

Laminar flow past the bottom with obstacles – a suspension approximation

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Abstract. From Albert Einstein's study (1905) it is known that suspension introduced to a fluid modifies its viscosity. We propose to describe the influence of obstacles on the Stokesian flow as a such modification. Hence, we treat the fluid flow through small obstacles as a flow with suspension. The flow is developing past the plane bottom under the gravity force. The spatial distribution of suspension concentration is treated as given, and is regarded as an approximation of different obstacles which modify the fluid flow and change its viscosity. The different densities of suspension are considered, beginning of small suspension concentration until 40%. The influence of suspension concentration on fluid viscosity is analyzed, and Brinkman's formula as fitting best to experimental data is applied.

Key words: Stokes' flow, Einstein's suspension, Brinkman's suspension, non-uniform viscosity, velocity distribution, wastes, plants on the bottom.

1. Introduction

1.1. Aims of the paper. Below we study Stokes' flow in a wide channel, when the significance of the channel banks can be neglected and the two-dimensional problem can be considered only. The viscosity of fluid is modified by the presence of suspension, whose concentration distribution is selected according to a given distribution of obstacles.

Suspension of even rarely dispersed particles has a higher viscosity than pure liquid, and in the case of spherical and rarely dispersed particles the viscosity of suspension is given by the Einsteinian formula. Mathematical formulae are known also for other shapes of the particles, for example ellipsoidal ones. To describe the viscosity for higher concentrations of the suspension Brinkman's relation is needed, but the steady flow of such a system is still described by Stokes' equation.

First, we look at flows, in which the water viscosity changes in a continuous manner with the depth of the fluid, as a result of existence of a suspension with the concentration varying in a given manner.

Also, two layer flow is analyzed. Both layers, denoted as A and B, are described by Stokes' equation, and the fluid A flows past the fluid B. The upper surface of the layer A is free and suffers only the air pressure. In both layers the suspension density is changing in different but in *a priori* given manners. Finally, in a limiting case the layer A does not contain any suspension, while the lower, layer B contains a suspension and imitates the influence of bottom obstacles.

1.2. Motivation of the paper. A suspension is a heterogeneous mixture containing solid particles that are sufficiently large for sedimentation. Usually they must be larger than 1 micrometer. The internal phase (solid) is dispersed throughout

the external phase (fluid) through mechanical agitation, with the use of certain excipients or suspending agents. Unlike colloids, suspensions will eventually settle. An example of a suspension would be sand in water. The suspended particles are visible under a microscope and may settle over time if left undisturbed. The influence of the temperature is here essential, cf. [1,2].

The problem because of its important applicability reaches up the Antiquity, cf. *The canals of the Nile will dry up, and the streams of Egypt will stink with rotting reeds and rushes*, (Isaiah 19:6). The history of construction Nile – Red Sea canal, a forerunner of the modern Suez Canal by the Persians in times of Darius the Great, and its reconstruction by Ptolemy II Philadelphus is marked by persistent accumulations of Nile silt, and the maintenance and repair of the canal became increasingly cumbersome over each passing century, [William Matthew Flinders Petrie, *A History of Egypt. Volume 3: From the XIXth to the XXXth Dynasties*, Adamant Media Corporation].

The Nîmes aqueduct, with the famous *Pont du Gard* was steadily under the threat posed by vegetation penetrating the stone lid of the channel. As well as obstructing the flow of the water, dangling roots introduced algae and bacteria that decomposed in a process called biolithogenesis.

In particular, in the lakes, streams, rivers and open channels grow different types of plants, e.g. Canadian waterweed (*Elodea canadensis*) and hydrilla (Esthwaite waterweed) *Hydrilla verticillata*. They grow rapidly in favourable conditions and can choke shallow ponds, canals, and the margins of some slow-flowing rivers, so, even the greatest water reservoirs are endangered. Silty sediments and water rich in nutrients favor the growth of the waterweed in nutrient-rich lakes. However, the plants will grow in a wide range of conditions, from very

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shallow to deep water, and in many sediment types. It can even continue to grow unrooted, as floating fragments.

Depending on the kind and obstacles, the effective viscosity of the bottom fluid may be modified, beginning of the density of suspension and finishing at the density of the particles creating an aggregate modelled by a porous medium. Such a system permits to study characteristic traits of the flows in beds of rivers (canals, pipes, lakes) with obstacles at the bottom (such as stones, plants or other structures), cf. [3–10].

The study of these processes requires cooperation of different branches of science, biology, chemistry, geophysics, and mathematics, and the results obtained in different branches can be readily exploited. One must not wonder that Albert Einstein and his son Hans Albert contributed significantly to the problem. For example, the change of fluid viscosity resulted from the presence of a suspension can be calculated after the formula found by Albert Einstein, [11–13], and their followers [14, 15], also [16–18].

1.3. Flowing waters in nature and their variability. It is truism that flowing waters met in the nature differ from those observed in hydraulic laboratory conditions, and are far more complicated aggregates. They carry stones, woods, leaves, sands and others suspensions.

A river is a natural watercourse, usually freshwater, flowing towards another river, a lake, a sea or an ocean. In some cases a river could flow into the ground and dry up completely at the end of its course, without reaching another body of water. Rivers also provide an easy means of disposing of waste-water and often other wastes. The coarse sediments, gravel, and sand, generated and moved by rivers are extensively used in construction.

The general pattern is that the first order streams contain particulate matter (decaying leaves from the surrounding forests), which is processed there by shredders like *Plecoptera* larvae. The leftovers of the shredders are utilized by collectors, such as *Hydropsychidae*, and further downstream algae that create the primary production become the main food source of the organisms.

The inorganic substrate of lotic systems is composed of the geologic material present in the catchment that is eroded, transported, sorted, and deposited by the current. Bacteria are present in large numbers in lotic waters. Emerging pathogens (bacteria, viruses, protozoa and cyanobacteria) influences the water quality.

Free-living forms are associated with decomposing organic material, biofilm on the surfaces of rocks and vegetation, in between particles that compose the substrate. Algae, consisting of phytoplankton and periphyton, are the most significant sources of primary production in most streams and rivers. Periphyton are typically filamentous and tufted algae that can attach themselves to objects to avoid being washed away by fast current. In places where flow rates are negligible or absent, periphyton may form a gelatinous, unanchored floating mat.

Plants exhibit limited adaptations to fast flow and are most successful in reduced currents. Plants, such as mosses and

liverworts attach themselves to solid objects. This typically occurs in colder headwaters where the mostly rocky substrate offers attachment sites. Some plants are free floating at the water's surface in dense mats like duckweed or water hyacinth. Others are rooted and may be classified as submerged or emergent. Rooted plants usually occur in areas of slackened current where fine-grained soils are found. These rooted plants are flexible, with elongated leaves that offer minimal resistance to current, [19–24]. However, in mass the plants may create an obstacle to the current, and the river flow responds to their appearing. Hence studying the flows of rivers through plant obstacles is one of the subjects of hydrology, and in broader plane, of geophysics.

1.4. Contaminant suspension in water bodies. The contamination of water bodies (e.g. lakes, rivers, oceans, aquifers and groundwater) by raw sewages and industrial waste, as well by storm waters washed off of parking lots, roads and highways leads to appearing of a suspension in the water body. Also releasing of chemical or radionuclide contaminants into soil (even located away from a surface water body) may contaminate the aquifer below, defined as a toxin plume.

Analysis of groundwater contamination may focus on the soil characteristics and site geology, hydrogeology, hydrology, and the nature of the contaminants. An additional example is the leaching out of nitrogen compounds from fertilized agricultural lands and nutrient runoff in storm water from “sheet flow” over agricultural fields or a forests, [25, 26]. In the most recent national report on water quality in the United States, 45% of assessed stream miles, 47% of assessed lake acres, and 32% of assessed bays and estuarine square miles were classified as polluted. In 2007, 1/4th the length of China's seven main rivers were so poisoned the water harmed the skin, [27].

1.5. Flow of the fluid with the free upper surface – fluid with the variable viscosity. Aquatic plants can only grow in water or in soil that is permanently saturated with water. They are therefore a common component of wetlands. To aquatic plants belong: *amphiphytes*, plants that are adapted to live either submerged or on land; and *elodeids*, stem plants that complete their entire life cycle submerged, or with only their flowers above the waterline, cf. [28].

In streams and rivers grow different types of plants. Frequently, they grow rapidly in favorable conditions and can choke shallow ponds, canals, and the margins of some slow-flowing rivers. In particular, they hinder the regulation as well as run-off and storage level control of water courses, flood control and lice-hazards abatement, as well as the effect of facilities situated in or aside the watercourse on its hydraulic regime. They are also important occasionally as an obstacle to flow to lake navigation. Formally, one can consider the existence of such obstacles either as a change of the fluid viscosity or the skeleton of the porous medium.

Here, we study the first possibility only. The laminar flow past the porous bottom will be the subject of the next paper. We consider flow of the fluid on an inclined plane under in-

fluence of the gravity. We discuss only central part of the fluid stream and neglect the influence of the stream banks, so the problem can be regarded as two-dimensional.

1.6. Large deflections of the underwater obstacles. An understanding of the behaviour of flexible obstacles, for example the vegetation in rivers is important for solving a broad range of problems in environmental hydraulics, flood control and water management, cf. recent study by Keramaris Evangelos, [29].

In assessing the vertical velocity distributions in flows through and above vegetation, the primary question is to quantify the deflection of vegetation elements. In the majority of studies, a relatively high stiffness of vegetation elements has been assumed and calculation of the deflection was based on the small deflection analysis of a cantilever beam. Observations of water vegetation suggest, however, that the assumed high stiffness in many cases is too unrealistic. For this reason, Elżbieta Kubrak, Janusz Kubrak and Paweł M. Rowiński rejected this assumption and proposed a method of calculation suitable for larger deflections of water plants, [30, 31].

1.7. Flow in the canal with obstacles at the bottom. The problem of flow past a half-space with obstacles (alive or of non-living matter) at the bottom of the water reservoir can be solved approximately in one of two ways.

1. If the obstacles are small and rare, the fluid flooding the obstacles at the bottom can be replaced by a layer of different viscous fluid, with viscosity changed according to Einsteinian theory of emulsion (or its developments).

2. If the obstacles at bottom are dense, they can be regarded as a porous medium, and the flow of fluid filling this medium can be treated by different averaging methods, in particular, by the homogenization theory, [32–54]. The theory of functions of a complex variable can be applied also. Namely, if the obstacles are cylinders arranged in some order, the potential theory of flow can be used, as it was indicated yet by Baron Rayleigh [55]. Rayleigh’s case deals with the cylindrical obstacles arranged in rectangular order at the bottom of canal with the flowing fluid. The theory is currently developed in the series of papers by Vladimir Mityushev and Sergei Rogosin, and their school, cf. [56–58].

1.8. Brinkman’s legacy. Two works of the huge multithematic legacy of Henri Coenraad Brinkman are important for the physics of fluid. The first one which is quoted and used in this paper discusses the influence of suspension density on the fluid viscosity for higher concentrations, above the Einsteinian limit, cf. [59]. Brinkman’s relation valid up to 40% concentration is referred to in Sec. 2. The second work proposes a correction to Darcy’s equation, [60]. Such a corrected equation is known also under Brinkman’s name, and is used, for example, to describe flow of a viscous fluid past the porous medium.

Anna Trykozko *et al.* proposed The Double Constraint method, with the aim is to find conductivities that satisfy

Darcy’s law, the continuity equation, as well as both the flux and the head boundary conditions, [61].

Wojciech Sobieski and Anna Trykozko, [62], investigated Forchheimer’s equation – a nonlinear extension of the linear Darcy’s law, which applies to a broader range of velocities for flows through porous media. They examined sensitivity of the Forchheimer model to permeability and a coefficient at nonlinear term, using both experimental and computational data for the validation.

2. Viscosity of suspensions: Brinkman’s relation

Let a solute-particle has a volume ω_0 and a spherical shape of radius R . Then $\omega_0 = (4/3)\pi R^3$. If such a particle were added to a pure solvent of volume V , the viscosity of such a suspension would be given by Einstein’s formula, cf. [11, 13],

$$\eta = \eta_0 \left(1 + \frac{5}{2} \frac{\omega_0}{V + \omega_0} \right), \quad (1)$$

where η_0 denotes the pure solvent viscosity. Now, let us assume that we have N solute particles, each with the volume ω_0 . Thus the concentration by volume is

$$c_v = \frac{N}{V} \omega_0.$$

If one can treat the contributions of the individual particles forming a suspension as independent, the Einstein formula for the effective viscosity of the suspension reads

$$\eta = \eta_0 \left(1 + \frac{5}{2} c_v \right). \quad (2)$$

The formula (2) was derived in the, so called, non-interaction approximation, it is under the assumption that the rigid spherical particles of the suspension are non-interacting, and the formula corresponds to summation of the viscosity contributions of individual particles. This means that (2) is valid only for small concentrations.

Brinkman derived an expression for the viscosity of suspensions of finite concentration

$$\eta = \frac{\eta_0}{(1 - c_v)^{5/2}}. \quad (3)$$

This equation agrees quite well with experimental results, [59]. Recently, Peng *et al.*, cf. [65] applied Brinkman’s formula to describe the viscosity of nanorefrigerants. Brinkman’s formula was used with success in [66, 67].

Behrouz Abedian and Mark Kachanov, [68, 69], discussed the problem of effective viscosity of a Newtonian fluid with rigid spherical particles. They found that the simple non-interaction approximation when formulated correctly yields an effective viscosity ratio for the suspension in the form

$$\frac{\eta}{\eta_0} = \frac{1}{1 - 2.5c_v} \quad (4)$$

that remains accurate at much higher volume fractions of particles c_v than the usual first-order approximation $\eta/\eta_0 = 1 + 2.5c_v$ resulting from Eq. (2).

Abedian and Kachanov insisted that Ford’s model is better than Einstein’s one when a high particle volume fraction

is considered. Those authors indicated also that the formula (2) does not satisfy the bounds predicted by Hashin and Shtrikman, [70], for composites, at any c_v , no matter how small.

Paradoxically, it is Einstein's formula which was used as the first approximation in Brinkman's study, and in deriving his relation (3).

The descriptions of Brinkman's and Abedian-Kachanov's are compared in Fig. 1.

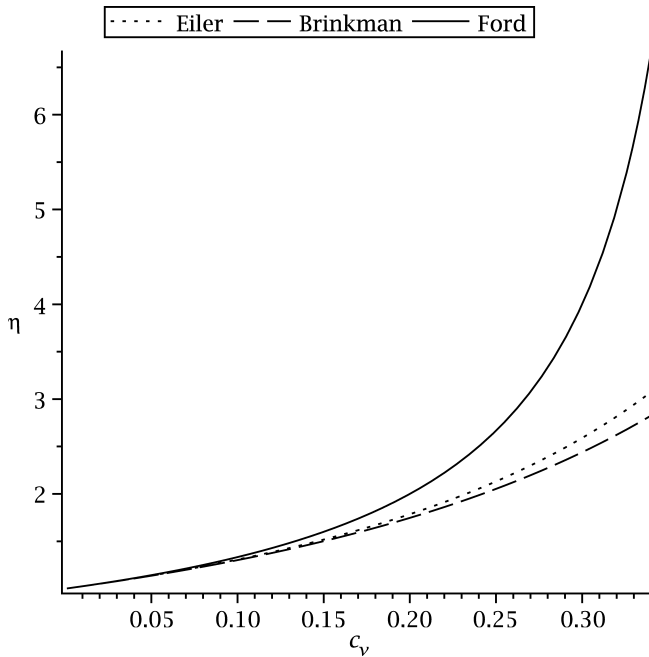


Fig. 1. The experimental viscosity (Eiler, dots) as the function of the concentration by volume c_v , compared with different theoretic formulae. The superiority of approximation the data by Brinkman's relation (dashed) in comparison with Ford's one (solid line) is visible. The tangent at $c_v = 0$ is common for all three curves and is given by Einstein's formula. The viscosity η_0 of pure solvent is the unit of viscosity on η axis. For the water $\eta_0 = 0.01 \text{ g/(cm s)}$

In this figure we see that Brinkman's relation fits well to Eilers' experimental data, while supported in Abedian and Kachanov's paper Ford's relation departs quickly from the experimental curve.

Examples of the volume concentrations. The volume inside a sphere inscribed in a cube can be approximated as 52.4% of the volume of the cube, since $\pi/6 \approx 0.5236$. For example, a sphere with diameter 1 cm has 52.4% the volume of a cube with edge length 1 cm, or about 0.524 cm^3 . In this case $c_v = 0.52$. A sphere with diameter 0.914 cm has 40% the volume of the 1 cm^3 cube, while a sphere with diameter 0.133 cm has about 1% the volume of such cube. This corresponds to $c_v = 0.4$ or $c_v = 0.01$, respectively.

3. Navier-Stokes equation for the steady flow

Let us take into account an incompressible viscous fluid of the density ρ and the viscosity η , cf. [13]. Let us consider the

steady flow of this fluid in the presence of the gravity field. The velocity field ruling in the fluid is given by the vector \mathbf{v} . The stress tensor in the fluid is

$$\sigma_{ij} = -p\delta_{ij} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (5)$$

while the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ in index notation reads

$$\frac{\partial v_i}{\partial x_i} = 0. \quad (6)$$

The equation of steady motion of an incompressible fluid (Stokes equation) under the pressure gradient ∇p and the gravitation force $\rho \mathbf{g}$ reads

$$\rho v_k \frac{\partial v_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} + \rho g_i. \quad (7)$$

Here, we admit possibility of dependence of the viscosity η on the position $\mathbf{x} = (x_1, x_2, x_3)$.

3.1. Two-dimensional cases. We can consider, as simpler, special cases, in which the field functions such as the velocity or the pressure do not depend on one of three spacial variables.

Two-dimensional Stokes' equation is a nexus which links Hele-Shaw's flow with our considerations below. In our case, in contrast to Hele-Shaw's flow, the upper surface of fluid is free; hence the boundary conditions are not symmetric in our case.

Two-dimensional Hele-Shaw flow. We take into account a flow between two flat parallel plates, separated by a small gap, and we neglect the influence of the gravity force.

Let x_1, x_2 be the directions parallel to the flat plates, and x_3 the perpendicular direction, with $2d$ being the gap between the plates (at $x_3 = \pm d$). When the flow between such plates is developed, its velocity profile in the x_3 direction is parabolic, with the symmetry plane at $x_3 = 0$, cf. [13]. This relation and the uniformity of the pressure in the narrow direction x_3 permits to average the velocity with regard to x_3 and thus to consider an effective velocity field in only the two dimensions x_1 and x_2 . When substituting this equation into the continuity equation and again averaging over x_3 , the equation of Hele-Shaw's flow is obtained, [71], cf. also [72, 73].

Asymptotic analysis of the flow passing over a small obstacle in the Hele-Shaw cell is recently performed by Gennady Mishuris, Sergei Rogosin and Michał Wróbel, [74].

Two-dimensional flow: one velocity component only. We change the description of coordinates, and substitute

$$x_1 = x, \quad x_2 = y \quad \text{and} \quad x_3 = z.$$

We consider a two-dimensional flow in the plane x, y . The fields in this case do not depend on z , but the field of body force does exist. We change the notation for the components of the body force also, writing

$$\rho \mathbf{g} = (g_x, -g_y, 0)$$

or

$$\rho g_1 = g_x, \quad \rho g_2 = -g_y \quad \text{and} \quad g_3 = 0.$$

If the velocity has only one component, say v_1 , it is when

$$\mathbf{v} = (v_1, 0, 0) \quad (8)$$

with

$$v_1 = v_1(x_2, x_3) \quad (9)$$

but not depending on x_1 , then left hand side of Eq. (7) vanishes and Stokes' equation becomes

$$-\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left\{ \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\} + \rho g_i = 0. \quad (10)$$

If additionally, $\eta = \eta(x_2, x_3)$ only, then

$$\begin{aligned} -\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2} \left(\eta \frac{\partial v_1}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(\eta \frac{\partial v_1}{\partial x_3} \right) + \rho g_1 &= 0, \\ -\frac{\partial p}{\partial x_2} + \rho g_2 &= 0, \\ -\frac{\partial p}{\partial x_3} + \rho g_3 &= 0. \end{aligned} \quad (11)$$

The first equation of this system, for $\eta = \text{constant}$, is typical equation describing Hagen-Poiseuille's flow by capillaries.

Notice that under the assumptions given by Eqs. (8) and (6), the incompressibility condition (6) is identically satisfied.

4. Flow of the fluid with the free upper surface

4.1. Fluid with the variable viscosity. In streams and rivers grow different types of plants, e.g. Canadian waterweed. Frequently, they grow rapidly in favorable conditions and can choke shallow ponds, canals, and the margins of some slow-flowing rivers. In particular, they hinder the regulation as well as run-off and storage level control of water courses, flood control and lice-hazards abatement, as well as the effect of facilities situated in or aside the watercourse on its hydraulic regime. They are also important occasionally as an obstacle to flow to lake navigation. Formally, one can consider the existence of such obstacles either as a change of the fluid viscosity or the skeleton of the porous medium.

This problem is important not only by its applications, but it possesses crucial scientific significance due to the simplicity of its mathematical description as well its experimental realisation. We consider flow of the fluid on an inclined plane under influence of the gravity. We discuss only central part of the fluid stream and neglect the influence of the stream banks, so the problem is two-dimensional.

Let a layer of an incompressible viscous fluid of thickness h be bounded above by a free surface and below by a fixed plane inclined at an angle α to the horizontal, see Fig. 2. Let the density ρ of the fluid be constant, while its viscosity η be dependent on the position according to a given law. Let us determine the steady flow due to gravity.

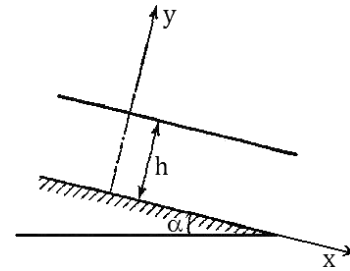


Fig. 2. Flow on a plane

We take the fixed plane as the xz -plane, with the x -axis in the direction of flow and y pointed upward in direction perpendicular to the bottom. Let the viscosity be function of y only and we seek a solution depending only on y . The Navier-Stokes Eqs. (11) with $v_1 = v(y)$ in a gravitational field reduce in our case to two equations

$$\begin{aligned} \frac{\partial}{\partial y} \left(\eta \frac{\partial v}{\partial y} \right) + g_x &= 0, \\ \frac{\partial p}{\partial y} + g_y &= 0. \end{aligned} \quad (12)$$

Here, see Fig. 1,

$$g_x = \rho g \sin \alpha \quad \text{and} \quad g_y = \rho g \cos \alpha \quad (13)$$

and $g = |\mathbf{g}|$ denotes the gravitational acceleration.

According to Eqs. (5), (8) and (9), the shear component in this flow is

$$\sigma_{xy} = \eta \frac{\partial v}{\partial y}. \quad (14)$$

From the first equation of the system (12) we get

$$\eta \frac{\partial v}{\partial y} = -g_x y + C \quad (15)$$

and after the second integration

$$v(y) - v_0 = \int_0^y \frac{1}{\eta(\tilde{y})} (-g_x \tilde{y} + C) d\tilde{y}. \quad (16)$$

The constants v_0 and C must be found from the boundary conditions. Integration of the second equation of the system (12) gives $p(y) - p_0 = -g_y y$ or

$$p(y) = -g_y y + p_0. \quad (17)$$

The constant p_0 must be found from the boundary condition. At the free surface ($y = h$) we must have $p = p_A$, where p_A denotes the atmospheric pressure. Hence $p_0 = p_A + g_y h$ and

$$p = (h - y)g_y + p_A. \quad (18)$$

This relation is independent of the eventual spatial variations of the fluid viscosity η .

At the free surface also $\sigma_{xy} = 0$, and according to Eq. (14) this boundary condition reads

$$\sigma_{xy}(h) = \left\{ \eta \frac{\partial v}{\partial y} \right\}_{y=h} = 0 \quad (19)$$

or after Eq. (15)

$$C = g_x h.$$

At the bottom, for $y = 0$ we must have $v = 0$. Hence

$$v_0 = 0.$$

Therefore

$$v(y) = g_x \int_0^y \frac{h - \tilde{y}}{\eta(\tilde{y})} d\tilde{y}. \tag{20}$$

This is the solution of the problem satisfying the boundary conditions.

4.2. Constant viscosity. For $\eta = \text{constant}$ the solution (20) reads

$$v(y) = \frac{1}{\eta} g_x \left(h - \frac{1}{2} y \right) y \tag{21}$$

with the maximum value of v at $y = h$.

What concerns the shear stress, after Eqs. (14) and (21),

$$\sigma_{xy} = g_x (h - y) \tag{22}$$

with the maximum value at $y = 0$.

An example of velocities. Let $g_x = 1 \text{ g/cm}^3 \cdot 981 \cdot 5 \cdot 10^{-5} \text{ cm/s}^2 \approx 0.05 \text{ g/cm}^2 \cdot \text{s}^2$; the value of gradient $5 \cdot 10^{-5}$ is taken approximately from the slope of *Pont du Gard*, which descends 2.5 cm in 456 m. According to Eq. (21), for the water with $\eta = 0.01 \text{ g/cm s}$ and at for $h = 1 \text{ cm}$ the fluid velocity at the free surface is $v = 2.5 \text{ cm/s}$ only, and for $h = 10 \text{ cm}$, it is just $v = 250 \text{ cm/s}$.

5. Fluid with Brinkman's viscosity

For the model derived in the previous Section we express the viscosity by the suspension concentration. To this end we substitute Brinkman's formula (3) to the integral (20) and obtain

$$v(y) = \frac{g_x}{\eta_0} \int_0^y (1 - c_v)^{5/2} (h - \tilde{y}) d\tilde{y}. \tag{23}$$

This relation will be exploited in subsequent calculations. From here, we suppress the index v at the concentration c_v and write simply

$$c \equiv c_v. \tag{24}$$

5.1. Linear variation of suspension concentration. For the linear dependence of suspension concentration, such that at the bottom, it is at $y = 0$, the density of the suspension is $c = c_0$ and just beneath the free surface, it is at $y = h$, it vanishes, $c = 0$, what is described by the formula

$$c = c_0 \left(1 - \frac{y}{h} \right) \tag{25}$$

and corresponding viscosity is

$$\eta(y) = \frac{\eta_0}{\left(1 - c_0 + c_0 \frac{y}{h} \right)^{5/2}}. \tag{26}$$

We have the following distribution of the fluid velocity along the y -axis

$$v(y) = \frac{g_x}{\eta_0} \int_0^y \left(1 - c_0 + c_0 \frac{\tilde{y}}{h} \right)^{5/2} (h - \tilde{y}) d\tilde{y} \tag{27}$$

$$\equiv \frac{g_x}{\eta_0} (I_0 \cdot h - I_1).$$

The dimension of the factor $[g_x/\eta_0] = \text{cm}^{-1} \text{s}^{-1}$ and the dimension of the integral is cm^2 ; this gives the dimension of the velocity cm/s . Above

$$I_0 = \frac{1}{(c_0/h) \cdot \frac{7}{2}} \left\{ \left(c_0 \frac{y}{h} + b \right)^{7/2} - b^{7/2} \right\},$$

$$I_1 = \frac{1}{(c_0/h)^2 \cdot \frac{9}{2}} \left\{ \left(c_0 \frac{y}{h} + b \right)^{9/2} - b^{9/2} \right\}$$

$$- \frac{b}{(c_0/h)^2 \cdot \frac{7}{2}} \left\{ \left(c_0 \frac{y}{h} + b \right)^{7/2} - b^{7/2} \right\}$$

with $b \equiv 1 - c_0$.

The velocity distribution $v(y)$ found from the relation (27) is shown in Fig. 3. Such is the velocity behaviour for linear distribution of the suspension concentration c with the depth, and for different bottom suspension concentration c_0 .

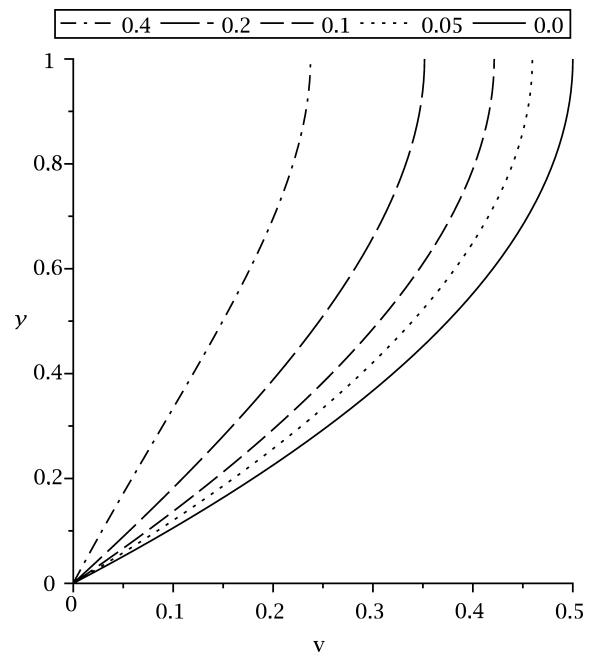


Fig. 3. The velocity v along the y -axis for the linear distribution of suspension concentration c with the depth, cf. Eq. (27). Numbers in the legend indicate corresponding values of the suspension concentration c_0 at the bottom, cf. Eq. (25). Obviously, with the growth of suspension density in the lower part of the fluid stream, for the same value of y , the fluid viscosity increases and the fluid velocity diminishes. As the unit of velocity the quantity $(g_x/\eta_0) \cdot h^2$ was accepted. For $g_x = 0.05 \text{ g/cm}^2 \cdot \text{s}^2$, $\eta_0 = 0.01 \text{ g/cm s}$ and $h = 1 \text{ cm}$ (see Example at the end of Sec. 5) this unit is equal 5 cm/s

5.2. Trapezoidal and triangular distribution of suspension. Now we investigate a situation in which the suspension concentration along the y -axis varies after a given broken line form, what means that it is given by piece-wise formulae. We divide the stream of fluid in two layers, the bottom $0 \leq y < h/2$ and the upper one $h/2 \leq y < h$, and we consider two cases.

In the first case, which we call the trapezoidal one, the suspension concentration c in the layer $0 \leq y < h/2$ is constant and in the upper layer $h/2 \leq y < h$ it diminishes linearly to zero at $y = h$, see Fig. 4a.

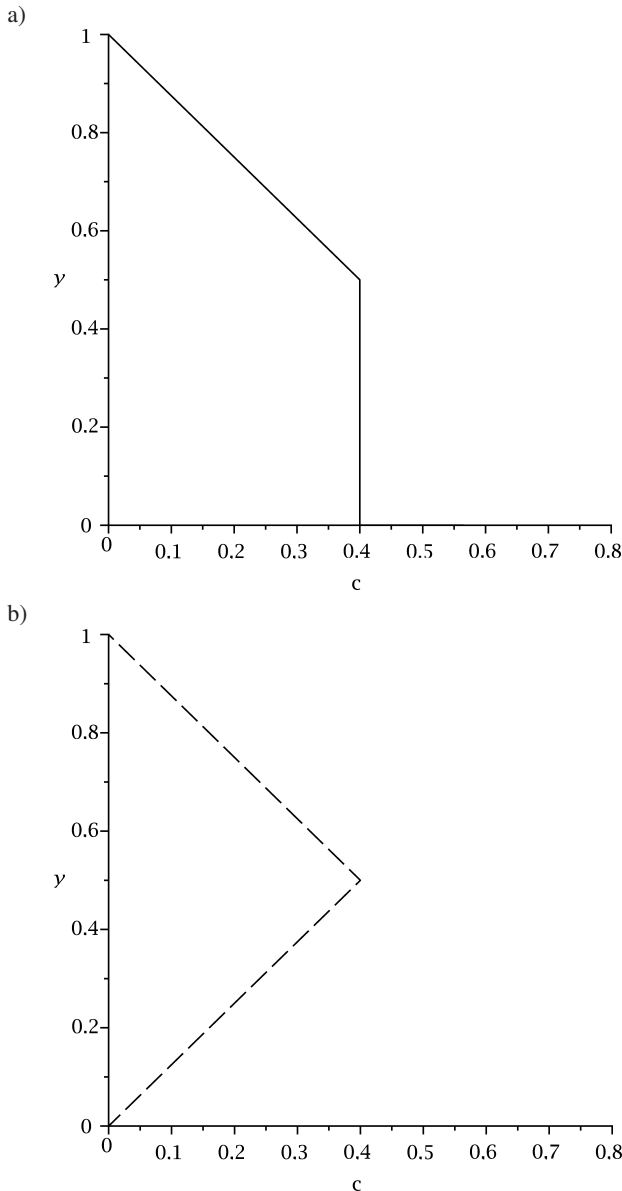


Fig. 4. Two types of the distributions of suspension concentration c considered in Subsec. 5.2. At the top (a) – the bottom concentration (for $y < 1/2$) is constant. At the bottom (b) – the bottom concentration growth linearly until $y = 1/2$

In the second case, the triangular one, the suspension concentration c in the layer $0 \leq y < h/2$ growth linearly with y ,

reaches its maximum at $y = h/2$ and subsequently, as in the first case, in the upper layer it diminishes linearly to zero at $y = h$, see Fig. 4b.

In the second case the suspension concentration reaches its greatest value not at the bottom but in the interior of the bulk of the fluid, near $y = h/2$. One may imagine that such behaviour approximates a flood flow past a wetland with bulrush or cattail, when some emergent plants become submerged ones.

Plants of genus *Typha*, such as broadleaf or common cattail *T. latifolia* or narrowleaf cattail *T. angustifolia* belong to “obligate wetland” species, meaning that they are always found in or near water. They generally grow in areas where the water depth does not exceed 2.6 feet (0.8 meters). However, it has also been reported growing in floating mats in slightly deeper water. One can expect that during a flood their upper parts with the flowering heads can give greater resistance to the flowing water than the lower parts.

In calculations we accept the suspension concentration value $c = 0.4$ as maximum, because this a maximal value for which Brinkman’s relation (3) holds true.

After the relation (23) we have

$$v(y) = \frac{g_x}{\eta_0} \int_0^y Y(\tilde{y})(h - \tilde{y})d\tilde{y}, \quad (28)$$

where

$$Y(y) = (1 - c)^{5/2}.$$

The concentration of suspension increases linearly from c_0 to c_1 in the lower part of the water, between the bottom ($y = 0$) and certain height $h_1 < h$, and next linearly decreases between h_1 and h , from c_1 to zero, it is

$$\begin{aligned} c &= \frac{c_1 - c_0}{h_1}y + c_0 \quad \text{for } 0 \leq y < h_1, \\ c &= \frac{c_1}{h - h_1}(h - y) \quad \text{for } h_1 < y \leq h. \end{aligned} \quad (29)$$

We write respectively

$$\begin{aligned} Y &= b - ay \quad \text{with } b = 1 - c_0 \\ \text{and } a &= \frac{c_{v1} - c_0}{h_1} \quad \text{for } 0 \leq y < h_1, \\ Y &= b + ay \quad \text{with } b = 1 - \frac{c_1}{h - h_1}h \\ \text{and } a &= \frac{c_1}{h - h_1} \quad \text{for } h_1 < y \leq h. \end{aligned} \quad (30)$$

Further, we have the integrals:

– for $0 \leq y < h_1$

$$\begin{aligned} \int_0^{h_1} Y^{5/2}dy &= \frac{1}{a \cdot \frac{7}{2}} (Y^{7/2} - b^{7/2}) \\ \int_0^{h_1} Y^{5/2}ydy &= \frac{1}{a \cdot \frac{9}{2}} (Y^{9/2} - b^{9/2}) \\ &+ \frac{B}{a^2 \cdot \frac{7}{2}} (Y^{7/2} - b^{7/2}), \end{aligned} \quad (31)$$

– for $h_1 < y \leq h$

$$\int_0^{h_1} Y^{5/2} dy = \frac{1}{a \cdot \frac{7}{2}} \left\{ Y^{7/2} - (b + ah_1)^{7/2} \right\}$$

$$\int_0^{h_1} Y^{5/2} y dy = \frac{1}{a \cdot \frac{9}{2}} \left\{ Y^{9/2} - (b + ah_1)^{9/2} \right\} \quad (32)$$

$$+ \frac{b}{a^2 \cdot \frac{7}{2}} \left\{ Y^{7/2} - (b + ah_1)^{7/2} \right\}.$$

The velocity distribution in the direction y for two suspension distributions given in Fig. 4 is presented in Fig. 5. The more dense suspension at the lower part ($y < 1/2$) of the fluid, the less is the velocity in the upper part ($y > 1/2$) of the fluid. Despite of the discontinuity of density gradients at $y = 1/2$ the velocity distribution is smooth together with the derivatives. It is conform with Eq. (23) as

$$\frac{\partial v}{\partial y} = \frac{g_x}{\eta_0} (1 - c)^{5/2} (h - y)$$

and the concentration c is a continuous function of y .

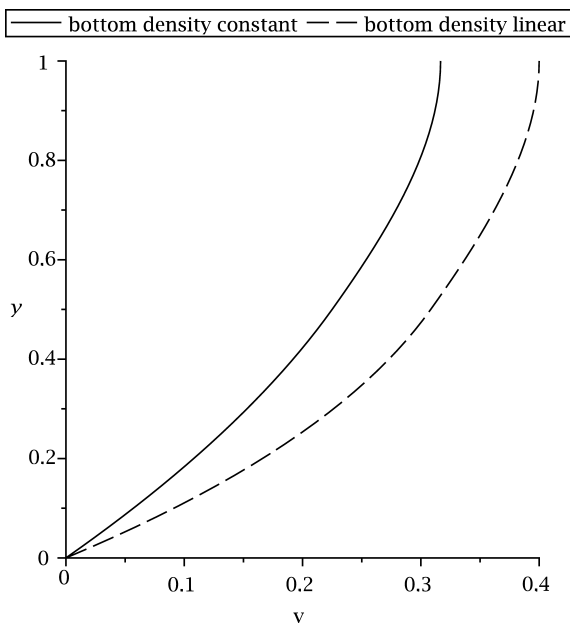


Fig. 5. The velocity distribution in the direction y for two suspension distributions given in Fig. 4. We accept $h = 1$, $g_x = 1$ and $\eta_0 = 1$. The solid line corresponds to the distribution at left (a) in Fig. 4, it is to the value $c_0 = 0.4$, while the dashed line corresponds to the distribution at right (b) in Fig. 4, where $c_0 = 0$. The velocity unit is the same as in Fig. 3

6. Fluid with two components of different viscosity

Consider flow of the fluid composed of two different immiscible parts: upper part for $h_1 < y < h$ has the viscosity η_A and the lower part for $0 \leq y < h_1$ has the viscosity η_B . The densities of the both parts are the same, cf. Fig. 6.

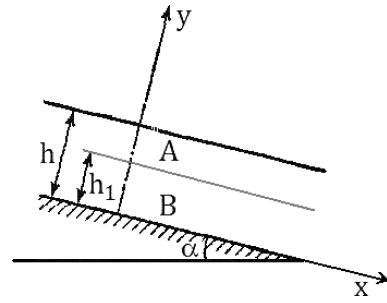


Fig. 6. Flow of the stream composed of two parts, A and B

We apply solutions (14) and (15) to both parts of the fluid. Namely, we have for the part A

$$\sigma_{xy}^{(A)} = \eta_A \frac{\partial v}{\partial y}, \quad (33)$$

$$\eta_A \frac{\partial v}{\partial y} = -g_x y + C_A$$

and for the part B

$$\sigma_{xy}^{(B)} = \eta_B \frac{\partial v}{\partial y}, \quad (34)$$

$$\eta_B \frac{\partial v}{\partial y} = -g_x y + C_B.$$

On the interface at $y = h_1$ we have equalities of the stresses

$$\sigma_{xy}^{(A)} = \sigma_{xy}^{(B)} \quad (35)$$

and the velocities

$$v^{(A)} = v^{(B)} \quad (36)$$

and at the bottom

$$v^{(B)} = 0. \quad (37)$$

Finally, we find for $0 \leq y < h_1$ (part B)

$$v = \frac{1}{\eta_B} g_x y \left(h - \frac{1}{2} y \right) \quad (38)$$

and for $h_1 < y < h$ (part A)

$$v = v_0^{(A)} + \frac{1}{\eta_A} g_x \left\{ h(y - h_1) - \frac{1}{2} (y^2 - h_1^2) \right\} \quad (39)$$

with

$$v_0^{(A)} = \frac{1}{\eta_B} g_x h_1 \left(h - \frac{1}{2} h_1 \right).$$

Examples of the dependence of velocity v upon the coordinate y are given in Fig. 7.

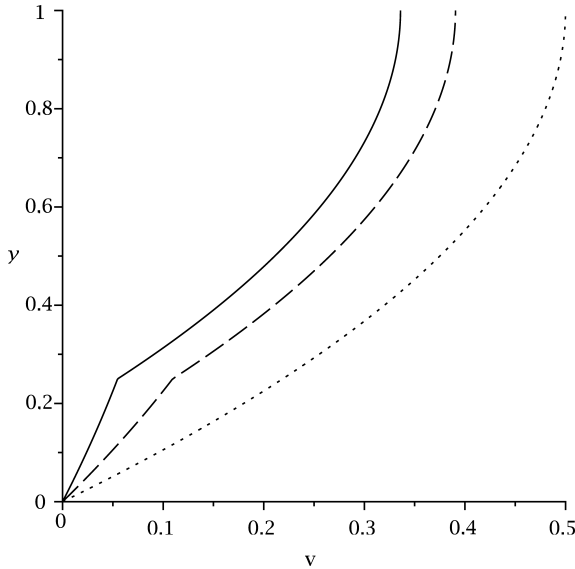


Fig. 7. Velocity v versus coordinate y for three cases: the whole stream has the viscosity η_A (dotted curve) and two component flow, with η_B at the bottom to η_A at the free surface. Example is given for: $\eta_A = 1$, $\eta_B = 2$, (dashed curve) and $\eta_B = 4$, (solid curve) with $g_x = 1$, $h = 1$, $h_1 = 1/4$. As the viscosity is not a continuous function of y at $y = 1/4$, the velocity derivative $\partial v/\partial y$ is not continuous in this point. The velocity unit is the same as in Fig. 3, but instead of η_0 the viscosity η_A is taken

7. Applicability of the results

The above results are obtained under the assumption of laminar flow. It is important to determine the ranges of parameters of the problem in which such flow can be developed. The Reynolds number is defined as the ratio of the inertial forces to the viscous forces.

$$Re \equiv \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho u^2 \ell^2}{\eta u \ell} \quad (40)$$

or

$$Re = \frac{\rho u \ell}{\eta} \quad (41)$$

Here ρ is the density of the fluid, u – its characteristic velocity, ℓ – its characteristic spatial dimension, and η – the (dynamical) viscosity. The number Re provides a convenient characteristics of the flow. For instance, the number Re is used to characterize different flow regimes, such as laminar or turbulent flow.

7.1. Transition and turbulent flow. For flow in a pipe of diameter D , experimental observations show that for fully developed flow, laminar flow occurs when $Re < 2100$ and turbulent flow occurs when $Re > 4000$. In the interval between 2300 and 4000, laminar and turbulent flows are possible and are called transition flows, depending on other factors, such as pipe roughness and flow uniformity, [75].

7.2. Reynold's number for open flow. The critical Reynolds number depends on geometry of the canal. For flow of liquid with a free surface the following Reynolds number should be taken, [75],

$$Re = 4 \cdot \frac{\rho u \ell}{\eta} \quad (42)$$

The characteristic velocity u is in our case the mean velocity \bar{v} of flow and the characteristic length ℓ it is – the depth h . For the constant viscosity, after (21) we have

$$\bar{v} = \frac{1}{\eta} g_x \frac{1}{h} \int_0^h \left(hy - \frac{1}{2} y^2 \right) dy$$

or

$$\bar{v} = \frac{1}{\eta} g_x \frac{1}{3} h^2 \quad (43)$$

Hence

$$Re = \frac{4}{3} \cdot \rho g_x \frac{h^3}{\eta^2} \quad (44)$$

Reynolds number Re calculated after the last formula versus the stream depth h for three different viscosities is shown in Fig. 8. Critical value of Reynolds number $Re = 2100$ is denoted by the horizontal dotted line. We see that the accessible stream depths for which the laminar flow can be developed are less than 6 cm. Such value of the channel depth is observed in experimental practice where vegetated stream beds are investigated, cf. Fig. 4 in [76].

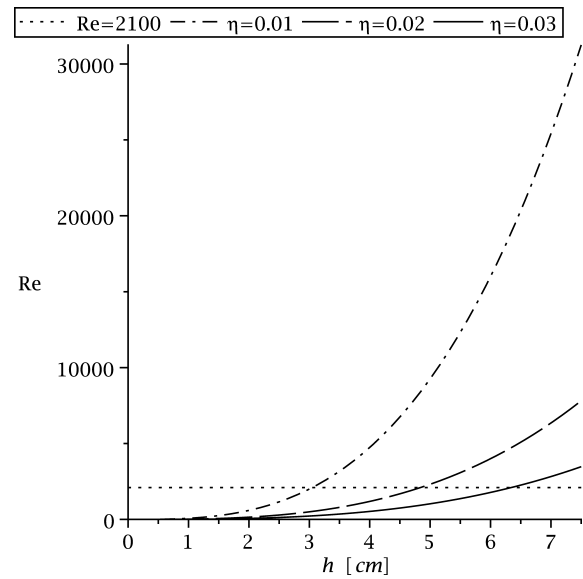


Fig. 8. Reynolds number Re versus the stream depth h for three cases of viscosity: $\eta = 0.01$, $\eta = 0.02$ and $\eta = 0.03$. Critical value of Reynolds number $Re = 2100$ is given by the horizontal dotted line. The figure is prepared for the example given at the end of Sec. 4, it is for: $\rho = 1 \text{ g/cm}^3$, $g_x = 0.05 \text{ g/(cm}^2 \text{ s}^2)$

8. Final remarks

We have studied Stokesian flow under the gravity force of the fluid containing suspensions with the various prescribed distributions of the concentration. In particular, we have analyzed the two-layer fluid flow on the plane, when the suspension modifies only the viscosity of the bottom layer of fluid. The suspension can model either the influence of the bottom plants or other obstacles. The influence of the contaminating suspension on the flow can also be modelled in this manner.

Above, we studied a case of stationary laminar flow with the different given distributions of suspension concentration. The laminarity idealization is suitable for the flow characteristics satisfying the constraint imposed by Reynolds' number $Re = 2100$. Such a limitation is widely observed in experimental hydrological practice, in which the influence of the stream bed vegetation is investigated.

As the Einsteinian theory of suspensions was built on the basis of macroscopic theory of liquids by Stokes, we believe that our results can be useful, at least, to the qualitative description of the influence of the various obstacles on the flow in different wide channels.

Acknowledgements. The authors gratefully acknowledge helpful suggestions from the unknown Referees.

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