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HOMOGENIZATION OF STRESS EQUATION OF MOTION IN LINEAR ELASTODYNAMICS

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Two methods of homogenization of elastic body with periodic heterogeneity based on a pure stress formulation of linear elastodynamics (cf [1 ÷ 4]) are presented. The methods are closely related to the two representations of the displacement field in terms of the stress field: the first derived from the geometrical relations and Hooke's law and the second obtained by means of the equation of motion. Both methods lead to the same homogenized form of the stress equation of motion, and the resulting homogenized coefficients are identical to those of a displacement homogenization procedure. Also, a theorem is proved in which it is shown that a mixed initial–boundary value problem for a homogenized medium can be characterized by a mean stress field only.

1. Introduction

The homogenization of a heterogenous elastic solid consists in replacing it by a homogeneous one physical properties of which are to some extent equivalent to those of the original solid. Thus within the frame of an initial–boundary value problem for the nonhomogeneous elastic solid, we replace the original initial–boundary value problem by a close one in which the elastic body is homogeneous.

An idea of homogenization is based on the extension of a domain of the homogenized solution in such a way that the solution can be represented by a small parameter series expansion. In the context of general equations of mathematical physics the concept of extension was proposed by Sandri [5], in sixties. Earlier, a homogenization procedure (based on averaging) was applied to dielectric and magnetic media composed of separated particles by Lorentz and and to conductors by Drude; both the authors obtained the electric and magnetic effective material coefficients which are included in basic text–books on electrodynamics (cf e.g. Suffczyński [6]). Also, a homogenization procedure in which microscopic properties of a solid composed of interacting Newtonian particles are replaced by macroscopic properties of an elastic continuum was proposed by Zorski [7].

Eimer [8,9] studied the behaviour of elastic materials with scattered cracks and obtained approximate bulk constitutive relations for an elastic material with random cracks. A number of homogenization procedures for periodic thermoelastic composites based on a nonstandard analysis were proposed by Woźniak [10 ÷ 12], Matysiak [13] and Mazur–Śniady [14].

The homogenization of a nonhomogeneous body can be also accomplished by using a two-scale asymptotic expansion method. For example, to study wave propagation in a periodically nonhomogeneous elastic body of period Y one can introduce Y–scale of the heterogeneity for material coefficients, and look for a homogenized wave length of which is much greater then Y. The two-scale method of homogenization based on the displacement equation of motion was discussed by Duvaut [15], Bensoussan, Lions and Papanicolaou [16], Sanchez–Palencia [17], and Bakhvalov and Panasenko [18]. For periodically nonhomogeneous thermoelastic, thermodiffusive–elastic, piezoelectro–thermoelastic bodies the two-scale homogenization procedures were proposed by Francfort, Lewiński, Telega, Bytner, Gambin and Galka (cf [19 ÷ 23]). Dual aspects of homogenization are discussed by Telega [24].

In the present[1] paper a two-scale homogenization method is used to homogenize a periodically nonhomogeneous elastic body in which pure stress waves are observed. By using a complete characterization of linear elastodynamics in terms of stresses only (cf Ignaczak [1,2]) we show that in contrast with the homogenized displacement procedure in which a homogenized displacement field $\mathbf{u}^{(0)}$ coincides with the first term of asymptotic expansion of a field $\mathbf{u}$ when the elementary cell tends to zero ($\varepsilon \to 0$), the homogenized stress field is identical to the mean stress $<\boldsymbol{\sigma}^{(0)}>$ over the cell, where $\boldsymbol{\sigma}^{(0)}$ is the first term of asymptotic expansion of a stress field $\boldsymbol{\sigma}$.

The paper is organized as follows. Section 2 presents a pure stress initial–boundary value problem of linear nonhomogeneous anisotropic elastodynamics with conventional and nonconventional tensorial initial data. In Section 3 the density ρ and elastic compliance K are assumed to be periodic functions of the space variable $\mathbf{x}$ period of which depends on a small parameter $\varepsilon > 0$. Such a nonhomogeneous body is then homogenized by the two methods in Sections 4 and 5. In the first method we expand a stress field satisfying Eq (2.1) into an asymptotic series of powers of ε; the associated displacement field now plays a role analogous to that of a potential in the homogenization of a body transmitting electromagnetic waves (cf [17,25]). In the second method (indirect approach) we use a definition of the displacement field based on the equation of motion. Such a displacement is next developed asymptotically in a way similar to the method of homogenization of the displacement problem of elasticity (cf [15,17]). Both

[1]The article is an extended version of the paper presented at a meeting of the Polish Society of Theoretical and Applied Mechanics – Łódź Branch, which was held on Dec.19, 1991 in the Łódź Polytechnic. The paper won an award from the organizers of the meeting.

methods lead to the same homogenized form of the stress equations of motion, and the resulting homogenized coefficients are identical to those of a displacement homogenization procedure, (cf [15,17]).

In Section 6 it is shown that a mixed initial boundary value problem for a homogenized medium can be characterized by the mean stress $<\sigma^{(0)}>$ only.

2. Field equations and initial boundary value problem

Let a nonhomogeneous anisotropic linear elastic body, occupying a three dimensional region B, be subject to a dynamic motion. Let the body forces be absent. Then the stress tensor σ_{ij} is a solution to the following initial–boundary value problem (cf Ignaczak, Gurtin [1 ÷ 3]).

Find the field $\sigma_{ij} = \sigma_{ij}(\mathbf{x},t)$ on $\bar{B} \times [0,\infty)$ that satisfies the equation

$$\frac{1}{2}\Big[\frac{\partial}{\partial x_j}\Big(\rho^{-1}\frac{\partial \sigma_{ik}}{\partial x_k}\Big) + \frac{\partial}{\partial x_i}\Big(\rho^{-1}\frac{\partial \sigma_{jk}}{\partial x_k}\Big)\Big] = K_{ijkl}\ddot{\sigma}_{kl} \qquad (\mathbf{x},t) \in B \times (0,\infty) \tag{2.1}$$

the initial conditions

$$\sigma_{ij}(\mathbf{x},0) = \sigma^{p}_{ij}(\mathbf{x}) \qquad \dot{\sigma}_{ij}(\mathbf{x},0) = \dot{\sigma}^{p}_{ij}(\mathbf{x}) \qquad \mathbf{x} \in B \tag{2.2}$$

and the boundary conditions

$$\begin{aligned} &\rho^{-1}t * \frac{\partial \sigma_{ik}}{\partial x_k}(\mathbf{x},t) = U_i(\mathbf{x},t) && \text{on} \quad \partial B_U \times (0,\infty) \\ &\sigma_{ij}n_j = F_i(\mathbf{x},t) && \text{on} \quad \partial B_F \times (0,\infty) \end{aligned} \tag{2.3}$$

Here $\rho = \rho(\mathbf{x})$ and $K_{ijkl} = K_{ijkl}(\mathbf{x})$ denote the density and the elastic compliance of the body, respectively; the upper dot ($\dot{}$) stands for the time derivative and the summation convention over repeated indices is implied; moreover, the star $(*)$ means convolution on the time axis, e.g.

$$t * f(\mathbf{x},t) = \int_0^t (t-\tau) f(\mathbf{x},\tau)\, d\tau$$

where f denotes a function of $\mathbf{x}$ and t. The fields σ^{p}_{ij}, $\dot{\sigma}^{p}_{ij}$, U_i and F_i are given functions, while ∂B_U and ∂B_F are complementary subsets of the boundary ∂B of B. The fields $\sigma^{p}_{ij}(\mathbf{x})$ and $\dot{\sigma}^{p}_{ij}(\mathbf{x})$ are determined by the initial displacement field $u_i(\boldsymbol{x},0) = u_i^p(\boldsymbol{x})$ and the initial velocity field $\dot{u}_i(\boldsymbol{x},0) = \dot{u}_i^p(\boldsymbol{x})$ through the relations

$$\begin{aligned} \sigma_{ij}^p(\boldsymbol{x}) &= C_{ijkl}(\boldsymbol{x})\frac{\partial u_k^p(\boldsymbol{x})}{\partial x_l} \\ & \qquad\qquad\qquad \text{on} \quad B \\ \dot{\sigma}_{ij}^p(\boldsymbol{x}) &= C_{ijkl}(\boldsymbol{x})\frac{\partial \dot{u}_k^p(\boldsymbol{x})}{\partial x_l} \end{aligned} \tag{2.4}$$

where C_{ijkl} is the elasticity tensor, i.e.

$$C_{ijkl}K_{klmn} = \delta_{i(m}\delta_{jn)} \qquad \text{on} \quad B$$

and δ_{ij} stands for the Kronecker symbol. Moreover, the field $U_i(\boldsymbol{x},t)$ is represented by the boundary displacement $\hat{u}_i(\boldsymbol{x},t)$ and the initial fields $u_i^p(\boldsymbol{x})$ and $\dot{u}_i^p(\boldsymbol{x})$ through the relation

$$U_i(\boldsymbol{x},t) = \hat{u}_i(\boldsymbol{x},t) - t\dot{u}_i^p(\boldsymbol{x}) - u_i^p(\boldsymbol{x}) \qquad \text{on} \quad \partial B_U \times [0,\infty) \tag{2.5}$$

Finally, the density ρ and the compliance K_{ijkl} satisfy the inequalities

$$\rho > 0 \qquad K_{ijkl}\sigma_{ij}\sigma_{kl} > 0 \qquad \forall \sigma_{ij} \in E_s^3 \qquad \boldsymbol{x} \in B \tag{2.6}$$

Remark 1

The problem (2.1) ÷ (2.6) is equivalent to a conventional mixed initial–boundary value problem of linear nonhomogeneous anisotropic elastodynamics in which an initial state of the body is described by the fields u_i^p and $\dot{u}_i^p$, the field $\hat{u}_i$ is given on $\partial B_U \times (0,\infty)$ and the traction F_i is prescribed on $\partial B_F \times (0,\infty)$. The boundary condition $(2.3)_1$ corresponds therefore to a displacement condition expressed in terms of stresses (cf Gurtin [3], p.222).

Remark 2

In the linear elastodynamics a stress tensor is characterized by a single tensorial equation (Eq (2.1)) while in the case of elastostatics a stress tensor satisfies, apart from 6 stress equations of compatibility also 3 equations of equilibrium (9 scalar equations totally).

Remark 3

Applying the double curl operator to Eq (2.1) we get

$$\varepsilon_{ipq}\varepsilon_{jrs}\frac{\partial^2}{\partial x_p \partial x_r}\Big(K_{qskl}\ddot{\sigma}_{kl}\Big) = 0 \qquad \text{on} \quad B \times (0,\infty) \tag{2.7}$$

where ε_{ijk} denotes the 3–dimensional alternator.

Integrating Eq (2.7) twice with respect to time we find

$$\varepsilon_{ipq}\varepsilon_{jrs}\frac{\partial^2}{\partial x_p \partial x_r}\Big[K_{qskl}(\sigma_{kl} - t\dot{\sigma}^p_{kl} - \sigma^p_{kl})\Big] = 0 \qquad \text{on} \qquad B \times [0,\infty) \tag{2.8}$$

Therefore, by virtue of (2.4), the tensorial equation (2.1) implies validity of the compatibility condition of linear elastodynamics for every $(\boldsymbol{x},t) \in B \times [0,\infty)$ (cf [3], p.40).

Remark 4

Since the problem (2.1) ÷ (2.6) is equivalent to a convential mixed problem of linear elastodynamics, there exists only one field σ_{ij} satisfying Eqs (2.1) ÷ (2.6). It was shown by Ignaczak [2] that if σ^p_{ij} and $\dot{\sigma}^p_{ij}$ are arbitrary second order symmetric tensor fields which in general do not satisfy the relations (2.4), and if $\partial B = \partial B_F$, then the associated pure stress problem has also the one and only one solution. Of course, the use of the arbitrary fields σ^p_{ij} and $\dot{\sigma}^p_{ij}$ means that the associated stress problem covers a class of stress fields that do not satisfy the compatibility conditions (2.8). Those non–compatible stress formulations are often called nonconventional formulations of linear elastodynamics.

3. Periodically nonhomogeneous elastic body

The initial–boundary value problem (Eqs (2.1) ÷ (2.6)) is parametrized with the aid of a small parameter $\varepsilon > 0$ in the following way. The fields $\rho(\boldsymbol{x})$ and $K_{ijkl}(\boldsymbol{x})$ are replaced by the fields

$$\begin{aligned} \rho^\varepsilon(\boldsymbol{x}) &\stackrel{\text{df}}{=} \rho\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \\ K^\varepsilon_{ijkl}(\boldsymbol{x}) &\stackrel{\text{df}}{=} K^\varepsilon_{ijkl}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big) \end{aligned} \qquad \boldsymbol{x} \in B \tag{3.1}$$

respectively, which are $\varepsilon \boldsymbol{Y}$ – periodic, i.e.

$$\begin{aligned} \rho^\varepsilon(\boldsymbol{x} + \varepsilon\boldsymbol{Y}) &= \rho^\varepsilon(\boldsymbol{x}) \\ K^\varepsilon_{ijkl}(\boldsymbol{x} + \varepsilon\boldsymbol{Y}) &= K^\varepsilon_{ijkl}(\boldsymbol{x}) \end{aligned}$$

where $\boldsymbol{Y}$ is a given vector. The vector $\boldsymbol{Y}$ describes an elementary cell of the body which is described by

$$Y = (0,Y_1) \times (0,Y_2) \times (0,Y_3) \tag{3.2}$$

The vector $Y = (Y_1, Y_2, Y_3)$ belongs to the first octant of the y–coordinate frame.
The mean value of a function $f(x,y,t)$ over Y is denoted below by

$$< f(x,y,t) >= \frac{1}{|Y|} \int_Y f(x,y,t)\, dy \tag{3.3}$$

As far as the remaining data of the initial–boundary value problem (2.1) ÷ (2.6) are concerned, we assume that they are independent of ε, e.g. the initial data of the problem are assumed to be ε - independent. Clearly, with such a parametrization a solution of the parametrized problem also depends on ε, and in the sequel it is denoted by $\sigma^{\varepsilon}_{ij}(x,t)$.

4. Direct homogenization of the stress equation

According to the 2–scale expansion method of homogenization, we assume that the field $\sigma^{\varepsilon}_{ij}$ can be represented by the series (cf e.g. [15,17])

$$\sigma^{\varepsilon}_{ij}(x,t) = \sum_{k=0}^{\infty} \varepsilon^k \sigma^{(k)}_{ij}(x,y,t) \tag{4.1}$$

where

$$y = \frac{x}{\varepsilon} \tag{4.2}$$

and the cofficients $\sigma^{(k)}_{ij}$ are periodic in y, i.e.

$$\sigma^{(k)}_{ij}(x, y + Y, t) = \sigma^{(k)}_{ij}(x,y,t)$$

The partial derivative $\partial/\partial x_i$ of a function $f(x,y,t)$ with $y = x/\varepsilon$ means that

$$\frac{\partial}{\partial x_i} \longrightarrow \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i} \tag{4.3}$$

Substituting the expansion (4.1) into the field equation (2.1) and using Eq (4.3) we get

$$\begin{aligned}
&\frac{1}{2}\Big\{\Big(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\Big)\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_k}\Big)(\sigma^{(0)}_{ik} + \varepsilon\sigma^{(1)}_{ik} + \varepsilon^2\sigma^{(2)}_{ik} + \ldots)\Big] + \\
&+\Big(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\Big)\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_k}\Big)(\sigma^{(0)}_{jk} + \varepsilon\sigma^{(1)}_{jk} + \varepsilon^2\sigma^{(2)}_{jk} + \ldots)\Big]\Big\} = \\
&= K_{ijkl}(y)(\ddot{\sigma}^{(0)}_{kl} + \varepsilon\ddot{\sigma}^{(1)}_{kl} + \varepsilon^2\ddot{\sigma}^{(2)}_{kl} + \ldots)
\end{aligned} \tag{4.4}$$

where

$$\rho^{-1} = \frac{1}{\rho(y)}$$

Now, equating coefficients of like powers of ε one obtaines the set of equations

ε^{-2}

$$\frac{\partial}{\partial y_j}\Big(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{ik}^{(0)}\Big) + \frac{\partial}{\partial y_i}\Big(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{jk}^{(0)}\Big) = 0 \tag{4.5}$$

ε^{-1}

$$\begin{aligned}&\frac{\partial}{\partial x_j}\Big(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{ik}^{(0)}\Big) + \frac{\partial}{\partial y_j}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{ik}^{(0)} + \frac{\partial}{\partial y_k}\sigma_{ik}^{(1)}\Big)\Big] + \\ &+\frac{\partial}{\partial x_i}\Big(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{jk}^{(0)}\Big) + \frac{\partial}{\partial y_i}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{jk}^{(0)} + \frac{\partial}{\partial y_k}\sigma_{jk}^{(1)}\Big)\Big] = 0\end{aligned} \tag{4.6}$$

ε^{0}

$$\begin{aligned}&\frac{1}{2}\Big\{\frac{\partial}{\partial x_j}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{ik}^{(0)} + \frac{\partial}{\partial y_k}\sigma_{ik}^{(1)}\Big)\Big] + \frac{\partial}{\partial y_j}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{ik}^{(1)} + \frac{\partial}{\partial y_k}\sigma_{ik}^{(2)}\Big)\Big] + \\ &+\frac{\partial}{\partial x_i}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{jk}^{(0)} + \frac{\partial}{\partial y_k}\sigma_{jk}^{(1)}\Big)\Big] + \frac{\partial}{\partial y_i}\Big[\rho^{-1}\Big(\frac{\partial}{\partial x_k}\sigma_{jk}^{(1)} + \frac{\partial}{\partial y_k}\sigma_{jk}^{(2)}\Big)\Big]\Big\} = \\ &= K_{ijkl}\ddot{\sigma}_{kl}^{(0)}\end{aligned} \tag{4.7}$$

Next, applying the operator

$$\varepsilon_{ipq}\varepsilon_{jrs}\Big(\frac{\partial}{\partial x_p} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_p}\Big)\Big(\frac{\partial}{\partial x_r} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_r}\Big)$$

to both sides of Eq (4.4) and using the compatibility conditions at the initial moment $t = 0$

$$\begin{aligned}&\varepsilon_{ipq}\varepsilon_{jrs}\Big(\frac{\partial}{\partial x_p} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_p}\Big)\Big(\frac{\partial}{\partial x_r} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_r}\Big)\Big[K_{qskl}\sigma_{kl}^{\varepsilon}(x, y, 0)\Big] = 0 \\ &\varepsilon_{ipq}\varepsilon_{jrs}\Big(\frac{\partial}{\partial x_p} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_p}\Big)\Big(\frac{\partial}{\partial x_r} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_r}\Big)\Big[K_{qskl}\dot{\sigma}_{kl}^{\varepsilon}(x, y, 0)\Big] = 0\end{aligned} \tag{4.8}$$

we get [2]

$$\varepsilon_{ipq}\varepsilon_{jrs}\Big(\frac{\partial}{\partial x_p} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_p}\Big)\Big(\frac{\partial}{\partial x_r} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_r}\Big)\cdot$$

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[2]The conditions (4.8) are satisfied, by virtue of (2.4), if the following asymptotic relations are met

$$\begin{aligned}&\sigma_{ij}^{\varepsilon}(x, y, 0) \to \sigma_{ij}(x, 0) = \sigma_{ij}^{p}(x) \qquad \text{as} \quad \varepsilon \to 0 \\ &\dot{\sigma}_{ij}^{\varepsilon}(x, y, 0) \to \dot{\sigma}_{ij}(x, 0) = \dot{\sigma}_{ij}^{p}(x) \qquad \text{as} \quad \varepsilon \to 0\end{aligned}$$

$$\cdot\Big[K_{qskl}(\sigma^{(0)}_{kl} + \varepsilon\sigma^{(1)}_{kl} + \varepsilon^2\sigma^{(2)}_{kl} + ...)\Big] = 0 \tag{4.9}$$

Considering the coefficients of ε^k $(k = -2, -1, ...)$ in Eq (4.9) yields

ε^{-2}

$$\varepsilon_{ipq}\varepsilon_{jrs}\frac{\partial^2}{\partial y_p \partial y_r}\Big(K_{qskl}\sigma^{(0)}_{kl}\Big) = 0 \tag{4.10}$$

ε^{-1}

$$\varepsilon_{ipq}\varepsilon_{jrs}\Big[\Big(\frac{\partial^2}{\partial x_r \partial y_p} + \frac{\partial^2}{\partial x_p \partial y_r}\Big)\Big(K_{qskl}\sigma^{(0)}_{kl}\Big) + \frac{\partial^2}{\partial y_p \partial y_r}\Big(K_{qskl}\sigma^{(1)}_{kl}\Big)\Big] = 0 \tag{4.11}$$

ε^{0}

$$\varepsilon_{ipq}\varepsilon_{jrs}\Big[\frac{\partial^2}{\partial x_p \partial x_r}\Big(K_{qskl}\sigma^{(0)}_{kl}\Big) + \Big(\frac{\partial^2}{\partial x_r \partial y_p} + \frac{\partial^2}{\partial x_p \partial y_r}\Big)\Big(K_{qskl}\sigma^{(1)}_{kl}\Big) + \tag{4.12}$$

$$+\frac{\partial^2}{\partial y_p \partial y_r}\Big(K_{qskl}\sigma^{(2)}_{kl}\Big)\Big] = 0$$

After integration of Eq (4.5) we get (cf Eq (A8) in Appendix A)

$$\frac{1}{\rho}\frac{\partial\sigma^{(0)}_{ik}}{\partial y_k} = b_i(\boldsymbol{x}, t)$$

where b_i is an unknown function depending on $\boldsymbol{x}$ and t only. Integrating both sides of this equation over the region of elementary cell Y, using the divergence theorem and periodicity of $\sigma^{(0)}_{ik}$ with regard to $\boldsymbol{y}$ we find that

$$b_i(\boldsymbol{x}, t) = 0 \qquad \text{for} \qquad (\boldsymbol{x}, t) \in B \times (0, \infty)$$

Therefore

$$\frac{\partial\sigma^{(0)}_{ik}}{\partial y_k} = 0 \qquad \text{for} \qquad (\boldsymbol{x}, t) \in B \times (0, \infty) \tag{4.13}$$

Substituting this result into (4.6) we get

$$\frac{\partial}{\partial y_j}\Big[\rho^{-1}\Big(\frac{\partial\sigma^{(0)}_{ik}}{\partial x_k} + \frac{\partial\sigma^{(1)}_{ik}}{\partial y_k}\Big)\Big] + \frac{\partial}{\partial y_i}\Big[\rho^{-1}\Big(\frac{\partial\sigma^{(0)}_{jk}}{\partial x_k} + \frac{\partial\sigma^{(1)}_{jk}}{\partial y_k}\Big)\Big] = 0$$

Clearly, the last equation has the same form as Eq (4.5) if the terms of type $(\rho^{-1}\partial\sigma^{(0)}_{ik}/\partial y_k)$ are identified with the expressions of type $[\rho^{-1}(\partial\sigma^{(0)}_{ik}/\partial x_k + \partial\sigma^{(1)}_{ik}/\partial y_k)]$; therefore, by analogy with (A8) we get

$$\frac{1}{\rho}\Big(\frac{\partial\sigma^{(0)}_{ik}}{\partial x_k} + \frac{\partial\sigma^{(1)}_{ik}}{\partial y_k}\Big) = B_i(\boldsymbol{x}, t) \qquad (\boldsymbol{x}, t) \in B \times (0, \infty) \tag{4.14}$$

where $B_i(x, t)$ is an unknown function depending on x and t only. Hence, we obtain

$$\frac{\partial \sigma_{ik}^{(0)}}{\partial x_k} + \frac{\partial \sigma_{ik}^{(1)}}{\partial y_k} = \rho(y) B_i(x, t) \tag{4.14)'$$

Averaging both sides of (4.14)' over the cell Y and using periodicity of $\sigma_{ik}^{(1)}$ with regard to y we find

$$\frac{\partial}{\partial x_k} < \sigma_{ik}^{(0)} > = < \rho > B_i(x, t) \tag{4.15}$$

where the operator $< >$ is defined by Eq (3.3). The last result shows that B_i represents an acceleration field corresponding to the average stress field $< \sigma_{ik}^{(0)} >$

Next, substituting (4.14) into (4.7) yields

$$\begin{gathered} \frac{1}{2} \Big\{ \frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i} + \frac{\partial}{\partial y_j} \Big[\rho^{-1} \Big(\frac{\partial \sigma_{ik}^{(1)}}{\partial x_k} + \frac{\partial \sigma_{ik}^{(2)}}{\partial y_k} \Big) \Big] + \\ + \frac{\partial}{\partial y_i} \Big[\rho^{-1} \Big(\frac{\partial \sigma_{jk}^{(1)}}{\partial x_k} + \frac{\partial \sigma_{jk}^{(2)}}{\partial y_k} \Big) \Big] \Big\} = K_{ijkl} \ddot{\sigma}_{kl}^{(0)} \end{gathered} \tag{4.16}$$

Averaging this equation over the cell Y, using the divergence theorem and the periodicity of $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ with regard to y lead to the result

$$\frac{1}{2} \Big(\frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i} \Big) = < K_{ijkl} \ddot{\sigma}_{kl}^{(0)} > \tag{4.17}$$

Hence, and according to Eq (4.15) we get

$$\frac{1}{2} \Big[\frac{\partial^2}{\partial x_j \partial x_k} < \sigma_{ik}^{(0)} > + \frac{\partial^2}{\partial x_i \partial x_k} < \sigma_{jk}^{(0)} > \Big] = < \rho > < K_{ijkl} \ddot{\sigma}_{kl}^{(0)} > \tag{4.18}$$

Now, we use the relation (B8) derived in Appendix B

$$< \sigma_{ij}^{(0)} > = C_{ijkl}^H < K_{klpq} \sigma_{pq}^{(0)} > \tag{4.19}$$

where C_{ijkl}^H denotes the homogenized elasticity tensor

$$C_{ijkl}^H = < C_{ijkl} + C_{klmn} \frac{\partial \chi_{mij}}{\partial y_n} > \tag{4.20}$$

in which χ_{mpq} satisfies the equation

$$\frac{\partial}{\partial y_j} \Big(C_{ijpq} + C_{ijmn} \frac{\partial \chi_{mpq}}{\partial y_n} \Big) = 0 \qquad \text{on} \quad Y \tag{4.21}$$

and the appropriate boundary conditions on ∂Y.

Assuming that C^H_{ijkl} is invertible, and denoting its inverse by K^H_{ijkl}, from (4.19) we obtain

$$< K_{ijkl}\sigma^{(0)}_{kl} >= K^H_{ijkl} < \sigma^{(0)}_{kl} > \tag{4.22}$$

Clearly, we have[3]

$$K^H_{ijmn}C^H_{mnlk} = C^H_{ijmn}K^H_{mnlk} = \delta_{i(l}\delta_{jk)} \tag{4.23}$$

Finally, substituting Eq (4.22) into right–hand side of Eq (4.18) yields

$$\frac{1}{2}\Big[\frac{\partial^2}{\partial x_j \partial x_k} < \sigma^{(0)}_{ik} > + \frac{\partial^2}{\partial x_i \partial x_k} < \sigma^{(0)}_{jk} >\Big] =< \rho > K^H_{ijkl} < \ddot{\sigma}^{(0)}_{kl} > \quad \text{on} \quad B \times (0,\infty) \tag{4.24}$$

To this equation the following initial and boundary conditions are adjoined

$$< \sigma^{(0)}_{ij} > (x,0) = \sigma^p_{ij}(x) \qquad < \dot{\sigma}^{(0)}_{ij} > (x,0) = \dot{\sigma}^p_{ij}(x) \qquad \text{on} B \tag{4.25}$$

$$\begin{cases} < \rho >^{-1} t * \frac{\partial}{\partial x_k} < \sigma^{(0)}_{ik} >= U_i(x,t) & \text{on} \quad \partial B_U \times (0,\infty) \\ < \sigma^{(0)}_{ij} > n_j = F_i(x,t) & \text{on} \quad \partial B_F \times (0,\infty) \end{cases} \tag{4.26}$$

The problem of finding an average stress field $< \sigma_{ij} >$ satisfying Eqs (4.24) ÷ (4.26) will be called a stress initial–boundary value problem of homogenized elastodynamics.

Remark

Homogenization procedure developed in this section can be also applied to the homogenization of the stress equations of elastostatics (Eqs (B1) and (B2) occur also in a statical problem).

5. Indirect homogenization of the stress equation

Assume that the fields $\sigma_{ij}(x,t)$, $u^p_i(x)$ and $\dot{u}^p_i(x)$ in the formulation (2.1) ÷ (2.6) are given, and define the function

$$u_i = u_i(x,t) = t * \Big(\rho^{-1}\frac{\partial \sigma_{ik}}{\partial x_k}\Big) + u^p_i(x) + t\dot{u}^p_i(x) \tag{5.1}$$

[3]The relations (4.23) are similar to those of the classical elasticity (cf Walpole [26] and Mehrabadi and Cowin [27]).

where

$$\frac{1}{2}\Big(\frac{\partial u_i^p}{\partial x_j}+\frac{\partial u_j^p}{\partial x_i}\Big)=K_{ijkl}\sigma_{kl}^p \tag{5.2}$$

$$\frac{1}{2}\Big(\frac{\partial \dot{u}_i^p}{\partial x_j}+\frac{\partial \dot{u}_j^p}{\partial x_i}\Big)=K_{ijkl}\dot{\sigma}_{kl}^p \tag{5.3}$$

(compare the initial conditions (2.2) and (2.4)).

Clearly, Eq (5.1) determines a displacement field obtained from the equation of motion by double integration with regard to time. Therefore, differentiating (5.1) twice with regard to time we get

$$\frac{\partial \sigma_{ik}}{\partial x_k}=\rho\ddot{u}_i \tag{5.4}$$

By virtue of Eqs (2.1) and (5.4) we obtain

$$\frac{1}{2}\Big(\frac{\partial \ddot{u}_i}{\partial x_j}+\frac{\partial \ddot{u}_j}{\partial x_i}\Big)=K_{ijkl}\ddot{\sigma}_{kl} \tag{5.5}$$

Integrating Eq (5.5) twice with regard to time and using (5.2) $\div$ (5.3) we get

$$\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)=K_{ijkl}\sigma_{kl} \tag{5.6}$$

or

$$\sigma_{ij}=C_{ijkl}\frac{\partial u_k}{\partial x_l} \tag{5.7}$$

Substituting Eq (5.7) into Eq (5.4) we arrive at the displacement equation of motion

$$\frac{\partial}{\partial x_j}\Big(C_{ijkl}\frac{\partial u_k}{\partial x_l}\Big)=\rho\ddot{u}_i \tag{5.8}$$

In the following we show how to apply Eq (5.8) for obtaining the homogenized stress problem Eqs (4.24) $\div$ (4.26). We expand u_i in the asymptotic series, (cf e.g. [15])

$$u_i^\varepsilon=\sum_{k=0}^{\infty}\varepsilon^k u_i^{(k)}(x,y,t) \tag{5.9}$$

Substituting Eq (5.9) into Eq (5.8) and following the procedure given in Section 4, we get

$$\begin{gathered}\Big(\frac{\partial}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\Big)\Big[C_{ijkl}(y)\Big(\frac{\partial}{\partial x_l}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_l}\Big)\cdot\\ \cdot(u_k^{(0)}+\varepsilon u_k^{(1)}+\varepsilon^2u_k^{(2)}+\ldots)\Big]=\rho(\ddot{u}_i^{(0)}+\varepsilon\ddot{u}_i^{(1)}+\varepsilon^2\ddot{u}_i^{(2)}+\ldots)\end{gathered} \tag{5.10}$$

Considering the coefficient ε^k ($k = -2, -1, ...$) in Eq (5.10) yields

(i) ε^{-2}

$$\frac{\partial}{\partial y_j}\Big(C_{ijkl}\frac{\partial u_k^{(0)}}{\partial y_l}\Big) = 0 \tag{5.11}$$

Eq (5.11), the positive definiteness of C_{ijkl} and the periodic boundary condition on ∂Y imply that $u_k^{(0)}$ cannot depend on y, i.e.

$$u_k^{(0)} = u_k^{(0)}(x,t) \tag{5.12}$$

Using this result we have

(ii) ε^{-1}

$$\frac{\partial}{\partial y_j}\Big[C_{ijkl}\Big(\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l}\Big)\Big] = 0 \tag{5.13}$$

This equation is satisfied by the function

$$u_k^{(1)} = \chi_{krs}(y)\frac{\partial u_r^{(0)}}{\partial x_s} + w_k(x) \tag{5.14}$$

where χ_{krs} satisfies equation (cf Eq (4.21))

$$\frac{\partial}{\partial y_j}\Big(C_{ijrs} + C_{ijkl}\frac{\partial \chi_{krs}}{\partial y_l}\Big) = 0 \tag{5.15}$$

and w_k is an arbitrary function of x (and eventually of t).

(iii) ε^{0}

$$\frac{\partial}{\partial x_j}\Big[C_{ijkl}\Big(\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l}\Big)\Big] + \frac{\partial}{\partial y_j}\Big[C_{ijkl}\Big(\frac{\partial u_k^{(1)}}{\partial x_l} + \frac{\partial u_k^{(2)}}{\partial y_l}\Big)\Big] = \rho \ddot{u}_i^{(0)} \tag{5.16}$$

Averaging Eq (5.16) over the cell Y gives

$$\frac{\partial}{\partial x_j} < C_{ijkl}\Big(\frac{\partial u_k^{(0)}}{\partial x_l} + \frac{\partial u_k^{(1)}}{\partial y_l}\Big)> = <\rho> \ddot{u}_i^{(0)} \tag{5.17}$$

Hence, using Eqs (5.12) and (5.14) we obtain

$$\frac{\partial}{\partial x_j}\Big(<C_{ijrs} + C_{ijkl}\frac{\partial \chi_{krs}}{\partial y_l}>\frac{\partial u_r^{(0)}}{\partial x_s}\Big) = <\rho> \ddot{u}_i^{(0)} \tag{5.18}$$

or

$$C_{ijkl}^{H}\frac{\partial^2 u_k^{(0)}}{\partial x_j \partial x_l} = <\rho> \ddot{u}_i^{(0)} \tag{5.19}$$

where (cf (4.20))

$$C^H_{ijkl} = < C_{ijkl} + C_{ijmn} \frac{\partial \chi_{mkl}}{\partial y_n} > \tag{5.20}$$

On the other hand, replacing in Eq (5.7) σ_{ij} by σ^ε_{ij}, (cf Eq (4.1)), and u_i by u^ε_i, (cf Eq (5.9)); and equating the coefficients of ε^0 on both sides we find that

$$\sigma^{(0)}_{ij} = C_{ijkl}\Big(\frac{\partial u^{(0)}_k}{\partial x_l} + \frac{\partial u^{(1)}_k}{\partial y_l}\Big) \tag{5.21}$$

or according to Eq (5.14) that

$$\sigma^{(0)}_{ij} = \Big(C_{ijkl} + C_{ijmn}\frac{\partial \chi_{mkl}}{\partial y_n}\Big)\frac{\partial u^{(0)}_k}{\partial x_l} \tag{5.22}$$

Next taking the average of Eq (5.22) and using Eq (5.20) we get

$$< \sigma^{(0)}_{ij} > = C^H_{ijkl}\frac{\partial u^{(0)}_k}{\partial x_l} \tag{5.23}$$

Hence

$$\frac{1}{2}\Big(\frac{\partial u^{(0)}_i}{\partial x_j} + \frac{\partial u^{(0)}_j}{\partial x_i}\Big) = K^H_{ijkl} < \sigma^{(0)}_{kl} > \tag{5.24}$$

where (cf Eq (4.23))

$$\mathbf{K}^H = (\mathbf{C}^H)^{-1} \tag{5.25}$$

The relations (5.19) and (5.23) imply that

$$\frac{\partial}{\partial x_j} < \sigma^{(0)}_{ij} > = < \rho > \ddot{u}^{(0)}_i \tag{5.26}$$

Hence operating on Eq (5.26) with the symmetric gradient operator and using Eq (5.24) yields the stress equation of motion of a homogenized medium (cf Eq (4.24))

$$\frac{1}{2}\Big(\frac{\partial^2}{\partial x_j \partial x_k} < \sigma^{(0)}_{ik} > + \frac{\partial^2}{\partial x_i \partial x_k} < \sigma^{(0)}_{jk} >\Big) = < \rho > K^H_{ijkl} < \ddot{\sigma}^{(0)}_{kl} > \tag{5.27}$$

Adjoining the initial and boundary conditions (4.25) ÷ (4.26) to Eq (5.27) leads to the stress formulation (4.24) ÷ (4.26).

6. Stress characterization of a homogenized mixed initial–boundary value problem

Using the results of Section 5 we are to prove the following counterpart of a classical theorem of elastodynamics (cf Gurtin [3], Theorem 4, p.222).

Theorem[4]

Let $<\sigma_{ij}^{(0)}>$ be a time – dependent admissible mean stress field on $B \times [0,\infty)$ and suppose that $<\dot{\sigma}_{ij}^{(0)}>$ is continuous on $\bar{B}\times[0,\infty)$. Then $<\sigma_{ij}^{(0)}>$ corresponds to a solution of the mixed problem of homogenized elastodynamics if and only if

$$<\sigma_{(ik}^{(0)}>_{,kj)} - <\rho> K_{ijkl}^{H} <\ddot{\sigma}_{kl}^{(0)}>= 0 \qquad \text{on} \quad B \times (0,\infty)$$

$$<\sigma_{ij}^{(0)}>\Big|_{t=0} = \sigma_{ij}^{p}(\boldsymbol{x}) \qquad <\dot{\sigma}_{ij}^{(0)}>\Big|_{t=0} = \dot{\sigma}_{ij}^{p}(\boldsymbol{x}) \qquad \text{on} \quad B$$

$$t * <\sigma_{ij}^{(0)}>_{,j} = <\rho> (\hat{u}_i - u_i^p - t\dot{u}_i^p) \qquad \text{on} \quad \partial B_U \times (0,\infty)$$

$$<\sigma_{ij}^{(0)}> n_j = F_i \qquad \text{on} \quad \partial B_F \times (0,\infty)$$

where

$$\sigma_{ij}^{p} = \sigma_{ij}^{p}(\boldsymbol{x}) \qquad \dot{\sigma}_{ij}^{p} = \dot{\sigma}_{ij}^{p}(\boldsymbol{x})$$
$$u_i^p = u_i^p(\boldsymbol{x}) \qquad \dot{u}_i^p = \dot{u}_i^p(\boldsymbol{x})$$
$$\hat{u}_i = \hat{u}_i(\boldsymbol{x},t) \qquad F_i = F_i(\boldsymbol{x},t)$$

are given functions of indicated arguments.

Proof

Assume that $<\sigma_{ij}^{(0)}>$ satisfies the above relations, and define

$$\varepsilon_{ij}^{(0)} = K_{ijkl}^{H} <\sigma_{kl}^{(0)}>$$

and

$$u_i^{(0)} = <\rho>^{-1} t * <\sigma_{ij}^{(0)}>_{,j} + u_i^{(0)}\Big|_{t=0} + t(\dot{u}_i^{(0)}\Big|_{t=0})$$

where

$$u_{(i,j)}^{(0)}\Big|_{t=0} = K_{ijkl}^{H} <\sigma_{kl}^{(0)}>\Big|_{t=0} \qquad (u_i^{(0)}\Big|_{t=0} \cong u_{i,}^{p})$$

$$\dot{u}_{(i,j)}^{(0)}\Big|_{t=0} = K_{ijkl}^{H} <\dot{\sigma}_{kl}^{(0)}>\Big|_{t=0} \qquad (\dot{u}_i^{(0)}\Big|_{t=0} \cong \dot{u}_{i,}^{p})$$

[4]Throughout this section index conventions are used; e.g. $A_{(ij)}$, denote symmetric part of A_{ij}, i.e. $A_{(ij)} \equiv \frac{1}{2}(A_{ij} + A_{ji})$ and $A_{ij,j} = \partial A_{ij}/\partial x_j$.

or

$$< \sigma_{ij}^{(0)} > \Big|_{t=0} = C_{ijkl}^{H} u_{k,l}^{(0)} \Big|_{t=0} \equiv \sigma_{ij}^{p}$$

$$< \dot{\sigma}_{ij}^{(0)} > \Big|_{t=0} = C_{ijkl}^{H} \dot{u}_{k,l}^{(0)} \Big|_{t=0} \equiv \dot{\sigma}_{ij}^{p}$$

Then it is easy to prove that

$$\varepsilon_{ij}^{(0)} = u_{(i,j)}^{(0)} \qquad \text{on} \qquad B \times [0, \infty)$$

and

$$u_i^{(0)} = \hat{u}_i^{(0)} \qquad \text{on} \qquad \partial B_U \times [0, \infty)$$

Therefore it follows that $[u_i^{(0)}, \varepsilon_{ij}^{(0)}, < \sigma_{ij}^{(0)} >]$ represents a solution to the mixed homogenized problem. Conversely, if $[u_i^{(0)}, \varepsilon_{ij}^{(0)}, < \sigma_{ij}^{(0)} >]$ is a solution to the mixed homogenized problem then the relations given in the theorem hold.

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Appendix A

Eq (4.5) written explicitly for $i,j = 1,2,3$ is equivalent to the following set of equations
for $i = j$

$$\frac{\partial}{\partial y_1}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{1k}^{(0)}\right) = 0$$

$$\frac{\partial}{\partial y_2}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{2k}^{(0)}\right) = 0 \qquad (A1)$$

$$\frac{\partial}{\partial y_3}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{3k}^{(0)}\right) = 0$$

and for $i \neq j$

$$\frac{\partial}{\partial y_2}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{1k}^{(0)}\right) + \frac{\partial}{\partial y_1}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{2k}^{(0)}\right) = 0$$

$$\frac{\partial}{\partial y_3}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{1k}^{(0)}\right) + \frac{\partial}{\partial y_1}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{3k}^{(0)}\right) = 0 \qquad (A2)$$

$$\frac{\partial}{\partial y_3}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{2k}^{(0)}\right) + \frac{\partial}{\partial y_2}\left(\rho^{-1}\frac{\partial}{\partial y_k}\sigma_{3k}^{(0)}\right) = 0$$

Integration of (A1) yields

$$\rho^{-1}\frac{\partial\sigma_{1k}^{(0)}}{\partial y_k} = f_{23}(x, y_2, y_3)$$

$$\rho^{-1}\frac{\partial\sigma_{2k}^{(0)}}{\partial y_k} = f_{13}(x, y_1, y_3) \qquad (A3)$$

$$\rho^{-1}\frac{\partial\sigma_{3k}^{(0)}}{\partial y_k} = f_{12}(x, y_1, y_2)$$

where f_{ij} are unknown functions depending on the indicated arguments, and time t (not shown). Substitution of (A3) into (A2) gives the following conditions for f_{ij}

$$\frac{\partial f_{23}}{\partial y_2} + \frac{\partial f_{13}}{\partial y_1} = 0 \qquad \frac{\partial f_{23}}{\partial y_3} + \frac{\partial f_{12}}{\partial y_1} = 0 \qquad \frac{\partial f_{13}}{\partial y_3} + \frac{\partial f_{12}}{\partial y_2} = 0 \quad (A4)$$

Eq $(A4)_1$ implies that f_{23} is a linear function of y_2, while f_{13} is a linear function of y_1

$$f_{23}(x, y_2, y_3) = y_2 g_{23}(x, y_3) + p_{23}(x, y_3) \qquad (A5)_1$$

$$f_{13}(x, y_1, y_3) = y_1 g_{13}(x, y_3) + p_{13}(x, y_3) \qquad (A5)_2$$

Similar analysis of the remaining equations in (A4) leads to

$$f_{23}(\mathbf{x}, y_2, y_3) = y_3 h_{23}(\mathbf{x}, y_2) + q_{23}(\mathbf{x}, y_2) \tag{A5$_3$}$$

$$f_{12}(\mathbf{x}, y_1, y_2) = y_1 h_{12}(\mathbf{x}, y_2) + q_{12}(\mathbf{x}, y_2) \tag{A5$_4$}$$

and

$$f_{13}(\mathbf{x}, y_1, y_3) = y_3 k_{13}(\mathbf{x}, y_1) + r_{13}(\mathbf{x}, y_1) \tag{A5$_5$}$$

$$f_{12}(\mathbf{x}, y_1, y_2) = y_2 k_{12}(\mathbf{x}, y_1) + r_{12}(\mathbf{x}, y_1) \tag{A5$_6$}$$

where the meaning of the new unknown functions is evident.

Comparing $(A5)_1$ with $(A5)_3$ we get

$$y_2 g_{23}(\mathbf{x}, y_3) + p_{23}(\mathbf{x}, y_3) = y_2 h_{23}(\mathbf{x}, y_3) + q_{23}(\mathbf{x}, y_3)$$

Hence

$$\begin{aligned} g_{23}(\mathbf{x}, y_3) &= a_1(\mathbf{x}) y_3 \\ h_{23}(\mathbf{x}, y_2) &= a_1(\mathbf{x}) y_2 \\ p_{23}(\mathbf{x}, y_3) &= q_{23}(\mathbf{x}, y_2) = b_1(\mathbf{x}) \end{aligned}$$

or

$$\begin{aligned} g_{23}(\mathbf{x}, y_3) &= c_1(\mathbf{x}) \qquad & g_{23}(\mathbf{x}, y_2) &= c_1(\mathbf{x}) y_2 \\ h_{23}(\mathbf{x}, y_2) &= d_1(\mathbf{x}) \qquad & p_{23}(\mathbf{x}, y_3) &= d_1(\mathbf{x}) y_3 \end{aligned}$$

Thus

$$f_{23} = a_1(\mathbf{x}) y_2 y_3 + b_1(\mathbf{x}) \tag{A6$_1$}$$

or

$$f_{23} = c_1(\mathbf{x}) y_2 + d_1(\mathbf{x}) y_3 \tag{A6$_2$}$$

In a similar way, by comparing $(A5)_2$ with $(A5)_5$, we get

$$f_{13} = a_2(\mathbf{x}) y_1 y_3 + b_2(\mathbf{x}) \tag{A6$_3$}$$

or

$$f_{13} = c_2(\mathbf{x}) y_1 + d_2(\mathbf{x}) y_3 \tag{A6$_4$}$$

while a comparison of $(A5)_4$ with $(A5)_6$ yields

$$f_{12} = a_3(\mathbf{x}) y_1 y_2 + b_3(\mathbf{x}) \tag{A6$_5$}$$

or

$$f_{12} = c_3(\mathbf{x}) y_1 + d_3(\mathbf{x}) y_2 \tag{A6$_6$}$$

From periodicity of $\sigma_{ik}^{(0)}$ with regard to y and from Eqs $(A6)_1 \div (A6)_6$ we conclude that

$$a_i = c_i = d_i = 0 \qquad i = 1,2,3 \tag{A7}$$

Thus Eqs (A3) take the form

$$\rho^{-1}\frac{\partial \sigma_{ik}^{(0)}}{\partial y_k} = b_i(x,t) \qquad i = 1,2,3 \tag{A8}$$

Appendix B

Eqs (4.10) and (4.13) can be rewritten in the form

$$\varepsilon_{ipq}\varepsilon_{jrs}\frac{\partial^2}{\partial y_p \partial y_r}\varepsilon_{qs}^{(0)} = 0 \tag{B1}$$

$$\frac{\partial}{\partial y_j}\left(C_{ijkl}\varepsilon_{kl}^{(0)}\right) = 0 \tag{B2}$$

where

$$\varepsilon_{ij}^{(0)} = K_{ijkl}\sigma_{kl}^{(0)} \Longleftrightarrow \sigma_{ij}^{(0)} = C_{ijkl}\varepsilon_{kl}^{(0)}$$

Since the mean value $<\varepsilon_{ij}^{(0)}>$ is independent of y, Eq (B1) can be rewritten as

$$\varepsilon_{ipq}\varepsilon_{jrs}\frac{\partial^2}{\partial y_p \partial y_r}\left(\varepsilon_{qs}^{(0)} - <\varepsilon_{qs}^{(0)}>\right) = 0$$

Hence by virtue of the compatibility theorem (cf Gurtin [3], p.40) there exists a vector field φ_i dependent on y, such that

$$\varepsilon_{ij}^{(0)} - <\varepsilon_{ij}^{(0)}> = \frac{1}{2}\left(\frac{\partial \varphi_i}{\partial y_j} + \frac{\partial \varphi_j}{\partial y_i}\right) \tag{B3}$$

Due to periodicity of $\varepsilon_{ij}^{(0)}$ with regard to y, the field φ_i is also a periodic function of y.

Now, let us introduce a function $\chi_{krs} = \chi_{krs}(y)$ which is y – periodic, symmetric in the indices r and s; and satisfies the equation

$$\frac{\partial}{\partial y_j}\left(C_{ijrs} + C_{ijkl}\frac{\partial \chi_{krs}}{\partial y_l}\right) = 0 \tag{B4}$$

Combining the formula

$$<\sigma_{ij}^{(0)}> = <C_{ijkl}\varepsilon_{kl}^{(0)}>$$

with (B3), integrating by parts and using y – periodicity of φ_i and C_{ijkl}, we get

$$<\sigma_{ij}^{(0)}>=<C_{ijkl}><\varepsilon_{kl}^{(0)}> - <\frac{\partial C_{ijkl}}{\partial y_l}\varphi_k> \tag{B5}$$

Rearrangement of the indices in (B4) yields

$$\frac{\partial}{\partial y_l}\Big(C_{klij} + C_{klmn}\frac{\partial \chi_{mij}}{\partial y_n}\Big) = 0$$

Hence

$$<\frac{\partial C_{ijkl}}{\partial y_l}\varphi_k>= - <\Big[\frac{\partial}{\partial y_l}\Big(C_{klmn}\frac{\partial \chi_{mij}}{\partial y_n}\Big)\Big]\varphi_k>$$

Integrating by parts and using the periodicity of the functions involved, we get

$$<\frac{\partial C_{ijkl}}{\partial y_l}\varphi_k>= - <\chi_{mij}\frac{\partial}{\partial y_n}\Big(C_{klmn}\frac{\partial \varphi_k}{\partial y_l}\Big)> \tag{B6}$$

On the other hand, from (B2) and (B3) we have

$$\frac{\partial}{\partial y_j}\Big[C_{ijkl}\Big(<\varepsilon_{kl}^{(0)}> + \frac{\partial \varphi_k}{\partial y_l}\Big)\Big] = 0$$

or

$$\frac{\partial}{\partial y_k}\Big(C_{klmn}\frac{\partial \varphi_m}{\partial y_n}\Big) = -\frac{\partial C_{klpq}}{\partial y_k} <\varepsilon_{pq}^{(0)}>$$

or

$$\frac{\partial}{\partial y_n}\Big(C_{klmn}\frac{\partial \varphi_k}{\partial y_l}\Big) = -\frac{\partial C_{mnpq}}{\partial y_n} <\varepsilon_{pq}^{(0)}>$$

Using the last relation, we reduce Eq (B6) to the form

$$<\frac{\partial C_{ijkl}}{\partial y_l}\varphi_k>= - <\chi_{mij}\frac{\partial C_{mnpq}}{\partial y_n}><\varepsilon_{pq}^{(0)}>$$

or

$$<\frac{\partial C_{ijkl}}{\partial y_l}\varphi_k>= - <C_{mnpq}\frac{\partial \chi_{mij}}{\partial y_n}><\varepsilon_{pq}^{(0)}> \tag{B7}$$

when integration by parts, and periodicity of χ_{mij} and C_{mnpq} are taken into account. Finally, substituting (B7) into (B5) we find that

$$<\sigma_{ij}^{(0)}>=<C_{ijkl}><\varepsilon_{kl}^{(0)}> + <C_{mnkl}\frac{\partial \chi_{mij}}{\partial y_n}><\varepsilon_{kl}^{(0)}>$$

or

$$<\sigma_{ij}^{(0)}>=<C_{ijkl} + C_{mnkl}\frac{\partial \chi_{mij}}{\partial y_n}><\varepsilon_{kl}^{(0)}>$$

or

$$<\sigma_{ij}^{(0)}>= C_{ijkl}^{H} <\varepsilon_{kl}^{(0)}> \tag{B8}$$

where C^H_{ijkl} stands for the homogenized tensor of elasticity

$$C^H_{ijkl} = < C_{ijkl} + C_{mnkl} \frac{\partial \chi_{mij}}{\partial y_n} > \tag{B9}$$

Homogenizacja naprężeniowego równania ruchu liniowej elastodynamiki

Streszczenie

Przedstawiono dwa sposoby homogenizacji ośrodka sprężystego z okresową niejednorodnością w oparciu o czysto naprężeniowe równanie ruchu ośrodka, por. [1 ÷ 4]. Wiążą się one z występującą w elastodynamice możliwością wyrażenia wektorowego pola przemieszczenia przez tensorowe pole naprężenia na dwa sposoby: albo przez związki geometryczne i prawo Hooke'a albo przez prawo ruchu. Oba sposoby prowadzą do tej samej postaci równań ruchu ośrodka zhomogenizowanego. Pokazano ponadto, że mieszany problem początkowo–brzegowy dla ośrodka zhomogenizowanego może być scharakteryzowany tylko poprzez naprężenie średnie.

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