Struct Multidisc Optim (2006) 32: 463–471
DOI 10.1007/s00158-005-0600-7

RESEARCH PAPER

C. Graczykowski · T. Lewiński

Michell cantilevers constructed within trapezoidal domains—Part II: virtual displacement fields

Received: 13 May 2005 / Revised manuscript received: 20 September 2005 / Published online: 10 June 2006

Abstract The present second part of the paper deals with the virtual displacement fields associated with the optimality conditions $\varepsilon_I(\bar{\mathbf{u}}) = 1$, $\varepsilon_{II}(\bar{\mathbf{u}}) = -\kappa$, $\kappa = \sigma_T \big/ \sigma_C$, where σ_T and σ_C represent the allowable values of the tensile and compressive stress, respectively. The displacement fields vanish along a straight segment of a line support and are constructed within an infinite domain bounded by two half-lines. The displacement fields are provided by the integral formulae involving the Lamé fields found in part I of this paper. All the results are expressed in terms of Lommel-like functions. These results make it possible to determine the volumes of the optimal cantilevers designs within the feasible domain considered. Computation of the volumes along with analyses of concrete cantilevers will be the subject of part IV of the present paper.

1 Introduction

The Hencky nets found in part I determine the geometry of the Michell cantilevers constructed within the infinite trapezoidal domain. The weight of these cantilevers can be computed by (I.2.12). This weight is proportional to the work of the concentrated force on the trial displacement satisfying the conditions (I.2.14) within the domain and (I.2.15) on its boundaries, along the reinforcing ribs. The subject of the present paper is a concatenating construction of these trial fields in the subdomains RBA, NAC, ABDC, CDG, BDH, DHJG, GJG_2, HJH_2 and $JH_2J_2G_2$ (see Fig. I.19). We apply two methods of construction of these fields: (a) by using the auxiliary virtual displacement fields u^0, v^0 satisfying the equations $Lu^0=0$, $Lv^0=0$ with the hyperbolic operator L defined by (I.6.3) or by (b) a direct integration along the α-lines or β-lines. All the integration operations are performed analytically using the Lommel-like functions (see the identities a.7–a.10) and the integration formulae set up in Lewiński et al. (1994a) (see a.16, a.31–a.36, a.38–a.41, a.A.1–a.A.12) and in Graczykowski and Lewiński (2006) in the Appendix, called Appendix I.A.

The results presented can be viewed as generalization of the previous results by Chan (1967) and Lewiński et al. (1994a) concerning the special case of $\sigma_\mathrm{T}=\sigma_\mathrm{C}$. The paper uses the notation and adopts the conventions of part I (see Graczykowski and Lewiński 2006). In particular, notation (a.101), Section a.2 means (101), Section 2 of the paper by Lewiński et al. (1994a). Similarly, (b.161), Section b.7 means (161), Section 7 of the paper by Lewiński et al. (1994b). We introduce now a new convention: (I.13.1) means (13.1) of part I or of Graczykowski and Lewiński (2006). Similarly, Section I.7, Fig. I.6 means Section 7, Fig. 6 in part I.

C. Graczykowski (✉)
Institute of Fundamental Technological Research,
Polish Academy of Sciences,
Świętokrzyska 21,
00-049 Warsaw, Poland
e-mail: Cezary.Graczykowski@ippt.gov.pl

T. Lewiński
Faculty of Civil Engineering, Institute of Structural Mechanics,
Warsaw University of Technology,
al.Armii Ludowej 16,
00-637 Warsaw, Poland
e-mail: T.Lewinski@il.pw.edu.pl

2 The methods of construction of the virtual displacement field by using the geometric characteristics of the Hencky nets

In part I the characteristics x, y, A, B of the lines α, β within the infinite domain Ω_0 (the domain bounded by the lines RR_1, NN_1 and the line segment RN; see Fig. I.1) have been found and analytically expressed by Lommel-like functions. Having these characteristics we shall find the components (u,v) of the virtual displacement field $\bar{\mathbf{u}}$ satisfying the optimality conditions (I.2.14). The components (u,v) represent virtual displacements along α and β lines, respectively. The strain components referred to the (α, β) coordinates are expressed by (a.62, a.63) or

$$\varepsilon_{11} = \frac{1}{A}\left(\frac{\partial u}{\partial \alpha} + \frac{1}{B}\frac{\partial A}{\partial \beta}v\right),$$

$$\varepsilon_{22} = \frac{1}{B}\left(\frac{\partial v}{\partial \beta} + \frac{1}{A}\frac{\partial B}{\partial \alpha}u\right),$$

$$\varepsilon_{12} = \frac{1}{2}(\omega_1 + \omega_2), \tag{2.1}$$

where

$$\omega_1 = \frac{1}{A}\left(\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial A}{\partial \beta}u\right), \ \omega_2 = \frac{1}{B}\left(\frac{\partial u}{\partial \beta} - \frac{1}{A}\frac{\partial B}{\partial \alpha}v\right) \tag{2.2}$$

and the quantity

$$\omega = \frac{1}{2}(\omega_1 - \omega_2) \tag{2.3}$$

represents the rigid rotation around the normal to the plane considered.

The lines (α, β) follow trajectories of virtual principal strains. According to the optimality conditions (I.2.14) we have

$$\varepsilon_{11} = 1\ ,\ \varepsilon_{22} = -\kappa\ ,\ \varepsilon_{12} = 0\ ,\ \omega_1 = \omega, \quad \omega_2 = -\omega. \tag{2.4}$$

The above conditions lead to the following formula for the virtual displacements

$$d\left[e^{i\phi}(u + iv)\right] = e^{i\phi}[A(1 + i\omega)d\alpha + B(-\omega - i\kappa)d\beta], \tag{2.5}$$

where $\phi(\alpha, \beta)=\beta-\alpha$. The right-hand side must be a complete differential. This condition gives the differential equations which govern the function $\omega(\alpha, \beta)$. Integration of these equations leads to the following formula (see Hemp 1973, (4.21)):

$$\omega = \omega_0 + (1 + \kappa) \cdot \left[\int_0^\alpha \frac{\partial \phi(\overline{\alpha}, 0)}{\partial \overline{\alpha}} d\overline{\alpha} - \int_0^\beta \frac{\partial \phi(\alpha, \overline{\beta})}{\partial \overline{\beta}} d\overline{\beta}\right], \tag{2.6}$$

where $\omega_0=\omega(0,0)$. Taking into account that $\phi(\overline{\alpha}, 0) = -\overline{\alpha}$ and $\phi(\alpha, \overline{\beta}) = \overline{\beta} - \alpha$, one finds

$$\omega(\alpha, \beta) = \omega_0 - (1 + \kappa)(\alpha + \beta). \tag{2.7}$$

Integration of (2.5) leads to the following integral formulae for the integration along

(a) the α-line

$$u(\xi, \eta) = \widehat{u}_\theta(\xi, \eta) + \int_\theta^\xi [\cos(\xi - \alpha) - \omega(\alpha, \eta) \times \sin(\xi - \alpha)]\, A(\alpha, \eta)d\alpha$$

$$v(\xi, \eta) = \widehat{v}_\theta(\xi, \eta) + \int_\theta^\xi [\sin(\xi - \alpha) + \omega(\alpha, \eta) \times \cos(\xi - \alpha)]\, A(\alpha, \eta)d\alpha \tag{2.8}$$

where $\omega(\alpha, \eta)$ is given by (2.7)

$$\widehat{u}_\theta(\xi, \eta) = \cos(\xi - \theta) \cdot u(\theta, \eta) - \sin(\xi - \theta) \cdot v(\theta, \eta)$$

$$\widehat{v}_\theta(\xi, \eta) = \sin(\xi - \theta) \cdot u(\theta, \eta) + \cos(\xi - \theta) \cdot v(\theta, \eta) \tag{2.9}$$

(b) the β-line

$$u(\lambda, \mu) = \breve{u}_\theta(\lambda, \mu) + \int_\theta^\mu [-\cos(\beta - \mu) \cdot \omega(\lambda, \beta) + \kappa \sin(\beta - \mu)]\, B(\lambda, \beta)d\beta$$

$$v(\lambda, \mu) = \breve{v}_\theta(\lambda, \mu) - \int_\theta^\mu [\kappa \cos(\beta - \mu) + \sin(\beta - \mu) \times \omega(\lambda, \beta)]\, B(\lambda, \beta)d\beta, \tag{2.10}$$

where

$$\breve{u}_\theta(\lambda, \mu) = \cos(\theta - \mu) \cdot u(\lambda, \theta) - \sin(\theta - \mu) \cdot v(\lambda, \theta)$$

$$\breve{v}_\theta(\lambda, \mu) = \sin(\theta - \mu) \cdot u(\lambda, \theta) + \cos(\theta - \mu) \cdot v(\lambda, \theta). \tag{2.11}$$

In the domains, where the rule (I.6.1) holds, the differential equations (I.6.2) are valid. Substitution of these equations into (2.1) gives

$$\varepsilon_{11} = \frac{1}{A}\left(\frac{\partial u}{\partial \alpha} + v\right), \quad \varepsilon_{22} = \frac{1}{B}\left(\frac{\partial v}{\partial \beta} + u\right). \tag{2.12}$$

Thus, the optimality conditions (2.4) lead to the equations

$$\frac{\partial u}{\partial \alpha} + v = A, \quad \frac{\partial v}{\partial \beta} + u = -\kappa B\,, \tag{2.13}$$

hence,

$$Lu = (\kappa + 1)B; \quad Lv = -(\kappa + 1)A\,, \tag{2.14}$$

where L is given by (I.6.3).

The displacement fields u, v can be found by the integration formulae (2.10), (2.11) or by solving the set of differential equations (2.14). This set can be simplified by a change of unknowns. Let us note that the auxiliary fields

$$u^0(\alpha, \beta) = u(\alpha, \beta) - (\kappa + 1)\alpha A(\alpha, \beta)$$

$$v^0(\alpha, \beta) = v(\alpha, \beta) + (\kappa + 1)\beta B(\alpha, \beta) \tag{2.15}$$

satisfy the hyperbolic equations

$$Lu^0 = 0; \quad Lv^0 = 0\,. \tag{2.16}$$

We remember that the fields $\overline{x}$, $\overline{y}$, A, B satisfy the same equation (see (I.6.2), (a.52) and (a.56)). Thus, the fields u^0, v^0 can be found by Riemann's method (see Section a.2.2). The formulae (2.15) are generalization of the formulae (a.66) for the case of $\kappa \neq 1$.

Upon finding the fields u^0, v^0 by Riemann's method one can easy find the fields u, v directly from (2.15).

The geometric analysis of the Hencky net within Ω_0 leads to a subdivision of this domain into the subdomains I(or RAN), II upper (RBA), II lower (NAC), III(ABDC), IV upper (BDH), IV lower (CDG), V(DHJG) etc. (see Fig. I.19). The functions x, y, $\overline{x}$, $\overline{y}$, A, B have been constructed step-by-step, starting from domain I and then extending the solution to further domains, which is typical for solving hyperbolic problems. The fields (u,v) will be constructed in the same sequence, starting from the domain RAN.

3 Displacement fields (u, v) in the domain RAN

The components $\left(u_{x_0}, u_{y_0}\right)$ of this field were found in Section I.4 (see (I.4.7)), referring to the coordinate system (x_0,y_0). It is helpful to find these components in the coordinate system (x,y) determined by the lines RA and AN (see Fig. I.4 and I.10). Instead of performing the transformation of the fields (I.4.7) we impose the conditions (2.4), assuming that (u,v) represent the components of the vector $\overline{\mathbf{u}}$ in the direction of the x, y lines (see Fig. I.10):

$$\frac{\partial u}{\partial x} = 1\,, \quad \frac{\partial v}{\partial y} = -\kappa\,, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0\,. \tag{3.1}$$

Moreover, the line RN is parameterized by the equation

$$y = -\frac{r_1}{r_2}x - r_1, \tag{3.2}$$

where $r_1 = NA$ and $r_2 = RA$. The fields u, v satisfying (2.4) and vanishing on the line (3.2) have the form

$$u = x + \frac{r_2}{r_1}y + r_2, \quad v = -\kappa y - \frac{r_2}{r_1}x - \kappa r_1\,. \tag{3.3}$$

This result is compatible with (I.4.7). Let us compute the rigid rotation

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{r_2}{r_1}. \tag{3.4}$$

By virtue of (I.4.10) we have

$$\omega = -\kappa^{1/2}. \tag{3.5}$$

4 Virtual displacement fields within the fan-like domains

The fan NAC. This domain is parameterized by (α, β_1). We substitute (I.5.4) into (2.1) and find the strain components

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{r_1\beta_1}\left[\frac{\partial u}{\partial \alpha} + v\right], \\ \varepsilon_{22} &= \frac{1}{r_1}\frac{\partial v}{\partial \beta_1}, \\ 2\varepsilon_{12} &= \frac{1}{r_1\beta_1}\frac{\partial v}{\partial \alpha} + \frac{1}{r_1}\left(\frac{\partial u}{\partial \beta_1} - \frac{u}{\beta_1}\right). \end{aligned} \tag{4.1}$$

The optimality conditions (2.4), along with the condition of vanishing of displacements at point N (or for $\beta_1 = 0$), yield

$$u = r_1[(\kappa+1)\alpha\beta_1 + C\beta_1], \qquad v = -r_1\kappa\beta_1\,, \tag{4.2}$$

where C is a constant. Adjustment of displacements (4.2) with the fields (3.3) along the line AN gives $C = r_2/r_1$. Thus, the fields (u,v) in the domain NAC read

$$u = (\kappa+1)\alpha\beta_1 r_1 + \beta_1 r_2, \qquad v = -r_1\kappa\beta_1\,. \tag{4.3}$$

Let us compute the rigid rotation

$$\omega = -\frac{1}{r_1}[r_2 + (\kappa+1)\alpha r_1]. \tag{4.4}$$

Along the line NA, where $\alpha = 0$, we have $\omega = -r_2/r_1$, which is compatible with (3.4).

The fan RBA. This domain is parameterized by (α_1, β). The Lamé coefficients are given by (I.5.2.) Proceeding as before, we make use of the optimality conditions (2.4) and find (u,v) from the conditions of vanishing these fields at point R and their compatibility with the displacement fields (3.3) along RA:

$$u = r_2\alpha_1\,, \quad v = -(\kappa+1)\alpha_1\,\beta\, r_2 - \kappa\alpha_1 r_1\,. \tag{4.5}$$

The rigid rotation ω along RA is constant and equals $\omega = -r_2/r_1$.

5 The Prager–Hill domain ABDC: construction of the virtual displacement fields

The geometric characteristics of the net were found in Section I.6. The formulae there reported make it possible to find the auxiliary fields u^0, v^0 by solving (2.16) by the Riemann method.

Along the line AB ($\alpha_1 = 1$, $\alpha = 0$; see Fig. 1), the values of v and B are known:

$$v(0,\beta) = -(\kappa+1)r_2\beta - \kappa r_1, \quad B(0,\beta) = r_2\,. \tag{5.1}$$

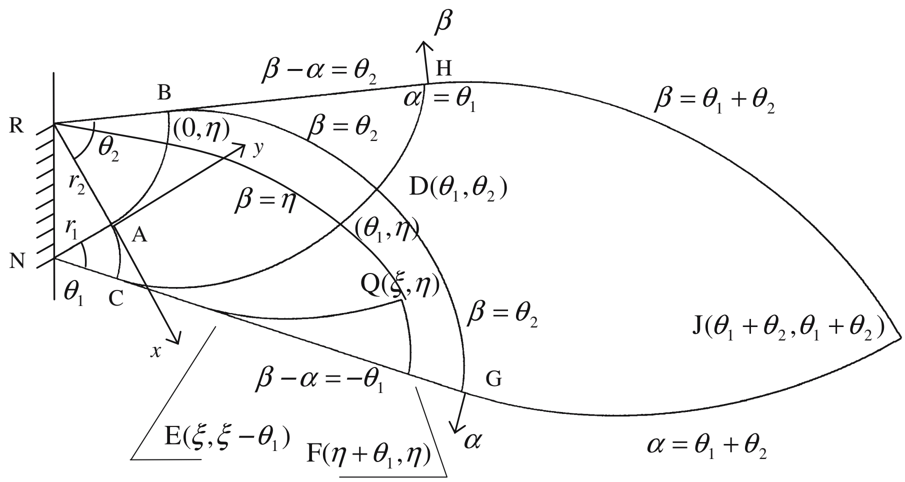

Fig. 1 Geometry of the Michell cantilever

Thus, along this line, we know the values of v^0 given by (2.12):

$$v^0(0, \beta) = -\kappa r_1. \tag{5.2}$$

Along the curve AC (β_1=1, β=0) we have

$$v(\alpha, 0) = -\kappa r_1. \tag{5.3}$$

Along this line β=0; hence, according to (2.12),

$$v^0(\alpha, 0) = -\kappa r_1. \tag{5.4}$$

Function v^0 satisfies (2.13). Thus, it can be expressed by Riemann's formula (a.17) for θ_1=θ_2=0. This formula reduces to the form

$$v^0(\lambda, \mu) = v^0(0, 0) \cdot G_0(\lambda, \mu), \tag{5.5}$$

since the functions $\alpha \rightarrow v^0(\alpha,0)$, $\beta \rightarrow v^0(0,\beta)$ are constant. Eventually,

$$v^0(\lambda, \mu) = -\kappa r_1 G_0(\lambda, \mu).$$

Substitution of the formula for B within ABDC (see I.6.8) into (2.15) gives

$$v(\lambda, \mu) = v^*(\lambda, \mu), \tag{5.6a}$$

where

$$v^*(\lambda, \mu) = -[\kappa r_1 + (\kappa + 1)\mu r_2]G_0(\lambda, \mu) \\ -(\kappa + 1)\mu r_1 G_1(\lambda, \mu). \tag{5.6b}$$

Similarly, one can find $u(\lambda, \mu)$. We note that along AB (α_1=1, α=0),

$$u(0, \beta) = r_2, \quad u^0(0, \beta) = r_2. \tag{5.7}$$

The following relations hold along AC (β_1=1, β=0):

$$u(\alpha, 0) = (\kappa + 1)\alpha r_1 + r_2, \quad A(\alpha, 0) = r_1; \tag{5.8}$$

hence,

$$u^0(\alpha, 0) = r_2. \tag{5.9}$$

Because the functions $\beta \rightarrow u^0(0, \beta)$, $\alpha \rightarrow u^0(\alpha, 0)$ are constant, the Riemann formula (a.17) reduces to its first term

$$u^0(\lambda, \mu) = u^0(0, 0)G_0(\lambda, \mu). \tag{5.10}$$

Thus, we have

$$u^0(\lambda, \mu) = r_2 G_0(\lambda, \mu),$$

and by using (2.15) and (I.6.7), we find the final result

$$u(\lambda, \mu) = u^*(\lambda, \mu), \tag{5.11a}$$

where

$$u^*(\lambda, \mu) = [r_2 + (\kappa + 1)r_1\lambda]G_0(\lambda, \mu) \\ +\lambda(\kappa + 1)r_2 G_1(\mu, \lambda). \tag{5.11b}$$

The results (5.11a, b and 5.6a, b) can be found by means of formulae (2.8)–(2.11). Let us explain here this alternative derivation. We apply the equations (2.8) in which integration runs over the α-line, starting from an arbitrary point (0, η) on the arc AB (see Fig. I.12). Function $\omega(\alpha, \beta)$ is given by (2.7), where constant ω_0 is given by (3.4) or (3.5); thus,

$$\omega(\alpha, \eta) = -\kappa^{1/2} - (1 + \kappa)(\alpha + \eta). \tag{5.12}$$

According to (4.5) we have

$$u(0, \eta) = r_2\,, \quad v(0, \eta) = -(\kappa + 1)r_2\eta - \kappa r_1\,. \tag{5.13}$$

Now we have θ=0, and (2.9) assumes the form

$$\widehat{u}_0(\xi, \eta) = r_2 \cos\xi + [(\kappa + 1)r_2\eta + \kappa r_1] \sin\xi \\ \widehat{v}_0(\xi, \eta) = r_2 \sin\xi - [(\kappa + 1)r_2\eta + \kappa r_1] \cos\xi. \tag{5.14}$$

We arrive at the following formulae for u:

$$u(\xi, \eta) = \widehat{u}_0(\xi, \eta) + u_1(\xi, \eta) \tag{5.15}$$

with

$$u_1(\xi,\eta) = \int_0^{\xi} U_\bullet(\xi,\eta;\alpha)[r_1G_0(\alpha,\eta) + r_2G_1(\eta,\alpha)]d\alpha \tag{5.16}$$

and

$$U_\bullet(\xi,\eta;\alpha) = \cos(\xi-\alpha) + \left(\kappa^{1/2} + (\kappa+1)(\alpha+\eta)\right) \times \sin(\xi-\alpha). \tag{5.17}$$

We have made use of (I.6.7) for the form of the function $A(\alpha,\eta)$ within the domain ABDC. The results reported in the Appendix (a.A) and (a.16) make it possible to perform integration in (5.16). By also using (a.7)–(a.14) we find

$$\begin{aligned} u_1(\xi,\eta) &= -r_2\cos\xi - \kappa r_1\sin\xi - (\kappa+1)\eta r_2\sin\xi \\ &\quad + r_2G_0(\xi,\eta) + (\kappa+1)r_1\xi G_0(\xi,\eta) \\ &\quad + (\kappa+1)\eta r_2 G_1(\xi,\eta). \end{aligned} \tag{5.18}$$

The results (5.14) and (5.18) give the formula (5.11a, b) obtained previously in a different manner.

We omit now a similar and equally lengthy alternative derivation of (5.6a,b) by using the second equation of (2.8). One should additionally use (a.40) and (a.41).

The fields u, v found here and the fields A, B found in Section I.6 are linked by (2.13), which can be confirmed by using the differentiation rules (a.4).

6 Domains of Chan

6.1 Lower domain of Chan (CDG): construction of virtual displacement fields

In the previous section two alternative methods of construction of the displacement fields within domain ABDC have been presented. The first of these methods, based on using auxiliary fields u^0, v^0, does not apply for the region considered here. We apply the second one based on representations (2.8). We consider the arc α starting at $(0,\eta)$ on the arc AB, going through point (θ_1,η) on arc CD and ending at point (ξ,η) within domain CDG (see Fig. 1). According to (2.8) we may write

$$u(\xi,\eta) = \widehat{u}_{\theta_1}(\xi,\eta) + \int_{\theta_1}^{\xi} U_\bullet(\xi,\eta;\alpha)A(\alpha,\eta)d\alpha, \tag{6.1}$$

where $\widehat{u}_{\theta_1}(\xi,\eta)$ is given by (2.9), and the function $U_\bullet(\xi,\eta;\alpha)$ is defined by (5.17). Although correct, (6.1) is not helpful. One should reinterpret (2.8) as follows:

$$u(\xi,\eta) = \widehat{u}_0(\xi,\eta) + \int_0^{\xi} U_\bullet(\xi,\eta;\alpha)A(\alpha,\eta)d\alpha, \tag{6.2}$$

where $\widehat{u}_0$ is given by (5.14), and the Lamé coefficient $A(\alpha,\eta)$ is expressed as follows:

$$A(\alpha,\eta) = \begin{cases} A^*(\alpha,\eta), & \text{if} \quad 0 \le \alpha \le \theta_1 \\ A^*(\alpha,\eta) + A_\bullet(\alpha,\eta) & \quad \alpha \ge \theta_1 \end{cases}, \tag{6.3}$$

where (see I.7.20)

$$\begin{aligned} A^*(\alpha,\eta) &= r_1G_0(\alpha,\eta) + r_2G_1(\eta,\alpha) \\ A_\bullet(\alpha,\eta) &= -r_1G_0(\alpha-\theta_1,\eta+\theta_1) - r_2G_1(\alpha-\theta_1,\eta+\theta_1). \end{aligned} \tag{6.4}$$

Consequently, the displacement u within CDG is represented by

$$u(\xi,\eta) = \widehat{u}_0(\xi,\eta) + \int_0^{\theta_1} U_\bullet(\xi,\eta;\alpha)A^*(\alpha,\eta)d\alpha + \int_{\theta_1}^{\xi} U_\bullet(\xi,\eta;\alpha)\left[A^*(\alpha,\eta) + A_\bullet(\alpha,\eta)\right]d\alpha. \tag{6.5}$$

Thus, using (5.15)–(5.18), we arrive at

$$u(\xi,\eta) = u^*(\xi,\eta) + \int_{\theta_1}^{\xi} U_\bullet(\xi,\eta;\alpha)A_\bullet(\alpha,\eta)d\alpha, \tag{6.6}$$

where function $u^*(\xi,\eta)$ is defined by (5.11b); the arguments of the function vary within the following limits:

$$\theta_1 \le \xi \le \eta + \theta_1, \quad 0 \le \eta \le \theta_2. \tag{6.7}$$

Using the results of integration reported in the Appendix I.A, one computes

$$\int_{\theta_1}^{\xi} U_\bullet(\xi,\eta;\alpha)A_\bullet(\alpha,\eta)d\alpha = u_\bullet(\xi-\theta_1,\eta+\theta_1) \tag{6.8}$$

with

$$\begin{aligned} u_\bullet(\xi,\eta)/r_1 &= \kappa\cdot F_1(\xi,\eta) + 2\kappa^{2/3}F_2(\xi,\eta) \\ &\quad -\kappa\cdot F_3(\xi,\eta) - (\kappa+1) \\ &\quad \times \xi(G_0(\xi,\eta) + \kappa^{1/2}G_1(\xi,\eta)). \end{aligned} \tag{6.9}$$

We eventually find

$$u(\xi,\eta) = u^*(\xi,\eta) + u_\bullet(\xi-\theta_1,\eta+\theta_1). \tag{6.10}$$

In the case of $\kappa=1$ or $\sigma_T/\sigma_C=1$, $r_1=r_2=r$, the formula above coincides with (a.119).

Proceeding similarly, we can find the formula for v within CDG. The second integral formula of (2.8) should be interpreted as follows:

$$v(\xi,\eta) = \widehat{v}_0(\xi,\eta) + \int_0^{\xi} V_\bullet(\xi,\eta;\alpha)A(\alpha,\eta)d\alpha \tag{6.11}$$

with $\widehat{v}_0$ given by (2.9) for $\theta=0$ and

$$V_\bullet(\xi,\eta;\alpha) = \sin(\xi-\alpha) - \left(\kappa^{1/2} + (\kappa+1)(\alpha+\eta)\right) \times \cos(\xi-\alpha). \tag{6.12}$$

The function $A(\alpha,\eta)$ is given by (6.3). Thus, we have

$$v(\xi,\eta) = v^*(\xi,\eta) + \int_{\theta_1}^{\xi} V_\bullet(\xi,\eta;\alpha)A_\bullet(\alpha,\eta)d\alpha \tag{6.13}$$

where v^* is given by (5.6b). Integration in (6.13) can be performed analytically by using the results put in the Appendix I.A to find

$$\int_{\theta_1}^{\xi} V_\bullet(\xi,\eta;\alpha)A_\bullet(\alpha,\eta)d\alpha = v_\bullet(\xi-\theta_1,\eta+\theta_1) \tag{6.14}$$

with

$$v_\bullet(\xi,\eta)/r_1 = 2\kappa F_2(\xi,\eta) + (\kappa+1)\eta\,(F_1(\xi,\eta)+F_3(\xi,\eta)) + (2\kappa+1)\,\kappa^{1/2}F_3(\xi,\eta) + (\kappa+1)\kappa^{1/2}\eta \times (F_2(\xi,\eta)+F_4(\xi,\eta)) + \kappa^{1/2}F_1(\xi,\eta) \tag{6.15}$$

or

$$v_\bullet(\xi,\eta)/r_1 = 2\kappa F_2(\xi,\eta) + (\kappa+1)\eta \times \left(G_1(\xi,\eta) + \kappa^{1/2}G_2(\xi,\eta)\right) + (2\kappa+1)\kappa^{1/2}F_3(\xi,\eta) + \kappa^{1/2}F_1(\xi,\eta) \tag{6.15'}$$

Eventually, we have

$$v(\xi,\eta) = v^*(\xi,\eta) + v_\bullet(\xi-\theta_1,\eta+\theta_1). \tag{6.16}$$

Functions A, B,u,v within CDG must be linked by the (2.13). To check it is sufficient to verify

$$\frac{\partial u_\bullet}{\partial\alpha} + v_\bullet = A_\bullet, \quad \frac{\partial v_\bullet}{\partial\beta} + u_\bullet = -\kappa B_\bullet \tag{6.17}$$

with

$$B_\bullet(\alpha,\eta) = -r_1G_1(\alpha-\theta_1,\eta+\theta_1) - r_2G_2(\alpha-\theta_1,\eta+\theta_1) \tag{6.18}$$

see the second equation of (I.7.20). Note that relations (2.13) were checked, as remarked in Section 5. Moreover, one can check that the fields $u_\bullet$, $v_\bullet$ vanish along CD.

6.2 The upper domain of Chan (BDH): construction of virtual displacement fields

To find the values of u at any point within domain BDH we apply the integration formula (2.10)

$$u(\xi,\eta) = \breve{u}_0(\xi,\eta) + \int_0^{\eta} U^\bullet(\xi,\eta;\beta)B(\xi,\beta)d\beta, \tag{6.19}$$

where $\breve{u}_0$ is given by (2.11) for $\theta=0$ and

$$U^\bullet(\xi,\eta;\beta) = \kappa\sin(\beta-\eta) + \left(\kappa^{1/2} + (\kappa+1)(\xi+\beta)\right) \times \cos(\beta-\eta) \tag{6.20}$$

$$B(\xi,\beta) = \begin{cases} B^*(\xi,\beta), & \text{if} \quad 0\le\beta\le\theta_2 \\ B^*(\xi,\beta)+B^\bullet(\xi,\beta) & \quad \beta\ge\theta_2 \end{cases} \tag{6.21}$$

with

$$B^*(\xi,\beta) = r_1G_1(\xi,\beta) + r_2G_0(\xi,\beta)$$
$$B^\bullet(\xi,\beta) = -r_1G_1(\beta-\theta_2,\xi+\theta_2) - r_2G_0(\beta-\theta_2,\xi+\theta_2). \tag{6.22}$$

Thus, we obtain

$$u(\xi,\eta) = u^*(\xi,\eta) + \int_{\theta_2}^{\eta} U^\bullet(\xi,\eta;\beta)B^\bullet(\xi,\beta)d\beta, \tag{6.23}$$

where u^* is defined by (5.11b). By using the results of the Appendix I.A we perform all the integrations analytically to find

$$\int_{\theta_2}^{\eta} U^\bullet(\xi,\eta;\beta)B^\bullet(\xi,\beta)d\beta = u^\bullet(\xi+\theta_2,\eta-\theta_2), \tag{6.24}$$

where

$$u^\bullet(\xi,\eta)/r_1 = -(2\kappa+1)F_3(\eta,\xi) - \kappa F_1(\eta,\xi) - 2\kappa^{1/2}F_2(\eta,\xi) - (\kappa+1) \times \xi\left(G_2(\eta,\xi) + \kappa^{1/2}G_1(\eta,\xi)\right) \tag{6.25}$$

Thus, we arrive at

$$u(\xi,\eta) = u^*(\xi,\eta) + u^\bullet(\xi+\theta_2,\eta-\theta_2). \tag{6.26}$$

Let us find now the field v in this domain. We use (2.10) with the help of (5.12) to obtain

$$v(\xi,\eta) = \breve{v}_0(\xi,\eta) + \int_0^{\eta} V^\bullet(\xi,\eta;\beta)B(\xi,\beta)d\beta, \tag{6.27}$$

where $\breve{v}_0$ is given by (2.11) for $\theta=0$ and

$$V^\bullet(\xi,\eta;\beta) = -\kappa\cos(\beta-\eta) + \left(\kappa^{1/2} + (\kappa+1)(\xi+\beta)\right) \times \sin(\beta-\eta). \tag{6.28}$$

Function B is given by (6.21) and (6.22). Thus, we obtain

$$v(\xi,\eta) = v^*(\xi,\eta) + \int_{\theta_2}^{\eta} V^\bullet(\xi,\eta;\beta)B^\bullet(\xi,\beta)d\beta\,, \tag{6.29}$$

where v^* is given by (5.6b). By applying the results of the Appendix I.A we express the integral in terms of Lommel functions

$$\int_{\theta_2}^{\eta} V^{\bullet}(\xi,\eta;\beta)B^{\bullet}(\xi,\beta)d\beta = v^{\bullet}(\xi+\theta_2,\eta-\theta_2) \tag{6.30}$$

where

$$v^{\bullet}(\xi,\eta)/r_1 = \kappa^{1/2}[F_3(\eta,\xi)-F_1(\eta,\xi)]-2F_2(\eta.\xi) + (\kappa+1)\eta\left(G_1(\eta,\xi)+\kappa^{1/2}G_0(\eta,\xi)\right). \tag{6.31}$$

Finally,

$$v(\xi,\eta) = v^*(\xi,\eta)+v^{\bullet}(\xi+\theta_2,\eta-\theta_2). \tag{6.32}$$

Functions $u^{\bullet}, v^{\bullet}, A^{\bullet}, B^{\bullet}$ are linked by the differential equations

$$\frac{\partial u^{\bullet}}{\partial\alpha}+v^{\bullet}=A^{\bullet} \qquad \frac{\partial v^{\bullet}}{\partial\beta}+u^{\bullet}=-\kappa B^{\bullet}, \tag{6.33}$$

where

$$A^{\bullet}(\alpha,\beta) = -r_1G_2(\beta-\theta_2,\alpha+\theta_2) - r_2G_1(\beta-\theta_2,\alpha+\theta_2). \tag{6.34}$$

Equation (6.33) follows from (2.13). Moreover, the functions u, v are continuous along BD.

In case of κ=1 or σ_T/σ_C=1, r_1=r_2=r, θ_1=θ_2=θ, the results found above coincide with those reported in Lewiński et al. (1994a). Moreover, the following relations hold:

$$\begin{aligned} u(\alpha,\beta)^{ABDC} &= -v(\beta,\alpha)^{ABDC} \\ u(\alpha,\beta)^{BDH} &= -v(\beta,\alpha)^{CDG}, \\ v(\alpha,\beta)^{BDH} &= -u(\beta,\alpha)^{CDG}. \end{aligned} \tag{6.35}$$

7 Construction of virtual displacement fields within the domain DHJG

The formulae (I.8.6) expressing the Lamé coefficients A, B in DHJG can be written by using the auxiliary functions $A^*, B^*, A_{\bullet}, A^{\bullet}, B_{\bullet}, B^{\bullet}$ introduced in the previous section by the formula

$$X(\alpha,\beta)_{|DHJG} = X^*(\alpha,\beta)+X_{\bullet}(\alpha,\beta)+X^{\bullet}(\alpha,\beta), \tag{7.1}$$

where X stands for A or B.

By using the formulae (2.8)–(2.11) one can show that the rule (7.1) determines the distribution of the displacement fields within DHJG. Thus, the substitution X=u and X=v into (7.1) provides the final analytical formulae for u and v within DHJG, respectively. We obtain

$$\begin{aligned} u(\xi,\eta)/r_1 = &\left[\kappa^{1/2}+(\kappa+1)\xi\right]G_0(\xi,\eta) \\ &+(\kappa+1)\kappa^{1/2}\xi G_1(\eta,\xi)+\kappa F_1(\xi-\theta_1,\eta+\theta_1) \\ &+2\kappa^{3/2}F_2(\xi-\theta_1,\eta+\theta_1)-\kappa F_3(\xi-\theta_1,\eta+\theta_1) \\ &-(\kappa+1)(\xi-\theta_1) \\ &\times\left[G_0(\xi-\theta_1,\eta+\theta_1)+\kappa^{1/2}G_1(\xi-\theta_1,\eta+\theta_1)\right] \\ &-(2\kappa+1)F_3(\eta-\theta_2,\xi+\theta_2) \\ &-\kappa F_1(\eta-\theta_2,\xi+\theta_2)-(\kappa+1)\cdot(\xi+\theta_2) \\ &\times\left[G_2(\eta-\theta_2,\xi+\theta_2)+\kappa^{1/2}G_1(\eta-\theta_2,\xi+\theta_2)\right] \\ &-2\kappa^{1/2}F_2(\eta-\theta_2,\xi+\theta_2) \end{aligned} \tag{7.2}$$

$$\begin{aligned} v(\xi,\eta)/r_1 = &-\left[\kappa+\kappa^{1/2}(\kappa+1)\eta\right]\cdot G_0(\xi,\eta) \\ &-(\kappa+1)\cdot\eta G_1(\xi,\eta) \\ &+2\kappa F_2(\xi-\theta_1,\eta+\theta_1)+(\kappa+1)(\eta+\theta_1) \\ &\times\left[G_1(\xi-\theta_1,\eta+\theta_1)+\kappa^{1/2}G_2(\xi-\theta_1,\eta+\theta_1)\right] \\ &+(2\kappa+1)\kappa^{1/2}F_3(\xi-\theta_1,\eta+\theta_1) \\ &+\kappa^{1/2}F_1(\xi-\theta_1,\eta+\theta_1) \\ &-2F_2(\eta-\theta_2,\xi+\theta_2)+(\kappa+1)(\eta-\theta_2) \\ &\times\left[G_1(\eta-\theta_2,\xi+\theta_2)+\kappa^{1/2}G_0(\eta-\theta_2,\xi+\theta_2)\right] \\ &-\kappa^{1/2}F_1(\eta-\theta_2,\xi+\theta_2)+\kappa^{1/2}F_3(\eta-\theta_2,\xi+\theta_2). \end{aligned} \tag{7.3}$$

In case of κ=1 or σ_T/σ_C=1, r_1=r_2=r, θ_1=θ_2=θ, the equations above reduce to

$$u(\xi,\eta) = u^{**}(\xi,\eta)\ ,\ \ v(\xi,\eta) = -u^{**}(\eta,\xi), \tag{7.4}$$

where

$$\begin{aligned} u^{**}(\xi,\eta)/r = &(1+2\xi)G_0(\xi,\eta)+2\xi G_1(\eta,\xi) \\ &-2(\xi-\theta)G_0(\xi-\theta,\eta+\theta) \\ &-(1+2\xi-2\theta)G_1(\xi-\theta,\eta+\theta) \\ &+2F_1(\xi-\theta,\eta+\theta)+2F_2(\xi-\theta,\eta+\theta) \\ &-[1+2(\xi+\theta)]G_1(\eta-\theta,\xi+\theta) \\ &-2(1+\eta-\theta)G_0(\eta-\theta,\xi+\theta) \\ &+2F_0(\eta-\theta,\xi+\theta)+2F_1(\eta-\theta,\xi+\theta). \end{aligned} \tag{7.5}$$

Function u assumes maximum at point J (see Fig. 2).

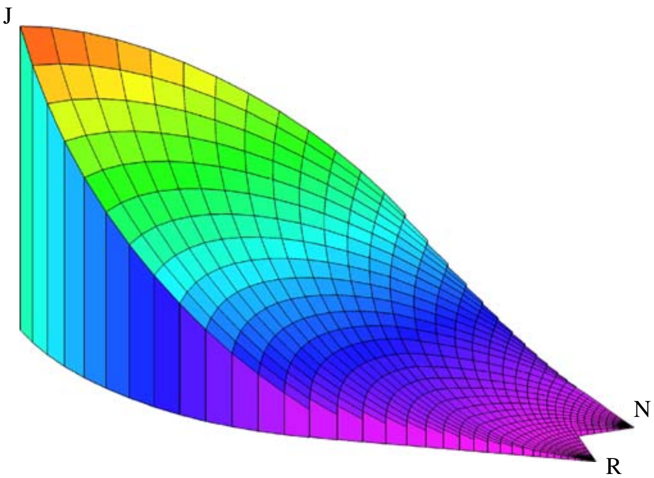

Fig. 2 Virtual displacement field $u(\alpha, \beta)$, κ=1, $\theta_1 = \theta_2 = \frac{13\pi}{40}$

8 Chan-like designs of second rank: virtual displacement fields for the symmetric case of κ=1

From now onward we shall concentrate on the special case of κ=1. The general formulae can be found (still by (2.8)–(2.11)), but the results are lengthy.

8.1 Domain GJG$_2$

In this domain the Lamé coefficient A is expressed by

$$A(\alpha, \eta) = A^*(\alpha, \eta) + A_\bullet(\alpha, \eta) + A^\bullet(\alpha, \eta) + A_{\bullet\bullet}(\alpha, \eta), \tag{8.1}$$

where A^*, $A_\bullet$, $A^\bullet$ are given in Section 6, whilst

$$A_{\bullet\bullet}(\alpha, \eta)/r = G_1(\alpha-2\theta, \eta+2\theta) + G_2(\alpha-2\theta, \eta+2\theta). \tag{8.2}$$

Function (5.17) now has the form

$$U_\bullet(\xi, \eta; \alpha) = \cos(\xi - \alpha) + (1 + 2(\alpha + \eta)) \sin(\xi - \alpha). \tag{8.3}$$

Similarly, as in Section 6, we write the representation

$$u(\xi, \eta) = u^{**}(\xi, \eta) + \int_{2\theta}^{\xi} U_\bullet(\xi, \eta; \alpha) A_{\bullet\bullet}(\alpha, \eta) d\alpha, \tag{8.4}$$

where $u^{**}(\xi, \eta)$ is given by (7.5).

The integral in (8.4) can be expressed in terms of the Lommel functions by using the results collected in the Appendix I.A. One finds

$$\int_{2\theta}^{\xi} U_\bullet(\xi, \eta; \alpha) A_{\bullet\bullet}(\alpha, \eta) d\alpha = u_{\bullet\bullet}(\xi - 2\theta, \eta + 2\theta), \tag{8.5}$$

where

$$u_{\bullet\bullet}(\xi, \eta)/r = 2\xi G_1(\xi, \eta) - (1 + 2\xi) G_2(\xi, \eta) - 4F_2(\xi, \eta) - 4F_3(\xi, \eta) \tag{8.6}$$

Finally, we have

$$u(\xi, \eta) = u^{**}(\xi, \eta) + u_{\bullet\bullet}(\xi - 2\theta, \eta + 2\theta). \tag{8.7}$$

Similarly, we find the field

$$v(\xi, \eta) = v^{**}(\xi, \eta) + \int_{2\theta}^{\xi} V_\bullet(\xi, \eta; \alpha) A_{\bullet\bullet}(\alpha, \eta) d\alpha \tag{8.8}$$

where $v^{**}(\xi, \eta) = -u^{**}(\eta, \xi)$

$$V_\bullet(\xi, \eta; \alpha) = \sin(\xi - \alpha) - (1 + 2(\alpha + \eta)) \cos(\xi - \alpha). \tag{8.9}$$

By using the results set up in the Appendix I.A we compute

$$\int_{2\theta}^{\xi} V_\bullet(\xi, \eta; \alpha) A_{\bullet\bullet}(\alpha, \eta) d\alpha = v_{\bullet\bullet}(\xi - 2\theta, \eta + 2\theta), \tag{8.10}$$

where

$$v_{\bullet\bullet}(\xi, \eta)/r = -2\xi G_0(\xi, \eta) - G_2(\xi, \eta) - 2(1+\xi) G_1(\xi, \eta) + 4F_1(\xi, \eta) + 4F_2(\xi, \eta) \tag{8.11}$$

Thus, we have

$$v(\xi, \eta) = v^{**}(\xi, \eta) + v_{\bullet\bullet}(\xi - 2\theta, \eta + 2\theta). \tag{8.12}$$

The function $v_{\bullet\bullet}$ can be also found from the equation

$$\frac{\partial u_{\bullet\bullet}}{\partial \alpha} + v_{\bullet\bullet} = A_{\bullet\bullet}. \tag{8.13}$$

8.2 Domain HJH$_2$

In this domain the function $B(\alpha, \beta)$ is expressed by

$$B(\alpha, \beta) = B^*(\alpha, \beta) + B_\bullet(\alpha, \beta) + B^\bullet(\alpha, \beta) + B^{\bullet\bullet}(\alpha, \beta), \tag{8.14}$$

where

$$B^{\bullet\bullet}(\alpha, \beta) = G_1(\beta - 2\theta, \alpha + 2\theta) + G_2(\beta - 2\theta, \alpha + 2\theta). \tag{8.15}$$

The function $U^\bullet$ given by (6.20) now has the form

$$U^\bullet(\xi, \eta; \beta) = \sin(\beta - \eta) - (1 + 2(\xi + \beta)) \cos(\beta - \eta). \tag{8.16}$$

By using (2.10) one finds

$$u(\xi, \eta) = u^{**}(\xi, \eta) + \int_{2\theta}^{\eta} U^\bullet(\xi, \eta; \beta) B^{\bullet\bullet}(\xi, \beta) d\beta. \tag{8.17}$$

We note now the analogy between the integral above and the integral in (8.10); hence, we find

$$\int_{2\theta}^{\eta} U^{\bullet}(\xi,\eta;\beta)B^{\bullet\bullet}(\xi,\beta)d\beta = -v_{\bullet\bullet}(\xi-2\theta,\eta+2\theta), \tag{8.18}$$

where $v_{\bullet\bullet}$ is given by (8.11). Thus, in the domain HJH_2, we have

$$u(\xi,\eta) = u^{**}(\xi,\eta) - v_{\bullet\bullet}(\eta-2\theta,\xi+2\theta). \tag{8.19}$$

Similarly, one can compute the field v within HJH_2. The function $V^{\bullet}$ given by (6.28) reduces to the form

$$V^{\bullet}(\xi,\eta;\beta) = -\cos(\beta-\eta) + (1+2(\xi+\beta))\sin(\beta-\eta). \tag{8.20}$$

According to (2.10) we have

$$v(\xi,\eta) = v^{**}(\xi,\eta) + \int_{2\theta}^{\eta} V^{\bullet}(\xi,\eta;\beta)B^{\bullet\bullet}(\xi,\beta)d\beta. \tag{8.21}$$

We note that

$$\int_{2\theta}^{\eta} V^{\bullet}(\xi,\eta;\beta)B^{\bullet\bullet}(\xi,\beta)d\beta = -u_{\bullet\bullet}(\eta-2\theta,\xi+2\theta), \tag{8.22}$$

where u is given by (8.6) or

$$v(\xi,\eta) = v^{**}(\xi,\eta) - v_{\bullet\bullet}(\eta-2\theta,\xi+2\theta). \tag{8.23}$$

9 Domain $JH_2J_2G_2$

We shall concentrate on the special case of κ=1. Let us note that

$$u(\xi,\eta)^{JH_2J_2G_2} = u^{**}(\xi,\eta) + u_{\bullet\bullet}(\xi-2\theta,\eta+2\theta) - v_{\bullet\bullet}(\eta-2\theta,\xi+2\theta) \tag{9.1}$$

$$v(\xi,\eta)^{JH_2J_2G_2} = v^{**}(\xi,\eta) + v_{\bullet\bullet}(\xi-2\theta,\eta+2\theta) - u_{\bullet\bullet}(\eta-2\theta,\xi+2\theta). \tag{9.2}$$

Thus, we see the relationship

$$v(\xi,\eta)^{JH_2J_2G_2} = -u(\eta,\xi)^{JH_2J_2G_2}. \tag{9.3}$$

For the sake of completeness we report the formula (9.1) explicitly:

$$\begin{aligned} u^{JH_2J_2G_2}(\xi,\eta)/r = {} & u^{**}(\xi,\eta)/r + 2(\xi-2\theta)G_1(\xi-2\theta,\eta+2\theta) \\ & +(1+2\xi-4\theta)G_2(\xi-2\theta,\eta+2\theta) \\ & -4F_2(\xi-2\theta,\eta+2\theta) - 4F_3(\xi-2\theta,\eta+2\theta) \\ & +2G_1(\eta-2\theta,\xi+2\theta) + 2(\eta-2\theta)[G_0(\eta-2\theta,\xi+2\theta) + G_1(\eta-2\theta,\xi+2\theta)] \\ & +G_2(\eta-2\theta,\xi+2\theta) - 4F_1(\eta-2\theta,\xi+2\theta) - 4F_2(\eta-2\theta,\xi+2\theta), \end{aligned} \tag{9.4}$$

where $u^{**}(\xi,\eta)$ is given by (7.5).

10 Final remarks

The results found in this part of the present work make it possible to compute the weights of the optimal Michell trusses considered directly by the formula (I.2.12). These formulae for the weights will be reported in the last part of the paper. However, to make the work complete, the third part of the work will concern the stress field analysis within the cantilevers and the analysis of the forces in the reinforcing ribs. In contrast to the present part of the work, which is independent of the loading applied, the results of the next part will be highly dependent on the point of application and on the direction of the concentrated force **P**.

References

Chan HSY (1967) Half-plane slip-line fields and Michell structures. Q J Mech Appl Math 20:453–469

Graczykowski C, Lewiński T (2006) Michell cantilevers constructed within trapezoidal domains—Part I: geometry of Hencky nets. Struct Multidisc Optim

Hemp WS (1973) Optimum structures. Oxford, Clarendon Press

Lewiński T, Zhou M, Rozvany GIN (1994a) Extended exact solutions for least-weight truss layouts—Part I: cantilever with a horizontal axis of symmetry. Int J Mech Sci 36:375–398

Lewiński T, Zhou M, Rozvany GIN (1994b) Extended exact solutions for least-weight truss layouts—Part II: unsymmetric cantilevers. Int J Mech Sci 36:399–419