XXIV Symposium *Vibrations in Physical Systems,* Poznan – Bedlewo, May 12-15, 2010

Numerical methods for vibration analysis of Timoshenko beam subjected to inertial moving load

Bartłomiej DYNIEWICZ
Institute of Fundamental Technological Research, Polish Academy of Sciences, Pawińskiego 5b, 02-106 Warsaw, Poland, bdynie@ippt.gov.pl

Czesław I. BAJER
Institute of Fundamental Technological Research, Polish Academy of Sciences, Pawińskiego 5b, 02-106 Warsaw, Poland, cbajer@ippt.gov.pl

Abstract

The paper deals with the problem of modelling of the moving mass particle in numerical computation by using the finite element method in one dimensional wave problems in which both the displacement and angle of the pure bending are described by linear shape functions. The analysis is based on the Timoshenko beam theory. We consider the simply supported beam, in a range of small deflections with zero initial conditions.

Keywords: numerical method, moving mass, moving inertial load, vibrations

1. Introduction

Rail and road transport development needs a closer understanding of phenomena accompanying travelling load. Most applications can be found in the interaction between railway wheels and rail or track, the effect of a moving vehicle on a bridge, interaction between rail power collector and traction power network, as well as magnetic rail, aerospace technology, automotive industry, and robotics. Despite of the wide interest in moving loads for more than a century, still many issues remain unresolved. In the case of non-inertial loads, for example the gravitational force or forces described by harmonic functions, complete analytical solutions in the series are known [1, 2]. Solutions differ in the case of inertial loads. A moving inertial load problem can not be solved fully analyticaly, except special cases such as the massless string [3]. There are semi-analytical solutions [4, 5, 6] which take into account the influence of a mass particle moving along the structure.

Modelling of the moving forces does not take into account the inertia of a moving point and is relatively simple. In practice it reduces to the modification of the right-hand-side vector at each time step. Inclusion of the inertia of a moving load requires the modification of the inertia, damping and stiffness matrices at every time step. A simple modification of the diagonal of the inertia matrix (Fig. 1), is incorrect and results in divergence of the solution. Errors, due to incorrect modeling, increase with increasing speed of a moving inertial load. According to the

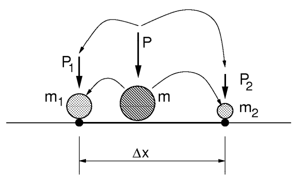

Figure 1: Ad hoc mass lumping in nodes.

Renaudot formula [7] the acceleration of the material point moving with a constant speed v, is composed of three elements:

$$\frac{\mathrm{d}^2 w(vt,t)}{\mathrm{d}t^2} = \left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=vt} + 2v\left.\frac{\partial^2 w(x,t)}{\partial x \mathrm{d}t}\right|_{x=vt} + v^2\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=vt} . \quad (1)$$

We can show the components corresponding to transverse acceleration, Coriolis acceleration and centrifugal acceleration.

There are numerous publications on numerical modelling of inertial moving load using the finite element method [8, 9, 10]. In most of them displacements and rotations are approximated as cubic functions. They can be applied to all the terms of the equation (1). In the case of wave problems in a string or the Timoshenko beam we have to use linear shape functions to describe independently displacements and rotations in pure bending. It entails mathematical consequences. We can not compute the second derivative of the displacement x. In such a case we should have to neglect the effect of centrifugal acceleration of the moving material point in the formula (1). It leads to incorrect solution.

Below we present recent results which enables us to solve the problem of a moving mass travelling on the Timoshenko beam with an arbitrary velocity. Numerical examples prove the efficiency of the proposed method.

2. Timoshenko beam theory

Let us consider the Timoshenko beam with the length l, mass density ρ, cross-sectional area A and moment of inertia I, subjected to the mass particle m accompanied by the force P, moving with the constant speed v. Denoting the transverse displacement by $w(x,t)$ and the pure bending angles by $\psi(x,t)$, the kinetic energy of the Timoshenko beam and moving material point with mass m is expressed by the equation

$$T = \frac{1}{2}\rho A\int_0^l \left[\frac{\partial w(x,t)}{\partial t}\right]^2 \mathrm{d}x + \frac{1}{2}\rho I\int_0^l \left[\frac{\partial \psi(x,t)}{\partial t}\right]^2 \mathrm{d}x + \frac{1}{2}mv^2 + \frac{1}{2}m\left[\frac{\mathrm{d}w(vt,t)}{\mathrm{d}t}\right]^2 . \quad (2)$$

The potential energy of the Timoshenko beam and a moving gravitational force is described as follows

$$U = \frac{1}{2}EI\int_0^l \left[\frac{\partial \psi(x,t)}{\partial x}\right]^2 \mathrm{d}x + \frac{1}{2}\frac{GA}{k}\int_0^l \left[\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right]^2 \mathrm{d}x - Pw(vt,t) . \quad (3)$$

E is the elastic modulus, G is the shear modulus and k is the cross-section shape ratio. Based on the second kind Lagrange equation, we determine two coupled equations describing the motion of the Timoshenko beam subjected to a moving load

$$\begin{cases} \rho A \frac{\partial^2 w(x,t)}{\partial t^2} - \frac{GA}{k}\left(\frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \psi(x,t)}{\partial x}\right) = \delta(x-vt)P - \delta(x-vt)m\frac{\mathrm{d}^2 w(vt,t)}{\mathrm{d}t^2}, \\ \rho I \frac{\partial^2 \psi(x,t)}{\partial t^2} - EI\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{GA}{k}\left(\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right) = 0\ . \end{cases} \tag{4}$$

The equations (4) can be transformed into one equation of motion. It depends only on displacements or rotations. Let us consider displacements first

$$\begin{gathered} \frac{\partial^4 w(x,t)}{\partial t^4} - \left(c_1^2 + c_2^2\right)\frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + \frac{A}{I}c_1^2\frac{\partial^2 w(x,t)}{\partial t^2} + c_1^2 c_2^2 \frac{\partial^4 w(x,t)}{\partial x^4} = \\ = c_1^2 c_2^2 q(x,t) - \frac{c_2^2}{\rho A}\frac{\partial^2 q(x,t)}{\partial x^2} + \frac{1}{\rho A}\frac{\partial^2 q(x,t)}{\partial t^2}\ , \end{gathered} \tag{5}$$

where the external load is given by the formula

$$q(x,t) = \delta(x-vt)P - \delta(x-vt)m\frac{\mathrm{d}^2 w(vt,t)}{\mathrm{d}t^2}\ . \tag{6}$$

$c_1 = \sqrt{G/(k\rho)}$ is the shear wave speed and $c_2 = \sqrt{E/\rho}$ is the bending wave speed. We assume a simply supported beam

$$w(0,t) = 0, \quad w(l,t) = 0, \quad \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0} = 0, \quad \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=l} = 0, \tag{7}$$

with zero initial conditions

$$w(x,0) = 0, \quad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0\,. \tag{8}$$

Equation (5) is a partial differential equation of the fourth order with respect to time. Its solution requires additional initial conditions

$$\left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{t=0} = \frac{1}{\rho A}q(x,0), \quad \left.\frac{\partial^3 w(x,t)}{\partial t^3}\right|_{t=0} = \frac{1}{\rho A}\left.\frac{\partial q(x,t)}{\partial t}\right|_{t=0}\ . \tag{9}$$

3. Semi-analytical solution

We can develop displacements of the beam into the sine Fourier series in a finite interval, which fulfil boundary conditions (7)

$$w(x,t) = \sum_{i=1}^{n} Q_i(t)\sin\frac{i\pi x}{l}\,. \tag{10}$$

By substituting the series (10) to the equation (5) we obtain a set of ordinary differential equations of the form

$$\mathbf{\Gamma}\begin{bmatrix}\dddot{Q}_1(t)\\ \dddot{Q}_2(t)\\ \vdots\\ \dddot{Q}_n(t)\end{bmatrix}+\mathbf{U}\begin{bmatrix}\dddot{Q}_1(t)\\ \dddot{Q}_2(t)\\ \vdots\\ \dddot{Q}_n(t)\end{bmatrix}+\mathbf{M}\begin{bmatrix}\ddot{Q}_1(t)\\ \ddot{Q}_2(t)\\ \vdots\\ \ddot{Q}_n(t)\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{Q}_1(t)\\ \dot{Q}_2(t)\\ \vdots\\ \dot{Q}_n(t)\end{bmatrix}+\mathbf{K}\begin{bmatrix}Q_1(t)\\ Q_2(t)\\ \vdots\\ Q_n(t)\end{bmatrix}=\mathbf{P}\,. \tag{11}$$

This method lead us to the system of differential equation of variable coefficients (11) solved by the Runge-Kutta 4 order method. We compute numerically the vector $\mathbf{Q}$ and then insert it to the resulting series (10).

4. Numerical solution by the finite element method

Let us consider the finite element of the length b of the Timoshenko beam. The element carries the inertial particle of the mass m, travelling with a constant velocity v. The equation of the virtual work which describes the influence of the inertial particle can be written in the following form

$$\int_0^b w^*(x)\delta(x-x_0-vt)\,m\,\frac{\mathrm{d}^2 w(vt,t)}{\mathrm{d}t^2}\mathrm{d}x=0\,. \tag{12}$$

We impose the linear shape function describing the transversal displacement in finite element nodes

$$w(x,t)\;=\;\left(1-\frac{x}{b}\right)w_1(t)\;+\;\frac{x}{b}w_2(t)\,. \tag{13}$$

Equation (1) describes the acceleration of a moving material point. It can be expressed in the form

$$\frac{\mathrm{d}^2 w(vt,t)}{\mathrm{d}t^2}\;=\;\left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=vt}+v\left.\frac{\partial^2 w(x,t)}{\partial x\partial t}\right|_{x=vt}+\;v\frac{\mathrm{d}}{\mathrm{d}t}\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=vt}\right]. \tag{14}$$

The third term of (14) is developed into the Taylor series in terms of the time increment $\Delta t=h$

$$\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=vt}\right]^{t+h}=\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=vt}\right]^{t}+\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=vt}\right]\right\}^{t}(1-\alpha)\,h+ \\ +\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=vt}\right]\right\}^{t+h}\alpha h. \tag{15}$$

Upper indices indicate time in which respective terms are defined. We assume the backward difference formula (α=1). In this case we have

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=vt}\right]\right\}^{t+h}=\frac{1}{h}\left[\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=vt}\right]^{t+h}-\frac{1}{h}\left[\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=vt}\right]^{t}. \tag{16}$$

The energy (12), with respect to (14) and (16) allows us to write the results in the matrix equation, after classical minimisation

$$\mathbf{M}_m \ddot{\mathbf{w}}^{i+1} + \mathbf{C}_m \dot{\mathbf{w}}^{i+1} + \mathbf{K}_m \mathbf{w}^{i+1} = \mathbf{e}_m^i \, . \tag{17}$$

where

$$\mathbf{M}_m = m \begin{bmatrix} (1-\kappa)^2 & \kappa(1-\kappa) \\ \kappa(1-\kappa) & \kappa^2 \end{bmatrix} , \quad \mathbf{C}_m = \frac{mv}{b} \begin{bmatrix} -(1-\kappa) & 1-\kappa \\ -\kappa & \kappa \end{bmatrix} ,$$
$$\mathbf{K}_m = \frac{mv}{bh} \begin{bmatrix} -(1-\kappa) & 1-\kappa \\ -\kappa & \kappa \end{bmatrix} \quad \mathbf{e}_m = \frac{mv}{bh} \begin{bmatrix} (1-\kappa)(u_2-u_1) \\ \kappa(u_2-u_1) \end{bmatrix} , \tag{18}$$

with the coefficient $\kappa = (x_0 + vh)/b$, $0 < \kappa \leq 1$. It determines the force equilibrium of the mass travelling over the finite element of a Timoshenko beam. Matrix factors $\mathbf{M}_m$, $\mathbf{C}_m$, and $\mathbf{K}_m$ can be called mass, damping, and stiffness matrices, since they have similar forms to matrices derived for pure finite element of the Timoshenko beam. The last term $\mathbf{e}_m$ describes nodal forces at the beginning of the time interval $[0; h]$. We must emphasise here that matrices (18) and the vector $\mathbf{e}$ contribute only the moving inertial particle effect. Pure classical matrices of the finite element of a string must be added to the global system of equations.

5. Examples

We choose the steel beam of the rectangular cross-section A=0.015 m^2 and the length l=2 m. We assume other data: ρ=7860 kg/m^3, I=0.0000281 m^4, m=200 kg, P=mg, g=9.81 m/s^2, E=2.1·10^5 MPa, G=8.1·10^4 MPa, k=1.2 . Fig. 2 shows a com-

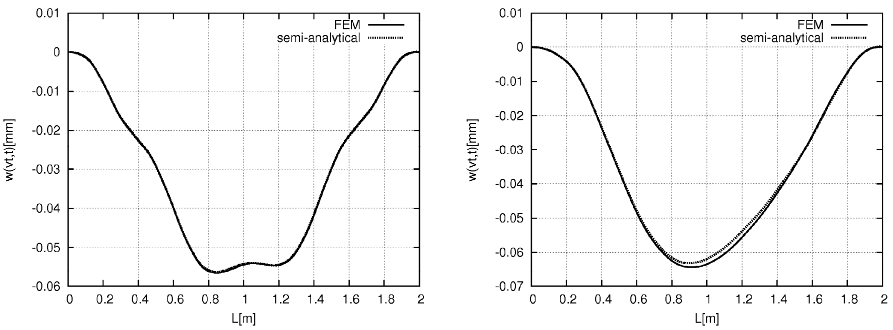

Figure 2: Trajectories of a mass particle travelling along the Timoshenko beam at the speed v=30 m/s (left picture) and v=60 m/s (right picture).

parison of the results obtained by semi-analytical method presented earlier, and the finite element method using matrices describing the moving material point of mass m. The obtained results confirm the correct way of modelling a moving mass particle.

6. Conclusions

The paper deals with the problem of vibrations of the Timoshenko beam subjected to a moving inertial particle. The presented approach allows accurate modeling of a mass particle travelling with a constant velocity in numerical computation by using finite element method. These matrices can be applied to every wave problem, where the displacement and rotations of the pure bending are described by linear shape functions.

References

1. L. Frŷba. *Vibrations of solids and structures under moving loads.* Thomas Telford House, 1999.
2. W. Szcześniak. Inertial moving loads on beams (in Polish). Scientific Reports, Warsaw University of Technology, Civil Engineering 112, 1990.
3. C.E. Smith. Motion of a stretched string carrying a moving mass particle. *J. Appl. Mech.*, 31(1):29–37, 1964.
4. B. Dyniewicz and C.I. Bajer. Paradox of the particle's trajectory moving on a string. *Arch. Appl. Mech.*, 79(3):213–223, 2009.
5. E.C. Ting, J. Genin, and J.H. Ginsberg. A general algorithm for moving mass problems. *J. Sound Vib.*, 33(1):49–58, 1974.
6. G. T. Michaltsos. Dynamic behaviour of a single-span beam subjected to loads moving with variable speeds. *J. Sound and Vibr.*, 258(2):359–372, 2002.
7. A. Renaudot. Etude de l'influence des charges en mouvement sur la resistance, des ponts metallique a poutres droites. *Annales des Ponts et Chausses*, 1:145–204, 1861.
8. C.I. Bajer and B. Dyniewicz. Numerical modelling of structure vibrations under inertial moving load. *Arch. Appl. Mech.*, 79(6-7):499 – 508, 2009.
9. F.V. Filho. Finite element analysis of structures under moving loads. *The Shock and Vibration Digest*, 10(8):27–35, 1978.
10. T. Borowicz. Strength of beams under moving load (in Polish). *Archives of Civil Engineering*, 24(2):219–235, 1978.

Metody numeryczne analizy drgań belki Timoshenki pod inercyjnym obciążeniem ruchomym

Praca omawia problem modelowania numerycznego poruszającej się cząstki masowej metodą elementów skończonych w zadaniu jednowymiarowym. Przemieszczenia i obroty opisano liniowymi funkcjami kształtu. Analizę oparto na teorii belki Timoshenki.