COMPLEX DIELECTRIC COEFFICIENTS AND MAXWELL'S AVERAGING

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Abstract. We apply Maxwell's averaging method to find effective values of electric permeability of two-phase composite one phase of which has a complex dielectric coefficient.

1 Introduction

The semi-classical Drude's dispersion relation indicates that for certain range of light frequencies the negative values of refraction index are possible. Victor Georgievich Veselago indicated that negative refraction can occur if both the electric permittivity and the magnetic permeability of a material are negative, [1]. This idea was confirmed when David Smith built a composite material with negative refractive index, and John Pendry showed that the planar lens proposed by Veselago is providing improved resolution, cf. [2] - [8].

There is also an unconventional alternative to a lens. Material with negative refractive index will focus light even when in the form of a parallel-sided slab of material. Snell's laws of refraction at the surface are still obeyed as light inside the medium makes a negative angle with the surface normal.

One obtains also an amplitude increase when light passes by a slab of thickness d. Not only the overall transmission coefficient is increasing but also the amplitude of the electric field vector. There is no back-reflected wave at the faces of the slab since the amplitude Fresnel coefficients for reflection are equal to zero and those for transmission equal to unity. The consequence of this is that for a macroscopic thickness of the slab and atomic size point sources the amplitude of the electric field can easily reach values beyond the breakdown of any material, cf. [1, 3].

In metamaterials i.e. artificially structured materials with negative refractive index, inclusions replace the atoms and molecules of conventional materials. The scale of these inclusions is smaller than that of the used electromagnetic wavelength, so that application of homogenisation is possible. Smith and Pendry presented a homogenisation technique in which macroscopic fields are determined *via* averaging the local fields.

In this contribution we propose another way of averaging, after Maxwell's original averaging method [9].

2 Maxwell's averaging

James Clerk Maxwell proposed the following method of determining properties of a substitute homogeneous material replacing the material with spherical inclusions.

After Maxwell, we take into account n_o spheres of radius a_1 and complex dielectric coefficient ε_1, placed in the medium whose complex dielectric coefficient is ε_2, at such distances from each other that their effects of disturbing the course of electrical lines may be taken as independent of each other. These spheres are all contained within a sphere of radius a_2. The ratio of volume of the n_o small spheres to that of the sphere which contains them is $p = n_o a_1^3/a_2^3$. The whole sphere of radius a_2 had been made of a material of effective dielectric coefficient ε. We find an expression for the dielectric coefficient of a compound medium consisting of a substance of dielectric coefficient ε_2, in which are disseminated small spheres of dielectric coefficient ε_1, the ratio of the volume of all the small spheres to that of the whole being p.

2.1 Dielectric sphere in uniform electric field

Let the permittivity of a dielectric sphere of radius a_1 be ε_1, and permittivity of its surrounding medium be ε_2. Moreover, let the dielectric sphere be immersed in an otherwise constant electric field $\mathbf{E}_o$, directed along the z-axis. The potential outside the sphere is given by

$$\phi = \left(E_o r + m\frac{1}{r^2}\right)\cos\theta \tag{1}$$

where

$$m = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} a_1^3 E_o \tag{2}$$

denotes an induced electric dipole moment of the sphere.

2.2 Maxwell's sphere

The small spheres, each of radius a_1 are all in number n_o contained within a sphere of radius a_2 and dielectric coefficient is ε_2.

By extension of (1), the potential of n dielectric spheres (contained within a sphere of radius a_2) in an otherwise uniform electric field E_o is

$$\Phi = \left(E_o r + n_o m \frac{1}{r^2}\right)\cos\theta \tag{3}$$

or

$$\Phi = E_o\left(r + n_o \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} a_1^3 \frac{1}{r^2}\right)\cos\theta \tag{4}$$

This expression is written under assumption that distances between small spheres are such that their effects of disturbing the course of electrical lines may be taken as independent of each other.

The ratio of volume of the n_o small spheres (radius a_1) to that of the sphere (radius a_2)

$$p = \frac{n_o a_1^3}{a_2^3} \tag{5}$$

Hence

$$a_1^3 = \frac{p a_2^3}{n_o} \tag{6}$$

Therefore

$$\Phi = E_o\left(r + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} p a_2^3 \frac{1}{r^2}\right)\cos\theta \tag{7}$$

Now, if the whole sphere of radius a_2 had been made of a material of dielectric constant ε^{eff}, we should have had

$$\Phi = E_o\left(r + \frac{\varepsilon^{\text{eff}} - \varepsilon_2}{\varepsilon^{\text{eff}} + 2\varepsilon_2} a_2^3 \frac{1}{r^2}\right)\cos\theta \tag{8}$$

That the one expression should be equivalent to the other one should put

$$\frac{\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_2} p = \frac{\varepsilon^{\text{eff}} - \varepsilon_2}{\varepsilon^{\text{eff}} + 2\varepsilon_2} \tag{9}$$

Hence

$$\varepsilon^{\text{eff}} = \frac{\varepsilon_1 + 2\varepsilon_2 + 2p(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_2 - p(\varepsilon_1 - \varepsilon_2)}\varepsilon_2 \tag{10}$$

This, therefore, is the effective dielectric coefficient of a compound medium consisting of a substance of dielectric coefficient ε_2, in which are scattered small spheres of dielectric coefficient ε_1, the ratio of the volume of all small spheres to that of whole being p. In order that the action of these spheres may not produce higher order effects depending on their interference, their radii must be small compared with their distances, and hence p must be a small fraction. If $p = 0$, $\varepsilon^{\rm eff} = \varepsilon_2$.

2.3 Effective coefficient for medium with complex dielectric inclusions

We assume that the dielectric coefficient ε_1 is a complex quantity, and we write

$$\varepsilon_1 = \Re\varepsilon_1 + i\Im\varepsilon_1 \tag{11}$$

From Maxwell's relation (10) we get

$$\varepsilon^{\rm eff} = \frac{\Re\varepsilon_1 + i\Im\varepsilon_1 + 2\varepsilon_2 + 2p(\Re\varepsilon_1 + i\Im\varepsilon_1 - \varepsilon_2)}{\Re\varepsilon_1 + i\Im\varepsilon_1 + 2\varepsilon_2 - p(\Re\varepsilon_1 + i\Im\varepsilon_1 - \varepsilon_2)}\varepsilon_2 \tag{12}$$

or

$$\varepsilon^{\rm eff} = \frac{(1+2p)\Re\varepsilon_1 + 2(1-p)\varepsilon_2 + i(1+2p)\Im\varepsilon_1}{(1-p)\Re\varepsilon_1 + (2+p)\varepsilon_2 + i(1-p)\Im\varepsilon_1}\,\varepsilon_2 \tag{13}$$

We remind that ε_2 is purely real dielectric coefficient, $\varepsilon_2 \equiv \Re\varepsilon_2$. After further transformations one gets

$$\varepsilon^{\rm eff} = \frac{\mathcal{L}}{\mathcal{M}} \tag{14}$$

where

$$\mathcal{L} = \Re\mathcal{L} + i\Im\mathcal{L} \tag{15}$$

and

$$\Re\mathcal{L} = (1-p)\left\{(1+2p)((\Re\varepsilon_1)^2 + (\Im\varepsilon_1)^2) + 2(2+p)\varepsilon_2^2\right\} + (4+p+4p^2)\Re\varepsilon_1\varepsilon_2 \tag{16}$$

$$\Im\mathcal{L} = 9p(\Im\varepsilon_1)\varepsilon_2 \tag{17}$$

$$\mathcal{M} = \{(1-p)\Re\varepsilon_1 + (2+p)\varepsilon_2\}^2 + (1-p)^2(\Im\varepsilon_1)^2) \tag{18}$$

We observe that the real part of numerator $\Re\mathcal{L}$ can be negative if the second term $(4+p+4p^2)\Re\varepsilon_1\varepsilon_2$ is negative and its modulus is sufficiently large.

3 Drude-Lorentz' description of dispersion in materials

Drude and Lorentz (*ca.* 1900) developed a classical theory to account for the complex index of refraction and dielectric coefficients of materials, as well as their variations with the frequency of light. The model is treating electrons as damped harmonically bound particles subject to external electric fields. The frequency dependence of the dielectric coefficient is as follows

$$\varepsilon(\omega) = 1 + 4\pi \frac{e^2}{m} \sum_{\nu} \frac{N_\nu}{\omega_\nu^2 - \omega^2 - i\gamma_\nu \omega} \tag{19}$$

Here e and m are the electrical charge and mass of electron, respectively, the sum is taken after different bonds of electrons in the substance, while N_ν denote number of electrons of belonging to the bond of type ν, cf. e.g. [10, 11]. For complex ε

$$\varepsilon(\omega) = [n(\omega) + ik(\omega)]^2 \tag{20}$$

where $n(\omega)$ and $k(\omega)$ stand for refractive and extinction coefficients, respectively.

It was shown that wire structures with lattice spacings of the order of a few millimeters behave like a plasma with a resonant frequency, ω_p, in the GHz region, [3]. The ideal dielectric response of a plasma is given by

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} \tag{21}$$

and takes negative values for $\omega < \omega_p$.

At optical frequencies several metals behave like a nearly perfect plasma with a dielectric function modelled by the last equation: silver, gold, and copper are perhaps the best examples. The magnetic properties of known materials are less obliging.

The dielectric coefficient of metals, such as silver, copper, and gold is described by a simplified formula, when only one type of electronic bond is taken into account

$$\varepsilon_1(\omega) = 1 - \frac{\omega_o^2}{\omega^2 + i\gamma\omega} \tag{22}$$

with parameters for gold: $\omega_o = 1.38 \cdot 10^{16} s^{-1}$ and $\gamma = 1.07 \cdot 10^{14} s^{-1}$, cf. [13].

Visible light has wavelength in a range from about 380 nanometres to about 740 nm, what corresponds to a frequency range of about 405 THz to 790 THz, (1 THz $= 10^{12}$ Hz $= 10^{12}$ 1/s). We observe that ω_o is more than one order greater than typical frequency of the visible light $\omega_{\rm mean} = 4 \cdot 10^{14}$ 1/s, and, we observe also that $\omega_{\rm o}$ is two order greater than γ.

The equation (22) can be reduced to the form

$$\varepsilon_1(\omega) = \frac{\omega^2 + \gamma^2 - \omega_{\rm o}^2}{\omega^2 + \gamma^2} + i \frac{\omega_{\rm o}^2}{\omega^2 + \gamma^2} \tag{23}$$

and when γ^2 can be neglected in comparison with ω^2 and $\omega_{\rm o}^2$, the real part of $\varepsilon_1(\omega)$ takes the form of (21).

From Eq.(23) we get

$$\Re \varepsilon_1 = \frac{\omega^2 + \gamma^2 - \omega_{\rm o}^2}{\omega^2 + \gamma^2} \quad \text{and} \quad \Im \varepsilon_1 = \frac{\omega_{\rm o}^2}{\omega^2 + \gamma^2} \tag{24}$$

and these expressions should be substituted to Eqs. (16) - (18) to obtain the description of the averaged permittivity $\varepsilon^{\rm eff}$ given by Eq. (14).

4 Plasma-type components

Let two components of composite have plasma-type dispersion characteristics, cf. Eq.(21),

$$\varepsilon_1 = 1 - \frac{\omega_1^2}{\omega^2} \quad \text{and} \quad \varepsilon_2 = 1 - \frac{\omega_2^2}{\omega^2} \tag{25}$$

After Eq.(10) we find

$$\varepsilon^{\rm eff} = \frac{1 - \frac{\omega_1^2}{\omega^2} + 2\left(1 - \frac{\omega_2^2}{\omega^2}\right) - \frac{2p}{\omega^2}(\omega_1^2 - \omega_2^2)}{1 - \frac{\omega_1^2}{\omega^2} + 2\left(1 - \frac{\omega_2^2}{\omega^2}\right) + \frac{p}{\omega^2}(\omega_1^2 - \omega_2^2)} \varepsilon_2 \tag{26}$$

or

$$\varepsilon^{\rm eff} = \left\{ 1 - 3p \frac{\omega_1^2 - \omega_2^2}{3\omega^2 - (1-p)\omega_1^2 - (2+p)\omega_2^2} \right\} \left(1 - \frac{\omega_2^2}{\omega^2} \right) \tag{27}$$

This expression is no more of simple structure of the plasma dispersion (21) and we see the interplay of positive and negatives values of above two factors, one in curly brackets and second in parentheses decide about sign of dielectric coefficient of composite. We observe that apart from the critical value ω_2 of the matrix, the second critical value appears $\omega_{\rm crit} = \sqrt{[(1+2p)\omega_1^2 + (2-p)\omega_2^2]/3}$ appears which decides about sign of expression in curly brackets.

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