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N-point Padé approximants and two-sided estimates of errors on the real axis for Stieltjes functions☆

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Abstract

Upper and lower estimates of Stieltjes function by N-point Padé approximants can be obtained using the new general inequality reported by Tokarzewski et al. (Arch. Mech. 54 (2002) 141–153) and rigorously proved in the present paper. In addition, we prove that the multipoint Padé approximants to Stieltjes function are symmetric with respect to the order of choice of the considered points.

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1. Introduction

Inequalities between the one-point classical Padé approximants to Stieltjes function f and f itself are presented in many monographs, cf. [1,2]. The order equilibrated, optimal inequalities for Padé approximant errors in the Stieltjes case were investigated in [3,4]. The classical inequalities for the two-point $\{0, \infty\}$ Padé approximants were derived in [10]. Optimal inequalities were recently obtained in [5,6].

☆ This paper is dedicated to the coauthor and our friend Maciek Pindor, who unexpectedly died on July 5, 2003, during his stay at the Astronomic Observatory in Nice, France.

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Bounding properties of N-point Padé approximants to Stieltjes functions were discussed in [8,9,11]. In this paper, in order to improve the bounds on Stieltjes functions we study the inequalities for the N-point ($N \geqslant 2$) Padé approximants (NPA) allowing to estimate f from top and from below.

Let f_1 be a Stieltjes function defined by

$$f_1(x) = \int_0^{1/R} \frac{\mathrm{d}\mu(u)}{1+xu} \tag{1}$$

and f be an analytic function at N different points $-R < x_1 < x_2 < \cdots < x_N < \infty$

$$f(x) = f(x_1) + (x - x_1) f_1(x) \tag{2}$$

having a finite limit at ∞ and the power expansions

$$\sum_{k=0}^{\infty} c_k(x_j)(x - x_j)^k, \quad j = 1, \ldots, N.$$

In practical situations, we only know a few first coefficients of each expansion and then we have to deal with the limited information characterized by the truncated power series

$$\sum_{k=0}^{p_j - 1} c_k(x_j)(x - x_j)^k + \mathrm{O}((x - x_j)^p_j), \quad j = 1, \ldots, N. \tag{3}$$

Remark. The reader may wonder why we always deal with functions defined by (2) instead of pure Stieltjes functions. In fact, our motivation is to apply the NPA in mechanics to estimate the effective moduli of inhomogeneous two-phase media which are represented precisely by functions of the form (2).

The aim of this paper is to present two new results. First, we give a complete revised proof of a general inequality between the N-point Padé approximants to f and f itself on $]-R, \infty]$ announced by the authors in [11]. This inequality allows to obtain lower and upper bounds of f on the real axis. Secondly, we generalize the known property of Thiele interpolants showing that N-point Padé approximants are symmetric with respect to all points x_j.

The N-point Padé approximant to f, if it exists, is a rational function $P_m/Q_n = [m/n]$ denoted, when needed, as follows:

$$[m/n]^{p_1 p_2 \ldots p_N}_{x_1 x_2 \ldots x_N}(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_n x^n}, \tag{4}$$

$$m + n + 1 = p = p_1 + p_2 + \cdots + p_N$$

satisfying the following relations:

$$f(x) - [m/n](x) = \mathrm{O}((x - x_j)^{p_j}), \quad j = 1, 2, \ldots, N. \tag{5}$$

Each p_j represents the number of coefficients $c_k(x_j)$ of expansion (3) actually used for the computation of NPA given by (4). In the following we deal only with diagonal $[n/n]$ and subdiagonal $[n+1/n]$ NPA, where

$$n = \mathscr{E}\left(\frac{p-1}{2}\right), \quad m = p - 1 - n, \tag{6}$$

$\mathscr{E}(x)$ denoting an entire part of x. All previous definitions of NPA are also valid for points in the complex domain, but here we are concerned only with the inequalities on the real axis.

Let us recall the usual notation of a continued fraction:

$$\mathop{K}_{k=1}^{r} \frac{a_k}{1} := \frac{a_1}{1+} \frac{a_2}{1+} \cdots \frac{a_r}{1}.$$

We begin by expanding the function f in an one-point continued fraction:

$$\begin{aligned} f(x) &= f(x_1) + (x - x_1) f_1(x) = f(x_1) + \frac{(x - x_1) f_1(x_1)}{1 + (x - x_1) f_2(x)} \\ &= \cdots = f(x_1) + \mathop{K}_{k=1}^{r-1} \frac{(x - x_1) f_k(x_1)}{1+} \frac{(x - x_1) f_r(x)}{1}. \end{aligned} \tag{7}$$

To expand f in a N-point continued fraction (NCF) we introduce a nondecreasing step-wise function L

$$L(x) = \sum_{j=1}^{N} p_j H(x - x_j), \qquad H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqslant 0, \end{cases} \tag{8}$$

where H is the Heaviside function. The value $L(x)$ denotes the total number of given coefficients of power expansions of f at all points $x_j \leqslant x$:

$$L(x_k) = p_1 + p_2 + \cdots + p_k, \quad L(x_N) = p = \sum_{j=1}^{N} p_j.$$

To construct a multipoint continued fraction of f, we begin by expanding f at x_1 as shown in (7), we follow by expanding $f_{L(x_1)=p_1}$ at x_2, and so on:

$$\begin{aligned} f(x) = f(x_1) &+ \mathop{K}_{k=1}^{L(x_1)-1} \frac{(x - x_1) f_k(x_1)}{1+} \frac{x - x_1}{x - x_2} \mathop{K}_{k=L(x_1)}^{L(x_2)-1} \frac{(x - x_2) f_k(x_2)}{1+} \\ &\cdots \frac{x - x_{N-1}}{x - x_N} \mathop{K}_{k=L(x_{N-1})}^{L(x_N)-1} \frac{(x - x_N) f_k(x_N)}{1+} \frac{(x - x_N) f_p(x)}{1}. \end{aligned} \tag{9}$$

One can readily verify that the diagonal or subdiagonal NPA to f is given by the truncation of the last term in (9)

$$\begin{aligned} [m/n]_{x_1 x_2 \ldots x_N}^{p_1 p_2 \ldots p_N}(x) &= f(x_1) \\ &+ \cdots + \mathop{K}_{k=1}^{L(x_1)-1} \frac{(x - x_1) f_k(x_1)}{1+} \frac{x - x_1}{x - x_2} \mathop{K}_{k=L(x_1)}^{L(x_2)-1} \frac{(x - x_2) f_k(x_2)}{1+} \\ &\cdots \frac{x - x_{N-1}}{x - x_N} \mathop{K}_{k=L(x_{N-1})}^{L(x_N)-1} \frac{(x - x_N) f_k(x_N)}{1}. \end{aligned} \tag{10}$$

2. Basic inequality

To prove the main inequality we need some preliminary lemmas.

Lemma 1 (*Baker Jr. [1, p. 244]*). *If F is a Stieltjes function defined by*

$$F(x) = \int_0^{1/R} \frac{\mathrm{d}\mu(u)}{1+xu},$$

then for the real $a \in]-R, \infty[$ *the function* $f(x) = F(x+a)$ *is also a Stieltjes function*:

$$f(x) = F(x+a) = \int_0^{1/(R+a)} \frac{\mathrm{d}\gamma(v)}{1+xv}.$$

Proof. If $\mathrm{d}\mu$ is a positive measure, then after the change of variables $u = v/(1-av)$ the measure $\mathrm{d}\gamma(v) = (1-av)\,\mathrm{d}\mu(v/(1-av))$ is also positive:

$$\int_0^{1/R} \frac{\mathrm{d}\mu(u)}{1+xu} = \int_0^{1/(R+a)} \frac{(1-av)\,\mathrm{d}\mu(v/(1-av))}{1+xv}.$$

The radius of convergence of F is at least R, while the radius of convergence of f is at least $R+a$. □

Lemma 2. *If f is a Stieltjes function, then the inverted Stieltjes function g given by the following shifted linear fractional transformation* (*LFT*):

$$f(x) = \frac{f(a)}{1+(x-a)g(x)} \tag{11}$$

is also a Stieltjes function.

Proof. The classical nonshifted LFT relates the Stieltjes function F to the inverted Stieltjes function G as follows:

$$F(x) = \frac{F(0)}{1+xG(x)}. \tag{12}$$

Defining $f(x+a) := F(x)$ and $g(x+a) := G(x)$, and changing now x by $x-a$ we obtain (11) using the Lemma 1. □

Lemma 3. *Let f and g are the Stieltjes functions with a radius of convergence at least R related by* (11), *then*

$$\forall x \in]-R, \infty[: \quad 1+(x-a)g(x) \geqslant 0. \tag{13}$$

Proof. $f(a) > 0$ because f, being Stieltjes function, is a positive decreasing function. Then the denominator in (11) cannot vanish in $]-R, \infty[$ and it is also positive. □

Now we are in a position to prove the main result of this paper.

Theorem 4. *Let* $f(x) = f(x_1) + (x - x_1) f_1(x)$, *where* f_1 *is a Stieltjes function defined by* (1). *Then the diagonal* $[n/n]$ *and subdiagonal* $[n+1/n]$ *N-point Padé approximants* $[m/n]^{p_1 p_2 \cdots p_N}_{x_1 x_2 \ldots x_N}(x)$ *to f obey the following inequality*:

$$x \in]-R, \infty[: \quad (-1)^{L(x)}[m/n](x) \geqslant (-1)^{L(x)} f(x), \tag{14}$$

where $L(x)$ *is defined by* (8).

Remark. The last theorem is also valid for the pure Stieltjes functions f_1 if one changes the sense of inequality (14).

Proof. Following Lemma 2 all f_k in (9) are Stieltjes functions, and so they are decreasing positive functions. Then, following Lemma 3 we have

$$\forall x \in]-R, \infty[, \quad \forall k, j: \quad 1 + (x - x_j) f_k(x_j) \geqslant 1 + (x - x_j) f_k(x) > 0. \tag{15}$$

Removing now $(x - x_N) f_p(x)$ in (9) we obtain the finite NCF equal to the NPA $[m/n]^{p_1 p_2 \cdots p_N}_{x_1 x_2 \ldots x_N}(x)$ to f. The last denominator of this NCF is

$$1 + (x - x_N) f_{p-1}(x_N) \geqslant 1 + (x - x_N) \frac{f_{p-1}(x_N)}{1 + (x - x_N) f_p(x)} = 1 + (x - x_N) f_{p-1}(x).$$

Then, this denominator is increased with respect to the actual denominator of $f(x)$. Then remounting the NCF, the previous denominator is decreased if $x > x_N$ $(L(x) = p)$ and increased if $x < x_N$ $(L(x) < p)$. Then the NPA $[m/n]$, that is a finite NCF, is increased if p is even and decreased if p is odd (cf. (14)) with respect to $f(x)$ which is identified with the nontruncated NCF. This completes the proof. □

Remark. We can observe that the parity of L controls the position of NPA with respect to *f* and so we can obtain two-sided estimates of *f* playing with this parity. It is illustrated by Fig. 1, where in each case the four-point PA is accompanied by "poorest" three-point PA which is calculated by removing the point x_1. This three-point PA bounds the given function on the opposite side with respect to the four-point PA. Consequently, one obtains a curious result giving the two-sided estimates by removing some part of information. We can also readily verify that for the one-point ($X_1 = 0$) classical PA, the inequality (14) reduces to the classical one: on $[-R, 0[$ all PA to *f* are greater than *f*, and on $]0, \infty[$ their positions with respect to *f* change with the parity of L used coefficients.

3. Symmetry property of multipoint Padé approximants

The symmetry of NCF and consequently of NPA is not necessarily a property of Stieltjes functions. The necessary condition requires that the Thiele continued fraction interpolating *f* be nondegenerate, that is, the Thiele interpolant must represent all initial nodes used to its construction. Let us analyze the following example of three-point Padé approximants and the corresponding continued fractions

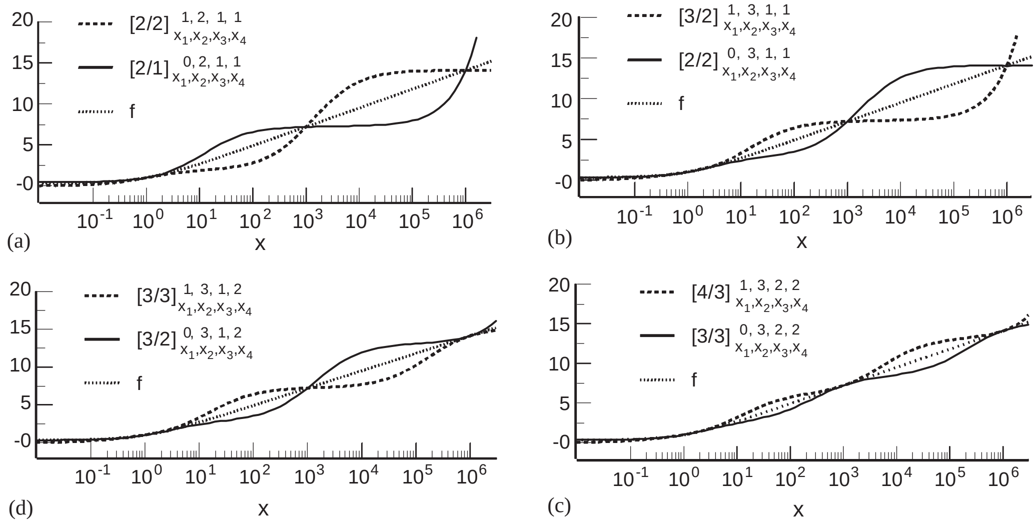

Fig. 1. Three- and four-point Padé approximants to the function $f(x) = 1 - \ln(2) + xf_1(x)$, where $x_1 = 0$, $x_2 = 1$, $x_3 = 10^3$, $x_4 = 10^6$, and where $f_1(x) = (1/x)\ln(1+x)$.

in the case of the nonStieltjes function $f(z) = z\mathrm{e}^z$:

$$
\begin{aligned}
& [2/1]^{1,2,1}_{-1,0,1}(z) = \frac{.368z}{1 - .632(1+z)/(1+.462z)} \\
= \quad & [2/1]^{1,2,1}_{1,0,-1}(z) = \frac{2.718z}{1 - 1.718(z-1)/(1+.462z)} \\
= \quad & [2/1]^{2,1,1}_{0,1,-1}(z) = \frac{z}{1 - .632z/(1+.316(z-1))} = \frac{z + .462z^2}{1 - .462z}.
\end{aligned}
$$

In each expansion in continued fraction the three points are taken in different order. However, the final result is the same. This interesting property is well-known for the Thiele interpolants (Thiele reciprocal differences method [7]), which are the NPA with all $p_j = 1$: $[m/n]^{1,1\ldots,1}_{x_1x_2\ldots x_N}(x)$. This property can readily be generalized to all N-point Padé approximants.

Theorem 5. *Let f be a function having a nondegenerate expansion in N-point continued fraction at the points* $x_1x_2\ldots x_N$ *and let* $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ *be an arbitrary permutation of* $(1, 2, \ldots, N)$. *Then all N-point Padé approximants* $[m/n]^{p_{\alpha_1}p_{\alpha_2}\ldots p_{\alpha_N}}_{x_{\alpha_1}x_{\alpha_2}\ldots x_{\alpha_N}}(x)$ *coincide.*

Proof. Let us consider the Thiele N-point continued fraction constructed with N nodes $(x_1, f(x_1) = a_1), \ldots, (x_N, f(x_N))$:

$$
f(x) = \frac{a_1}{1+}\,\frac{(x-x_1)a_2}{1+}\cdots\frac{(x-x_{N-1}a_N)}{1+}\,\frac{(x-x_N)f_{N+1}(x)}{1}.
$$

Taking $f_{N+1}(x) = 0$ we obtain the Thiele interpolant, or the NPA $[m/n]^{1,1\ldots,1}_{x_1x_2\ldots x_N}(x)$. If this interpolant is nondegenerate and conserves this property for arbitrary points x_j (it is, for instance, the case of Stieltjes functions where all a_j are positive numbers) we can proceed as follows: If r first x_j tend y_1 and $s=N-r$ last points to y_2, we obtain the NPA $[m/n]^{r,\,s}_{y_1,\,y_2}$. The symmetry of Thiele interpolants allows us to interchange the position of the first r points with the last s points without changing the interpolant. Then, after the above coalescence we obtain the NPA $[m/n]^{s,\,r}_{y_2,\,y_1} = [m/n]^{r,\,s}_{y_1,\,y_2}$. Similar statement is also true for more limit points y_j, what proves the theorem. □

Our next aim will be the derivation of the inequalities between different NPA and also optimal inequalities for the errors in the Stieltjes case, like for the one- and two-point PA.

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