BOUNDARY-VALUE PROBLEMS IN CYCLIC PLASTICITY

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SUMMARY

In order to predict the critical number of cycles before rupture, the stress distribution within the structure should be known under cyclic loading. Generally, a steady cyclic state occurs before onset of failure. For moderate temperatures, the viscous effects can be neglected and only instantaneous plastic deformations need to be considered.

To describe analytically a steady cyclic state for moderate temperatures, a simplified approach has been used. It is assumed that the material possesses a discrete memory of past loading history and only maximal values of stresses from the past affect the actual behaviour which is piecewise analytical. Principal shear stress rate reversals define consecutive portions of stress paths along which finite stress-strain relations are valid.

Particular constitutive relations are discussed which are based on the concept of field of hardening moduli (see: Z. Mróz, On the description of anisotropic workhardening, *J. Mech. Phys. Sol.*, vol. 15, 1967). Several boundary-value problems are treated for disks, tubes and circular plates subjected to cyclic loading. For non-linear homogeneous stress-strain relations, the dissipation per cycle can be simply calculated and stress distribution during proportional cyclic loading can be obtained by proper scale mapping of the solution for monotonic loading.

For high temperatures, the viscous strain components should be accounted for. The total strain is assumed to be composed of elastic, thermal, instantaneous plastic, and viscous terms. The viscous strain rates are determined using a creep hardening law and accounting for the anisotropy of viscous hardening.

The initial transient cyclic state is most difficult to analyse and rate constitutive relations should be used with additional state variables defining the state of hardening. The cyclic solution can be determined by step-by-step integration of constitutive rate equations.

1. Introduction

When a metal structure is subjected to cyclic loading, a steady state may set in after an initially transitory period. In some cases, the steady cyclic state corresponds to purely elastic behaviour /elastic shake-down/, whereas for more intensive loading, this state corresponds to an elastic-plastic cycle. For a large class of viscoelastic materials or plastic materials it can be proved that the steady plastic cycle is uniquely defined by the loading conditions and is independent of the prior deformation history or initial state. Experimental observations confirm that this property is typical of many structural materials.

The steady cyclic state is followed by the onset of failure and is most important in understanding the cyclic behaviour of structures and formulating proper fatigue criteria. However, no general theory of steady plastic cycles is available so far, and the problem of cyclic loading cannot be cast into a standart routine of solution of boundary-value problems. The aim of this paper is to consider some fairly simple material models that can be used in treating boundary-value problems for steady cyclic behaviour. We shall introduce the concept of material with discrete memory which is sensitive to stress rate reversals and other details of the loading history except for the maximum stress and points of stress rate reversals do not affect the actual behaviour. For surface structures, such hardening rules can be directly described in terms of generalized stresses and strains and thus applied in solving particular structural problems.

2. Stress-strain relations for cyclic loading

2.1. Uniaxial loading

Comprehensive review of plastic behaviour of metals under uniaxial loading /tension-compression/ has recently been given by Burbach /1/. It has been demonstrated that under symmetric or almost symmetric cyclic loading, a closed steady loop is obtained under transitory period, Fig. 1a. For asymmetric loading, a progressive cycle is observed and plastic strain accumulation occurs, Fig. 1b.

Consider first the case of uniaxial loading for which the hardening curve is described by the relation

$$\sigma > 0,\ \dot{\sigma} > 0; \qquad \sigma = f(\varepsilon) \quad \text{or} \quad \varepsilon = \phi(\sigma), \qquad /1/$$

whereas the reverse loading curve form A is given by

$$\dot{\sigma} < 0; \qquad \sigma - \sigma_A = f_1(\varepsilon_A, \varepsilon - \varepsilon_A), \quad \varepsilon - \varepsilon_A = \phi_1(\sigma_A, \sigma - \sigma_A) \qquad /2/$$

where σ_A and ε_A denote the values of stress and strain at the point A of stress rate reversal, Fig. 1a. When upon reaching B, the stress rate $\dot{\sigma}$ is reversed, the stress strain curve BA is generally described by

$$\dot{\sigma} > 0; \quad \sigma-\sigma_B = f_2(\varepsilon_B, \varepsilon-\varepsilon_B), \quad \varepsilon-\varepsilon_B = \phi_2(\sigma_B, \sigma-\sigma_B). \qquad /3/$$

For further cycling between A and B, the relations /2.2/ and /2.3/ are valid and steady cyclic state occurs for which hysteresis loop is described by these relations. Thus, we neglect the transitory period during which cycle stabilizes to its steady shape.

It has been demonstrated that for numerous metals, a simple relationship occurs between monotonic hardening curve or skeleton curve and hysteresis curves in the steady state. Assume first that the functions f_1 and f_2 are identical and the hysteresis curve is derived by proper scaling of variables occuring in the skeleton curve. Then, instead of /2/ and /3/, we shall have

$$\dot{\sigma} < 0; \quad \frac{\sigma-\sigma_A}{\delta} = f\left(\varepsilon_A, \frac{\varepsilon-\varepsilon_A}{\delta}\right), \quad \frac{\varepsilon-\varepsilon_A}{\delta} = \phi\left(\sigma_A, \frac{\sigma-\sigma_A}{\delta}\right). \qquad /4/$$

and

$$\dot{\sigma} > 0; \quad \frac{\sigma-\sigma_B}{\delta} = f\left(\varepsilon_B, \frac{\varepsilon-\varepsilon_B}{\delta}\right), \quad \frac{\varepsilon-\varepsilon_B}{\delta} = \phi\left(\sigma_B, \frac{\sigma-\sigma_B}{\delta}\right) \qquad /5/$$

where δ is a scaling factor. A particular form of /4/ is obtained when the values of stress and strain for which reversal of loading occurs, do not affect the hysteresis curve. Then, closed loops are described and there is no progressive plastic deformation during cyclic loading. Thus, we have

$$\dot{\sigma} < 0; \quad \frac{\sigma-\sigma_A}{\delta} = f\left(\frac{\varepsilon-\varepsilon_A}{\delta}\right), \quad \frac{\varepsilon-\varepsilon_A}{\delta} = \phi\left(\frac{\sigma-\sigma_A}{\delta}\right). \qquad /6/$$

and identical relations for loading from B to A. When $\delta=2$, eq. /6/ correspond to Masing relations which can be derived by considering an assemblage of elastic-plastic elements with different yield points, connected in series or in parallel. Relations /6/ have been shown to provide fairly good approximation to actual steady cyclic states for symmetric or nearly symmetric loading. Let us exemplify our assumptions by assuming that the skeleton curve is approximated as follows

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n, \quad \dot{\sigma} > 0, \qquad /7/$$

and the cyclic curve is expressed in the form

$$\varepsilon-\varepsilon_R = \frac{\sigma-\sigma_R}{E} + \left(\frac{\sigma-\sigma_R}{2B}\right)^n \qquad \begin{array}{l} \dot{\sigma} < 0, \ \sigma < \sigma_A, \\ \dot{\sigma} > 0, \ \sigma > \sigma_B \end{array} \qquad /8/$$

where E denotes the Young modulus, B is the material constant and σ_R is either σ_A or σ_B; n is an odd integer. When upon reaching A, stress further increases, the stress-strain curve follows eq. /7/.

Let us note that in the uniaxial case, eqs. /1-8/ can be regarded as piecewise elastic relations and the material after unloading possesses a memory of the last reversal stress and the maximum stress of the previous loading history. In fact, if σ_A is the maximum stress, then any cyclic loading between σ_A and $-\sigma_A$ is described by /6/, whereas for $\sigma > \sigma_A$ or $\sigma < \sigma_A$, the equation of the initial hardening curve /1/ applies.

Following this property, let us introduce the strain and stress potentials in the form

$$V(\varepsilon, \varepsilon_R) = \int_{\varepsilon_r}^{\varepsilon} (\sigma - \sigma_R)\, d\varepsilon \,, \quad W(\sigma, \sigma_R) = \int_{\sigma_R}^{\sigma} (\varepsilon - \varepsilon_R)\, d\sigma \qquad /9/$$

where σ_R, ε_R are the values of stress and strain corresponding to last reversal of the stress rate. For the eqs. /7/ and /8/, the stress potentials are given by

$$W(\sigma) = W(\sigma, 0) = \frac{1}{2}\frac{\sigma^2}{E} + \frac{1}{n+1}\frac{\sigma^{n+1}}{B^n} \,,$$

$$W(\sigma, \sigma_R) = W(\sigma - \sigma_R) = \frac{1}{2}\frac{(\sigma - \sigma_R)^2}{E} + \frac{1}{n+1}\left(\frac{\sigma - \sigma_R}{2^{\frac{n-1}{n+1}}}\right)^{n+1}\frac{1}{B^n} \qquad /10/$$

The total strains are derived as follows

$$\varepsilon = \frac{\partial W(\sigma, 0)}{\partial \sigma} \,, \ \dot{\sigma} > 0 \,; \quad \varepsilon - \varepsilon_A = \frac{\partial W(\sigma, \sigma_A)}{\partial \sigma} \,, \ \dot{\sigma} < 0 \,, \qquad /11/$$

and similar relations for further cycles between σ_A and σ_B. The work dissipated on a stress cycle ABA is given by the expression

$$W_d = V(\varepsilon_B - \varepsilon_A) - W(\sigma_A, \sigma_B) \,. \qquad /12/$$

2.2 Multiaxial stress state

We shall generalize this simple model to multiaxial stress state by assuming that the material exhibits limited path dependence: it behaves as non-linearly elastic for some portions of the stress path belonging to specified sub-domains and stress-strain relations change when the path enters new sub-domain in the stress space. In this way, only a discrete set of points of the loading history affects the actual behaviour.

Consider a stress trajectory OABCD with a set of discrete points A, B, C, D which are used as switching points for different stress-strain relations. Similarly as in /9/, define the stress potentials $W(\sigma_A, 0)$, $W(\sigma_A, \sigma_B)$ as follows

$$W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A) = \int_{\underset{\sim}{\sigma}_A}^{\underset{\sim}{\sigma}_B} (\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}_A) \cdot d\underset{\sim}{\sigma}, \qquad V(\underset{\sim}{\varepsilon}_B, \underset{\sim}{\varepsilon}_A) = \int_{\underset{\sim}{\varepsilon}_A}^{\underset{\sim}{\varepsilon}_B} (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_A) \cdot d\underset{\sim}{\varepsilon} \qquad /13/$$

and the stress strain relations are

$$\underset{\sim}{\varepsilon}_A = \frac{\partial W(\underset{\sim}{\sigma}_A, 0)}{\partial \underset{\sim}{\sigma}_A}, \qquad \underset{\sim}{\varepsilon}_B - \underset{\sim}{\varepsilon}_A = \frac{\partial W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A)}{\partial \underset{\sim}{\sigma}_B}, \qquad /14/$$

and the dissipation for any closed stress cycle with switching points A, B, C is expressed as follows

$$W_d(\underset{\sim}{\sigma}_A, \underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_C) = \oint \underset{\sim}{\sigma} \cdot d\underset{\sim}{\varepsilon} = W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A) + W(\underset{\sim}{\sigma}_C, \underset{\sim}{\sigma}_B) - V(\underset{\sim}{\varepsilon}_A, \underset{\sim}{\varepsilon}_C) \qquad /15/$$

To determine the material behaviour completely, we should specify the switching points on the stress trajectory. To this end, let us divide the stress space into several subregions.For any given point on the stress trajectory, the subsequent behaviour depends on the direction of stress increment since in each subregion different stress-strain relations occur. To define these suregions, let us introduce a vector function of stress , $g_i = g_i(\sigma_{ij})$, $i = 1, 2, 3, \ldots$ such that stress potentials are expressed in terms of components of g_i, $W(\sigma_{ij}) = W(g_i)$. Let us define the corresponding strain measures, such that

$$\gamma_1 = \frac{\partial W(g_i)}{\partial g_1} \qquad \gamma_2 = \frac{\partial W(g_i)}{\partial g_2}, \qquad \gamma_3 = \frac{\partial W(g_i)}{\partial g_3}, \ldots \qquad /16/$$

and

The relations /16/ occur when $\dot{g}_1$ $\dot{g}_2$ and $\dot{g}_3$ have some fixed signs. When the sign of $\dot{g}_1$ changes, the stress potential takes the form

$$W' = W\left(\frac{g_1 - g_{1A}}{2^{\frac{n-1}{n+1}}}, \; g_2, \; g_3\right), \qquad /17/$$

and the respective stress strain relations are

$$\gamma_1 - \gamma_{1A} = \frac{\partial W'}{\partial g_1}, \qquad \gamma_2 = \frac{\partial W'}{\partial g_2}, \qquad \gamma_3 = \frac{\partial W'}{\partial g_3}, \qquad /18/$$

where n denotes the order of homogeneity of the stress potential or its additive terms.

Let us exemplify these relations by assuming that g_1, g_2, g_3 coincide with principal stresses defined as follows

$$\tau_1 = \tfrac{1}{2}(\sigma_2 - \sigma_3), \quad \tau_2 = \tfrac{1}{2}(\sigma_3 - \sigma_1), \quad \tau_3 = \tfrac{1}{2}(\sigma_1 - \sigma_2), \qquad /19/$$

and the corresponding shear strains are

$$\gamma_1 = \tfrac{2}{3}(\varepsilon_2 - \varepsilon_3), \quad \gamma_2 = \tfrac{2}{3}(\varepsilon_3 - \varepsilon_1), \quad \gamma_3 = \tfrac{2}{3}(\varepsilon_1 - \varepsilon_2). \qquad /20/$$

For the power law, assume that the initial loading corresponds to some signs of

$\dot{\tau}_i$, say , $\dot{\tau}_1 > 0 \; \dot{\tau}_2 > 0 \; \dot{\tau}_3 < 0$. The stress potential is expressed in the form

$$W = \frac{2^n}{n+1}\left[c_1 \tau_1^{n+1} + c_2 \tau_2^{n+1} + c_3 \tau_3^{n+1} \right] , \qquad /21/$$

and in view of /16/ we have

$$\gamma_1 = 2^n c_1 \tau_1^n , \quad \gamma_2 = 2^n c_2 \tau_2^n , \quad \gamma_3 = 2^n c_3 \tau_3^n ; \qquad /22/$$

assume now that at A the rate of $\dot{\tau}_1$ changes its sign, so that for subsequent stage we have $\dot{\tau}_1 < 0 , \; \dot{\tau}_2 > 0 \; \dot{\tau}_3 < 0$. Then

$$W(\tau_1 - \tau_{1A} , \tau_2 , \tau_3) = \frac{2^n}{n+1}\left[c_1 \left(\frac{\tau_1 - \tau_{1A}}{2^{\frac{n-1}{n+1}}} \right)^{n+1} + c_2 \tau_2^{n+1} + c_3 \tau_3^{n+1} \right] , \qquad /23/$$

$$\gamma_1 - \gamma_{1A} = 2 c_1 (\tau_1 - \tau_{1A})^n , \quad \gamma_2 = 2^n c_2 \tau_2^n , \quad \gamma_3 = 2^n c_3 \tau_3^n . \qquad /24/$$

Equations /21-24/ can easily be generalized to any loading program. Stress-strain relations are piecewise finite as for elastic materials and change only when the rate of any of maximum shear stresses changes its sign. Obviously, more complex hardening rules and non-stationary hardening processes can be incorporated in this model by formulating proper stress-strai relations between τ_i and γ_i , i= 1, 2, 3.

3. Extremem principle and bounds on dissipation

Consider now the power hardening law defined by the stress potential /21/ and limit our analysis to the case of proportionally varying load between to extreme values. In view of homogeneity of stress potential, the radial stress path are induced at each point of the body and switching points A and B correspond to points of load reversal on the structure boundary. For the steady cyclic state and Masing hardening rule, we have

$$W(\underset{\sim}{\sigma}_A , \underset{\sim}{\sigma}_B) = W(\underset{\sim}{\sigma}_B , \underset{\sim}{\sigma}_A) = W(\underset{\sim}{\sigma}_B - \underset{\sim}{\sigma}_A) ,$$
$$V(\underset{\sim}{\varepsilon}_A , \underset{\sim}{\varepsilon}_B) = V(\underset{\sim}{\varepsilon}_B , \underset{\sim}{\varepsilon}_A) = V(\underset{\sim}{\varepsilon}_B - \underset{\sim}{\varepsilon}_A) , \qquad /25/$$

and the specific energy dissipation per cycle and unit volume is expressed as follows

$$W_d = (\underset{\sim}{\sigma}_B - \underset{\sim}{\sigma}_A) \cdot (\underset{\sim}{\varepsilon}_B - \underset{\sim}{\varepsilon}_A) - 2W(\underset{\sim}{\sigma}_A , \underset{\sim}{\sigma}_B) = (n-1) W(\underset{\sim}{\sigma}_B - \underset{\sim}{\sigma}_A)$$
$$W_d = \frac{n-1}{n} V(\underset{\sim}{\varepsilon}_B - \underset{\sim}{\varepsilon}_A) \qquad /26/$$

The relations /26/ imply that bounds on W_d can be derived using minimum principles of the non-linear elasticity theory. In fact, we have

$$\frac{n-1}{n}\left[\int V(\varepsilon_B^k - \varepsilon_A^k)\, dV - \int \dot{T} \cdot \Delta u^k \, dS_T \right] \leqslant \int W_d \, dV \leqslant (n-1) \int W(\sigma_B^s - \sigma_A^s)\, dV , \qquad /27/$$

where $\Delta \underset{\sim}{\sigma}^{s} = \underset{\sim}{\sigma}_B^{s} - \underset{\sim}{\sigma}_A^{s}$, $\Delta \underset{\sim}{\varepsilon}^{k} = \underset{\sim}{\varepsilon}_B^{k} - \underset{\sim}{\varepsilon}_A^{k}$, $\Delta \underset{\sim}{u}^{k} = \underset{\sim}{u}_B^{k} - \underset{\sim}{u}_A^{k}$, $\Delta \underset{\sim}{T} = \underset{\sim}{T}_B - \underset{\sim}{T}_A$

denote statically and kinematically admissible stress, strain and displacement amplitudes. Thus solution for monotonic loading between extreme values of load implies the solution for cyclic loading.

For more general case when loading is not proportional and more than two switching points occur on the loading path, the stationarity of more general functional can be proved. Using /13/ and /14/, let us write the generalized complementary energy functional

$$\Pi_\sigma = \int \left[W(\underset{\sim}{\sigma}_A, 0) + W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A) + W(\underset{\sim}{\sigma}_C, \underset{\sim}{\sigma}_B) \right] dv - \int \underset{\sim}{T}_A \cdot \underset{\sim}{u}_A^{o}\, dS_u - \int \underset{\sim}{T}_B \cdot (\underset{\sim}{u}_B^{o} - \underset{\sim}{u}_A^{o})\, dS_u - \int \underset{\sim}{T}_C \cdot (\underset{\sim}{u}_C^{o} - \underset{\sim}{u}_B^{o})\, dS_u \qquad /28/$$

where $\underset{\sim}{u}_A^{o}, \underset{\sim}{u}_B^{o}, \underset{\sim}{u}_C^{o}$ denote displacements prescribed on the portion of the boundary S_u. This functional not only depends on the value of stress at the end point of stress path but also on values of stresses at switching points. The extremum of /28/ occurs provided variation is performed with respect to first argument in each term, that is

$$\delta W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A) = \delta_B W(\underset{\sim}{\sigma}_B, \underset{\sim}{\sigma}_A) = \frac{\partial W}{\partial \underset{\sim}{\sigma}_B} \cdot \delta \underset{\sim}{\sigma}_B \, . \qquad /29/$$

References

1. J. Burbach, Zum zyklischen Verformungsverhalten einiger technisher Werkstoffe, Techn. Mitt. Krupp Forsh. Ber., Band 28, 1970

2. Z. Mroz and N. C. Lind, On simplified theories of ycyclic plasticity, Univ. of Waterloo Rep. no. 20, 1972.

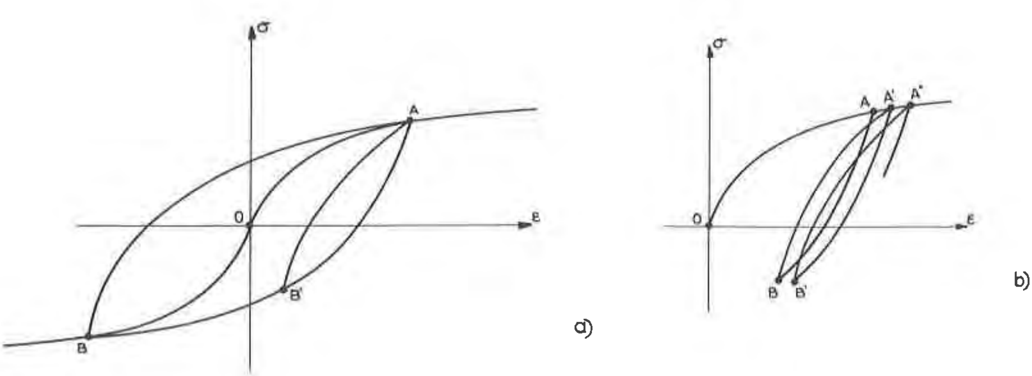

Fig.1.