T., et al., eds.), pp. 497–516. Amsterdam: Elsevier (1997)
[53] Gabriel, G., Bathe, K.J.: Some computational issues in large strain elasto-plastic analysis. Comput. Struct. **56**, 249–267 (1995)
[54] Bathe, K.J.: Finite element method. In: ISBN 0-13-301458-4. Prentice-Hall, Englewood Cliffs (1996)

[55] Ogden, R.W.: Large deformation isotropic elasticity i. on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. Lond. Ser. **A326**, 565–584 (1972)
[56] Ogden, R.W.: Large deformation isotropic elasticity i. on the correlation of theory and experiment for incompressible rubberlike solids. Proc. R. Soc. Lond. Ser. **A328**, 567–583 (1972)
[57] Ogden, R.W.: Volume changes associated with the deformation of rubberlike solids. J. Mech. Phys. Solids **24**, 323–338 (1976)