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Numerical and experimental investigation of prestress effect on natural frequencies of composite beams

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Abstract

The paper analyzes the influence of prestressing force on the natural frequencies on Glass Fibre Reinforced Polymer (GFRP) composite beams. Prestress is introduced by applying initial tensile forces to the fibres embedded in selected layer of the composite material during the manufacturing process. Release of prestressing forces results in deformation and self-equilibrated state of stress of the entire composite which changes both its static properties and dynamic characteristics. The paper is focused on analysis of shifts in natural frequencies corresponding to initial bending modes of the composite beams of various fibre volume fraction and prestressed layer location. The problem is analyzed with the use of finite element simulations and experimental modal analysis. The conducted numerical and experimental work reveals that shifts in the natural frequencies caused by non-axial prestressing can be significant and they are important phenomenon which has to be taken into account during design of the composite material.

1 Introduction

Initial prestressing is effectively used in reinforced concrete structures, such as girders and slabs, in order to minimize tensile stresses generated during bending and to overcome low tensile strength and brittle behaviour of concrete. Large number of applications of prestressed concrete structures in civil engineering proves that such technique is efficient and allows to design lighter and more durable structures. In the recent years the concept of prestressing was generalized and applied to fibre reinforced laminates of polymeric matrix. In such case prestressing refers to initial pre-tensioning of the reinforcing fibres of selected layers of the laminate, which produces clamping load generating compressive stresses in the surrounding matrix and self-equilibrated state of stress of the entire laminate. Depending on the location and orientation of the layer the introduced prestress may evoke various state of deformation and state of stress of composite material. Therefore, prestressing provides wide possibilities for modifying initial state of composite elements and enables influencing their response to external loading.

From technological point of view the process of composite prestressing is relatively simple. The first step is pre-tensioning of the fibres, which is conducted after moulding but before curing of the matrix. In this stage the external tensile force is applied to the fibres of selected composite layer causing their elongation and corresponding tensile stresses. Since polymeric matrix remains in uncured state and it is not yet solidified, it is not affected by applied pre-load. Further, the laminate is cured in a standard manner, causing the matrix to harden, to adhere and bond to prestressing fibres. Once the composite is entirely cured and cooled down to room temperature the external tensile force is released. The pre-tensioned fibres tend to contract and static friction forces cause that compressive stresses are locally induced to the surrounding matrix. This results in generation of the initial state of deformation and initial state of stress in the entire composite specimen.

Most of the publications devoted to prestressed fibre reinforce polymer (FRP) composites analyze the influence of prestress on their static properties. The researchers had reported significant advantages of prestressing such as improvement of load-carrying capacity [1, 2, 3], reduction of thermally induced residual stresses and increased durability to fatigue [4], reduction of fibre misalignment [5] and increase of resistance to stiffness degradation [6]. The paper [7] presents simple analytic model of the prestressed laminated composite, which is verified by standard numerical model and experimental analysis of prestressed composites samples, and further used for optimization of the applied prestressing forces.

In contrast, the number of publications related to dynamic properties of prestressed structures is very limited. In particular, the influence of the prestress on modal characteristics and mode shapes is not well recognised. In fact, there seems to be certain contradiction in conclusions drawn by particular authors. Several researchers observe decrease in natural frequencies of prestressed concrete structures [8, 9, 10], which corresponds to "compression-softening" effect occurring in externally, axially loaded homogenous beams. Other researchers suggest that natural frequencies of prestressed concrete structures are unaffected by prestressing forces [11, 12]. Finally, there are papers reporting that prestressing causes increase of natural frequencies of the reinforced concrete structures [13, 14, 15]. The extensive experimental investigation of the natural frequencies of prestressed composites beams with various location of the prestressing strand and various magnitude of prestressing force is presented in [16]. Unfortunately, the analysis does not reveal any non-random, statistically relevant correlation between application of prestressing force and direction of the shift of natural frequencies. According to above discussion the problem of the influence of the prestressing force magnitude and location on the natural frequencies of the prestressed FRP composites remains an open question.

The paper attempts to fill the above gap in the theory of prestressed structures by detailed numerical and experimental analysis of fibre reinforced composite polymer beams of various fibre volume fraction, location and magnitude of the prestressing force. The remainder of the paper is organized as follows. The second section briefly describes fundamentals of the finite element modelling of prestressed composites. The third section presents results of finite element simulations aimed at determination of natural frequencies of composite beams subjected to non-axial prestressing. In the fourth section the modal characteristics of the same composite beams are analyzed with the use of Experimental Modal Analysis. Finally, the conclusion section attempts to identify two phenomena responsible for shifts in natural frequencies of prestressed structures and shed a new light on interpretation of the prestressing effect.

2 Finite element model of prestressed composite

The complete derivation of the equations governing the finite element model of the prestressed composites was presented in previous paper of the authors [17]. Therefore, herein we will discuss exclusively the basic assumptions of the analysis and final form of the equations describing small-displacement static model, large-displacement static model and large-displacement dynamic model of the composite structure. The proposed model will be based on layer-wise (LW) method, which utilizes three dimensional modeling of each layer of the composite and does not directly apply classical kinematic hypotheses of the plate theory.

In all considered problems the material will be assumed as linear elastic. We will apply standard constitutive equation for the prestressed material, which links the strain tensor $\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathbf{o}}$ with the stress tensor $\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\mathbf{o}}$. The upper index "zero" indicates initial strains and stresses.

In the case of assumption of small displacement theory the relation between strains and stress will be expressed in a standard manner, with the use of first order linear differential operator including derivatives with respect to subsequent space variables. The corresponding equation of the Finite Element Method describing static equilibrium of the composite takes the form:

$$\mathbf{K_0 q} + \mathbf{R_0} = \mathbf{P} \tag{1}$$

where $\mathbf{q}$ are nodal displacements, $\mathbf{P}$ are nodal forces, $\mathbf{K_0}$ is the global linear stiffness matrix composed of element matrices $\mathbf{K}^{(e)}$:

$$\mathbf{K}^{(\mathbf{e})} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \tag{2}$$

where $\mathbf{B}$ is the so-called strain matrix involving derivatives of elements' shape functions and V is the volume of the finite element. Finally $\mathbf{R_0}$ expresses internal elastic bonds resulting from prestress and is composed of element vectors $\mathbf{R_0^{(e)}}$ given in the form:

$$\mathbf{R_0^{(e)}} = \int_V \mathbf{B}^T \boldsymbol{\sigma}_\mathbf{0} \, dV \tag{3}$$

Eq.1 can be solved with the use of standard methods of linear statics in order to obtain displacement field caused by initial prestress and applied external loading.

In contrast, in the case of adopting the large displacement theory the relation between strains and stresses is expressed with the use of nonlinear differential operator, which includes the products of derivatives with respect to subsequent space variables. In this case, the equation of the Finite Element Method describing equilibrium of the composite takes the form:

$$\left(\mathbf{K_{NL}} + \mathbf{K_\sigma}\right)\mathbf{q} + \mathbf{R_0} = \mathbf{P} \tag{4}$$

where $\mathbf{K_{NL}}$ is the "geometrically nonlinear" global stiffness matrix corresponding to current configuration of the structure, composed of elements' stiffness matrices $\mathbf{K_{NL}^{(e)}}$:

$$\mathbf{K_{NL}^{(e)}} = \int_v \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \tag{5}$$

in which the integration is performed over actual (deformed) volume of the finite element, while $\mathbf{K_\sigma}$ is global stress stiffness matrix composed of elementary stress stiffness matrices $\mathbf{K_\sigma^{(e)}}$:

$$\mathbf{K_\sigma^{(e)}} = \int_v \mathbf{B}^\mathbf{T} \boldsymbol{\sigma} \mathbf{B} \, dV \tag{6}$$

where $\boldsymbol{\sigma}$ is the tensor of Cauchy stresses and integration is also performed over actual volume of the finite element. Let us note that matrix $\mathbf{K_{NL}}$ reflects large deformation of the prestressed structure, while matrix $\mathbf{K_\sigma}$ describes the so-called "stress-stiffening" effect. Eq. 4 can be solved with the use of standard iterative methods of non-linear statics in order to obtain displacement field caused by initial prestress and applied external loading.

When large displacement theory is assumed the finite element equation governing the dynamics of prestressed structure takes the form:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \left(\mathbf{K_{NL}} + \mathbf{K_\sigma}\right)\mathbf{q}(t) + \mathbf{R_0}(t) = \mathbf{P}(t) \tag{7}$$

where $\mathbf{M}$ is the global mass matrix composed of elementary mass matrices $\mathbf{M}^{(\mathbf{e})}$:

$$\mathbf{M}^{(\mathbf{e})} = \int_v \mathbf{N}^T \rho \mathbf{N} \, dV \tag{8}$$

where ρ is material density, $\mathbf{N}$ is the shape functions matrix, whereas $\mathbf{K_{NL}}$, $\mathbf{K_\sigma}$ and $\mathbf{R_0}$ are defined in the same way as previously. By assuming that there are no external forces acting on the system, we obtain homogenous equation describing free vibrations of the structure. Further, by assuming that harmonic free vibrations in the form $\mathbf{q}(t) = \tilde{\mathbf{q}} \sin(\omega t + \phi)$, where $\tilde{\mathbf{q}}$ is vector of vibration amplitudes, we obtain system of linear homogeneous algebraic equations:

$$\left(\mathbf{K_{NL}} + \mathbf{K_\sigma} - \boldsymbol{\omega}^2 \mathbf{M}\right)\tilde{\mathbf{q}} = \mathbf{0} \tag{9}$$

which has non-zero solutions when:

$$\det\left(\mathbf{K}_{\mathbf{NL}} + \mathbf{K}_{\boldsymbol{\sigma}} - \boldsymbol{\omega}^2\mathbf{M}\right) = 0 \tag{10}$$

We obtain a polynomial of degree n in respect of $\boldsymbol{\omega}^2$, where $\boldsymbol{\omega}$ are natural frequencies of vibrations of the prestressed structure. The corresponding eigenvectors indicate its mode shapes.

Let us once again highlight the fact that components of the stiffness matrices $\mathbf{K}_{\mathbf{NL}}$ and $\mathbf{K}_{\boldsymbol{\sigma}}$ depend on deformation of the structure and state of stress resulting from prestress. Therefore, the modal analysis of the prestressed structure has to be always preceded by nonlinear static analysis. Otherwise, the effect of prestress will not be observed.

3 Numerical simulations of prestress effect

The model of the prestressed composite analyzed the numerical tests was uni-directionally reinforced 5-layer composite beam with geometrical dimensions: length L=0.2 [m], height h=0.00131 [m] and width w=0.01 [m]. Each layer of the composite was constructed of the resin (ρ=1200 kg/m^3, E=3.5 GPa, v=0.35) and reinforcing fibres (ρ=2600 kg/m^3, E=60 GPa, v=0.22). The fibre volume fraction in the subsequent analysis was modified in the range between 20% to 90%. The finite element analysis always consisted of two separate steps:

- static prestressing of the lowest layer of simply supported composite beam by forces of a various magnitudes, aimed at determination of the stiffness matrix for further computations, Fig. 1a;
- modal analysis of prestressed composite with modified boundary conditions (static scheme of a cantilever), aimed at finding natural frequencies corresponding to initial bending modes and their change caused by prestressing, Fig. 1b.

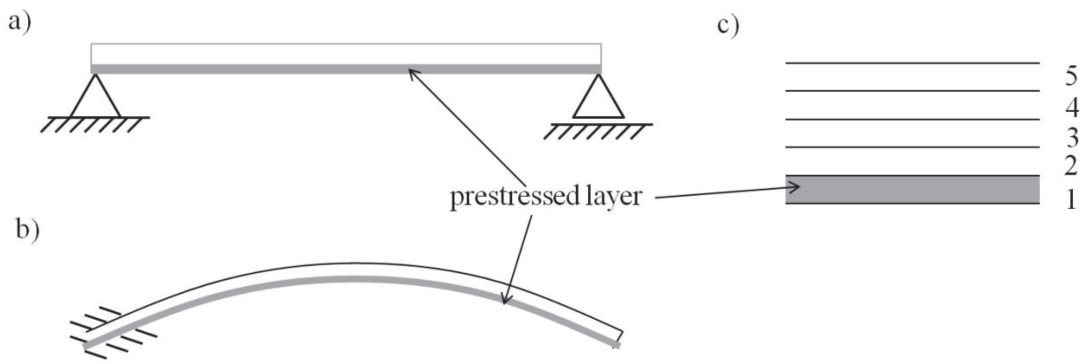

Figure 1: Boundary conditions applied in simulation: prestressing step (a), modal analysis step (b), scheme of the layers numbering (c)

The static analysis of the prestressing process enabled comparison of the deflection obtained with the use linear and nonlinear layer-wise (LW) model, as well as linear and nonlinear equivalent single layer (ESL) model. In all cases comparable values of maximal displacement were obtained (Fig. 2a). However, it was observed that both nonlinear model results in slightly smaller value of displacement.

Maximal deflection difference between linear and nonlinear LW models was analyzed for various fibre volume fractions and magnitudes of prestressing forces (Fig. 2b). The smallest difference was obtained for the highest fibre volume fraction (below 1% for prestressing force of 200N), while the largest discrepancy was observed for the smallest fibre volume fraction (above 8% for prestressing force of 200N). This confirms intuitive conclusion that linear model provides much better estimation of stiff prestressed structure with high volume fraction then compliant structure with small fibre volume fraction.

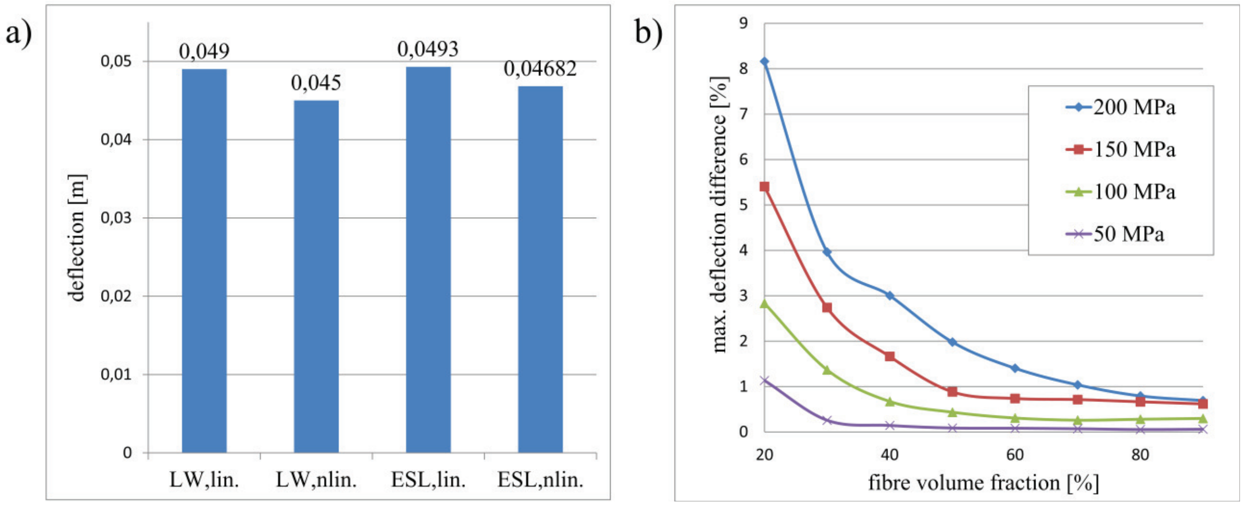

Figure 2: Comparison of deflection obtained from linear and nonlinear LW and ESL models (a), comparison of linear and nonlinear LW models for various FVF and magnitude of prestressing force (b)

The elaborated numerical model also allows to analyze change of longitudinal strains and stresses in prestressed beam. Fig. 3a shows that applied non-axial prestress being a superposition of compression and bending causes that maximal shortening of the lowest layer is significantly larger than maximal elongation of the upper one. Moreover, it reveals that dependence between the absolute values of the longitudinal strains and fibre volume fraction of the composite is nonlinear. In turn, Fig. 3b shows exemplary distribution of stresses which clearly corresponds to applied loading combining compression and bending. It includes transparent shift of stress at the boundary of the prestressed layer.

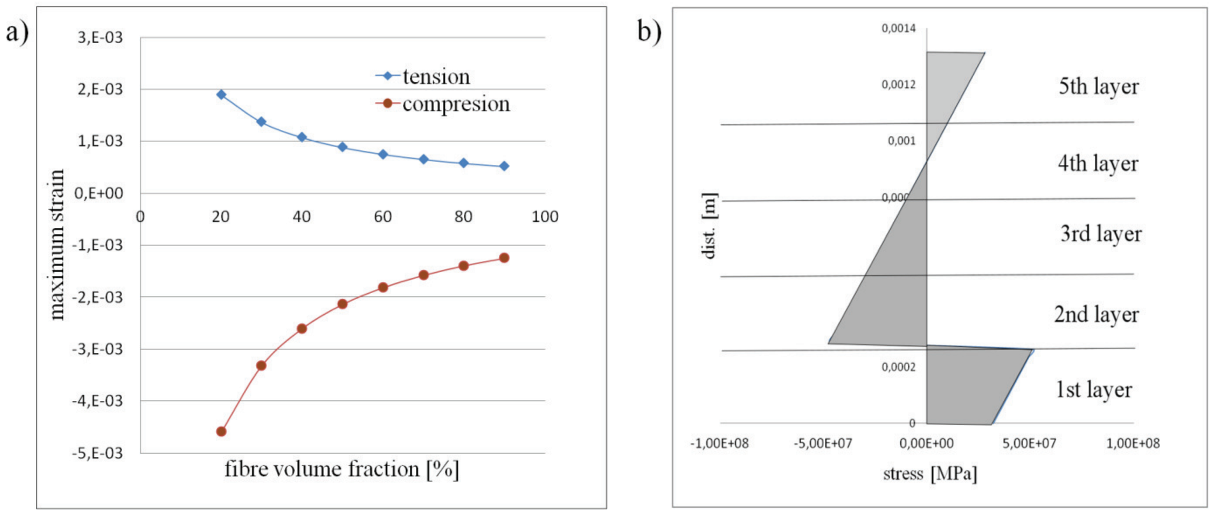

Figure 3: Comparison of maximum strain obtained for various fibre volume fractions (a), exemplary stress distribution in the middle cross section of the beam (b)

In the following stage of numerical simulations the modal analysis was used to compute natural frequencies and modal shapes of prestressed composite beams and their shifts caused by change of magnitude of the applied prestressing force. In order to observe the effect of prestress the stiffness matrix used in computations included both the effect of prestressing and structure deformation (cf. Eq.7, Sec.2). The analysis was focused exclusively on bending modes, which are the most sensitive to prestress. The initial bending modes of the prestressed structure are presented in Fig. 4. They appear to be qualitatively similar to the bending modes of straight non-prestressed beam, however they are imposed on deformed configuration of the structure.

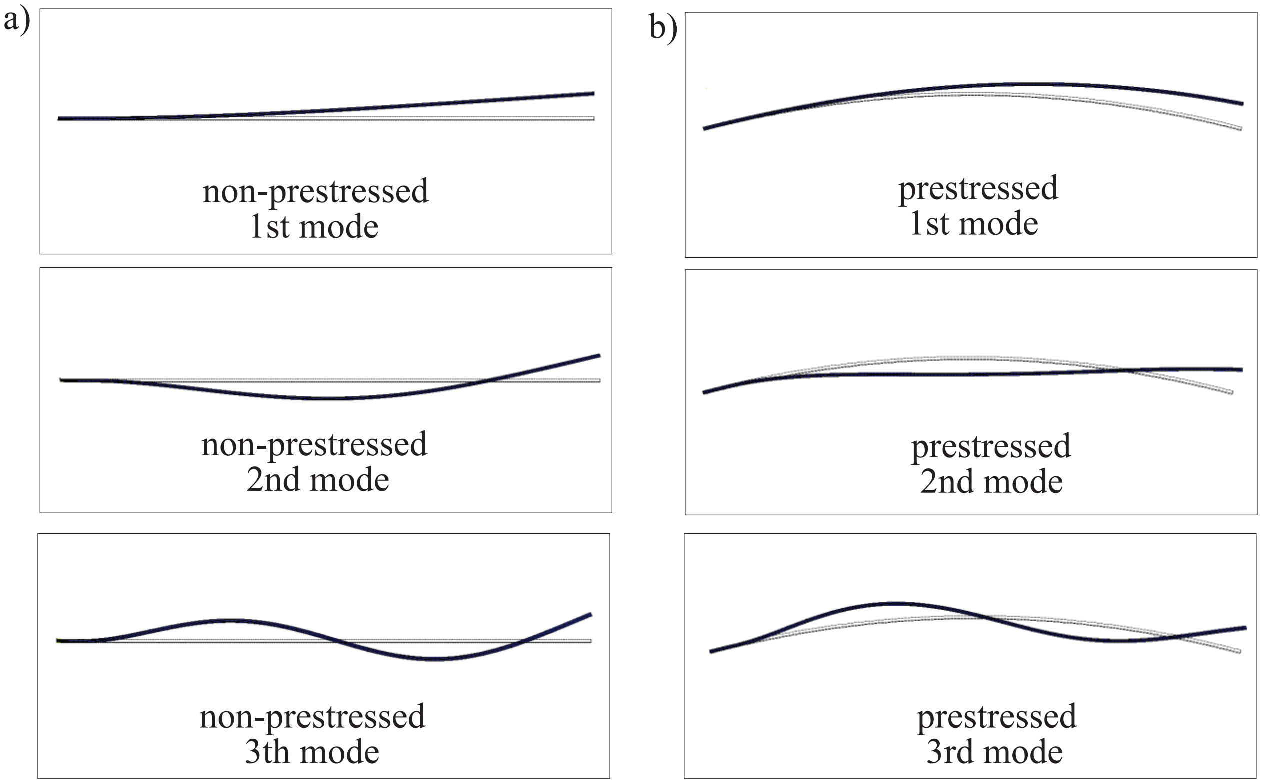

Figure 4: Three initial bending modes of the non-prestressed straight composite beam (a) and deformed prestressed composite beam (b)

The comparison of natural frequencies corresponding to three initial bending modes of prestressed composite beams with various fibre volume fraction is presented in Table 1. It is observed that for the first bending mode the applied prestressing force causes increase of the natural frequency. The effect is more transparent for the fibre volume fraction equal to 20%, which is associated with larger compliance and deformation of the composite structure. Moreover, the effect of increase of the first natural frequency gradually decreases when the prestressed layer is located closer to the natural axis of the beam, which corresponds to smaller flexural deformation caused by prestress.

The influence of prestress for the second and third bending mode is reversed, i.e. prestressing force causes decrease of the natural frequency of these modes. Similarly as in previous case, the effect is more transparent for smaller fibre volume fraction and it decreases when prestressed layer is located closer to the natural axis of the beam.

	0 MPa (20%)	100 MPa 1st layer (20 %)	100 MPa 2nd layer (20%)	100 MPa 3rd layer (20%)	0 MPa (50%)	100 MPa 1st layer (50 %)	100 MPa 2nd layer (50 %)	100 MPa 3rd layer (50 %)
1 mode	**18,62**	19,00	18,70	18,63	**24,07**	24,19	24,11	24,09
2 mode	**116,61**	106,94	114,20	117,06	**150,72**	147,72	150,16	150,99
3 mode	**326,25**	315,29	324,13	327,43	**421,50**	418,18	421,24	422,25

Table 1: Natural frequencies [Hz] of three initial bending modes for non-prestressed and prestressed composite beams of fibre volume fraction 20% and 50%.

The last investigated topic was detailed analysis of the influence of prestressing force of the same magnitude, applied to various layers of the beam. The conducted simulations confirm that the largest shift in natural frequency occurs in case of prestressing of the lowest layer of the composite (Fig. 5). The corresponding frequency shift is the largest in case of 20% fibre volume fraction. It exceeds 2% for the first mode, -8% for the second mode and -3,4% for the third mode. Although the frequency shifts corresponding to fibre volume fraction 50% are significantly smaller, they are also the largest in case of prestressing of the lowest layer. When prestressing is applied to the layer located closer to the middle surface of the composite the relative change in each natural frequency decreases. In particular, in the case of axial prestressing the shifts in frequency are almost negligible. An illustrative example can be the frequency shift observed in the most compliant, axially prestressed composite with FVF=20%, which does not exceed 0,4%. Therefore, it can be concluded that the factor which has the largest influence on modal characteristics of prestressed composite is the change of composite shape.

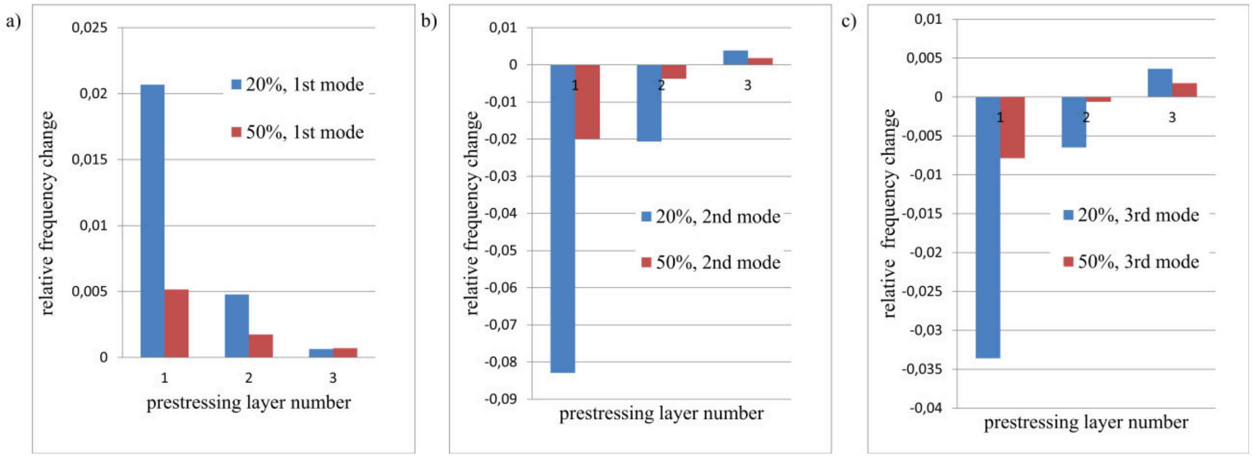

Figure 5: Relative change of frequency caused by prestressing force of magnitude 260 N applied to different layers of composite. Results presented for the first (a), second (b) and third (c) bending mode

4 Experimental verification

In order to identify influence of the prestress on the modal behaviour of the composite beams, two types of composite plates were prepared:

- one plate composed of 5-layer of bidirectional E-glass fabric and epoxy resin with no external force applied during fabrication,
- one plate composed of 5-layer of bidirectional E-glass fabric and epoxy resin with tension applied to the outer layer during fabrication.

Both plates were manufactured simultaneously by the vacuum bag technique and cured in identical conditions. The prestressed plate was manufactured using a so-called prestressing bed presented in Fig. 6a. Methodology of preparation the prestressed composite with the use of such device was described by Krishnamurthy in [4]. The main part of the prestressing bed was a plane table, made of two blocks: one of them was fixed to a base plate and the other one was movable. A load screw was installed in the moving block in that way, that the screw end was blocked in the fixed part of the prestressing bed and the movable part was shifted by the rotation of the screw. For the purpose of alignment of the moving and fixed blocks two guide bars were used. The glass fabric was mounted on the prestressing bed by the use of clamps placed at its opposite ends and prestress was applied by screw rotation. Prestressing force was measured by the load cell installed in the fixed part of the prestressing bed. The force acting on the sensor was a reaction force due to the tensioning of the fabric. After finishing the plates preparation, each plate was cut into beam samples with 0.02 [m] width and 0.2 [m] length and tested by the use of experimental modal analysis procedure.

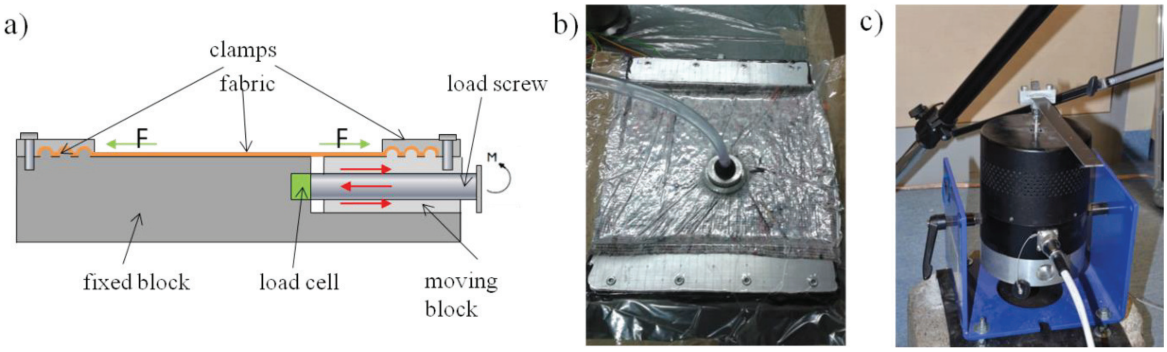

Figure 6: Prestressing bed scheme (a), view of the prestressing bed during samples preparation (b), view of the modal shaker with specimen during modal test experiment (c).

The experimental test-stand for modal analysis research was composed of modal shaker ModalShop 2100E11, Polytec Scanning Vibrometer PSV-400-3D and acceleration sensor B&K 4507 B 004. The modal shaker created wide band pseudorandom excitation according to control signal produced by Polytec PSV-400-3D internal signal generator. Velocity of vibrations of the specimen were measured by the Scanning Vibrometer in 1D configuration, in direction normal to surface of the specimen. The acceleration sensor B&K 4507 B 004 was used as a reference sensor measuring the vibrations in direction of shaker excitation. Signals from the scanning head and the accelerometer as well as from the generator were recorded by PSV-400-3D acquisition system for purpose of further analysis.

During modal experiments, the tested specimens were mounted in special mounting clamp, attached to the shaker, which allowed to create repetitive boundary conditions for all experiments. The reference sensor was constantly mounted on clamp not influencing vibrations of the specimens. It has been decided to focus only on identification of bending modes. Due to this, measurement points created a straight line on the surface of the specimen.

For each specimen the test procedure was repeated ten times. Between repeating measurement, the sample was removed from mounting clamp and mounted again before start of a test. This approach allowed to identify dispersion of experiment. The frequencies measured for successive tests of given beam did not differ more than 0,8%. Averaged values of identification of modal parameters as well as relative differences between frequencies measured for prestressed and non-prestressed sample are presented in Tab. 2. Fig. 7 shows Complex Mode Indicator Function for examined prestressed and non-prestressed composite beams.

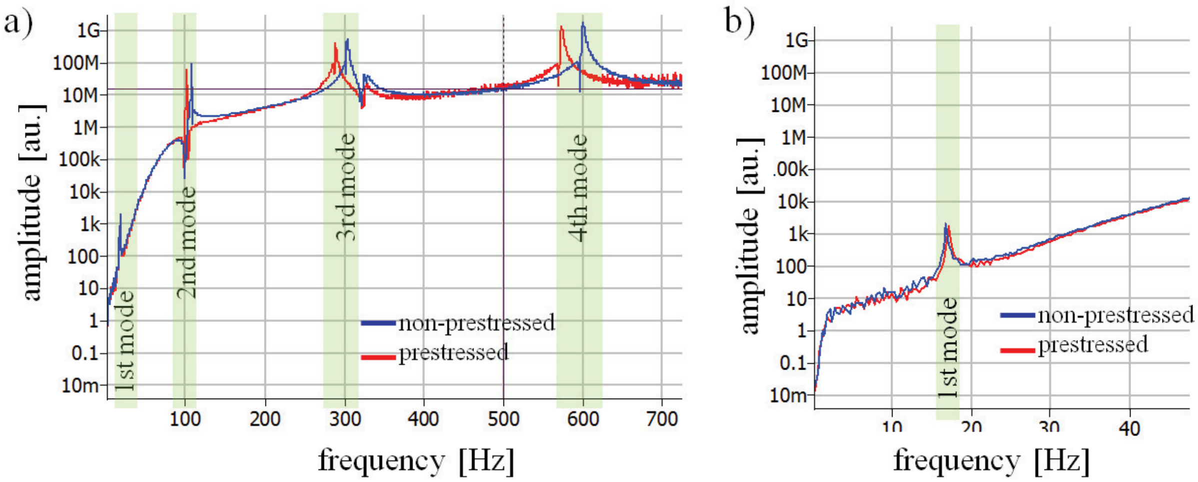

Figure 7: CMIF functions for examined prestressed and non-prestressed composite beams: for frequency range 0 -700 Hz (a), for frequency range 0-50 Hz (b).

Mode number	natural frequencies of non-prestressed sample [Hz]	natural frequencies of prestressed sample [Hz]	relative frequency change [%]
1st	17	17.3	1.8
2nd	107.5	101.5	-5.6
3rd	303.4	288.2	-5.0

Table 2: Results of experimental measurements.

The increase of fundamental frequency was observed for the prestressed samples of composite cantilever beam. In turn, the second and third bending frequency for those beam decreased with prestress application. Character of frequency change observed in the experiment reflect the overall character of frequency change revealed by numerical computations.

5 Conclusions

Prediction of changes of the natural frequencies of fibre reinforced polymer composites caused by applied prestressing force appeared to be relatively complex numerical and experimental problem. Conducting numerical simulation requires taking into account the effect of structure deformation and the effect of generated non-uniform state of stress, and their influence on stiffness matrix of the model. In turn, the challenging problems during experimental analysis include obtaining required thickness and stiffness of the specimens, as well as taking into account the processes such as relaxation and creep. The presented study clearly shows that non-axial prestressing of the fibre reinforced polymer beams significantly changes their dynamic characteristics including natural frequencies and mode shapes. The most important conclusions from above numerical and experimental work are as follows:

- the influence of prestress on natural frequencies of the composite beams is significant when prestressing causes their large flexural deformation;
- prestressing causes increase of the natural frequency associated with the first bending mode of the composite beam;
- prestressing causes decrease of the natural frequencies associated with the second and third bending mode of the composite beam (and probably the same holds for subsequent modes);

Conducted research indicates that shifts in the natural frequencies of prestressed composite beams are caused by two concurrent but counteracting effects: i) the stress-stiffening effect and ii) prestress-induced change of beam geometry. The following hypothesis can be drawn. In the case of the first bending mode the stress-stiffening effect prevails over the geometry-change effect and thus the natural frequency increases. In turn, for the subsequent bending modes the stress-stiffening effect gradually diminishes while geometry-change effect raises and thus the corresponding natural frequencies decrease.

The entire analysis reveals that in case of non-axial prestressing the resulting frequency changes can be significant. Therefore the observed frequency shift it is a practically relevant phenomenon, which has to be taken into account during potential application of prestressing technology in FRP composite structures.

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