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Invariant Geodetic Problems on the Projective Group $\mathbf{Pr}\,(\boldsymbol{n},\mathbb{R})$

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The concept of n-dimensional projectively-rigid body is introduced and its connection to the concept of $(n+1)$-dimensional incompressible affinely-rigid body is analysed. The equations of geodetic motion for such a projectively-rigid body are obtained. As an instructive example, the special case of $n=1$ is investigated.

1 Introduction

The concept of the metrically-rigid body have played a very important role in the theoretical and applied mechanics (see e.g. [1]), mainly because our macroscopic environment is dominated by objects which are approximately rigid. But what would happen if we went forward and got rid of the metrical properties keeping the concept of rigidness? Such problems in the application to the theory of continuous media were investigated in [2–10], where the concept of an affinely-rigid body as a medium the deformative behaviour of which is restricted to performing homogeneous deformations only was developed. Other applications are also possible, e.g. in the theory of large oscillations of molecules, small mono-crystals, atomic nuclei and even in the theory of elementary particles. In fact, an affinely-rigid body in an amorphous affine space is an obvious counterpart of the usual metrically-rigid body in a Euclidean space. But we need not to get rid of the metric once and for all, we may introduce it in our consideration at any step, and that is what makes this approach attractive. For example, to be able to introduce the notion of the kinetic energy, we should have some fixed Euclidean metric.

Let us explain this more precisely and consider the classical system of points (discrete or continuous), which is placed in the physical space M. We assume that the material points are distinguishable and label them by means of points of an auxiliary space N, which is called the material space (e.g. we may choose these labels as initial positions of all points at the moment t_0). Let V and U be the linear spaces of translations (free vectors) in M and N, respectively, then $(M,V,\rightarrow)$ and $(N,U,\rightarrow)$ are affine spaces. The position of the a-th material point at the time instant t is denoted by $x(t,a)$ ($x\in M$, $a\in N$). If the system is continuous, the label a and position x become the Lagrangian and Eulerian radius-vectors (material and physical variables), respectively. If the dimensions of M and N are equal, then we can impose such constraints of affine rigidness that the connection between material and physical variables is as follows:

$$x^i(t,a)=\varphi^i_B(t)a^B+r^i(t),\qquad i,B=\overline{1,n},\qquad n=\dim(M)=\dim(N). \tag{1}$$

Now we have two opposite possibilities to introduce the metric: a) in the physical space (then $g\in V^*\otimes V^*$ is the metric tensor and $(M,V,\rightarrow,g)$ is the corresponding Euclidean space) or b) in the material space (then $\eta\in U^*\otimes U^*$ is the metric tensor and $(N,U,\rightarrow,\eta)$ is the corresponding Euclidean space). In the case a) distances in N are measured by means of the configuration-dependent Green deformation tensor $G=\varphi^*g$ and in the case b) in M they are measured by means of the configuration dependent Cauchy deformation tensor $C=\varphi_*\eta$. Although the first possibility is more physical, the second one also has attractive features (e.g. then we have the situation very similar to the one in the general relativity, i.e. in the physical space there is no fixed metric geometry at all and the components of the metric tensor are included in the physical degrees of freedom and dynamically coupled with matter distribution).

In the present paper we would like to consider even more "amorphous" case, which we obtain from (1) by generalizing our affine constraints to the projective ones:

$$x^i(t,a) = \frac{A^i_B(t)a^B + b^i(t)}{c_D(t)a^D + d(t)}, \qquad Ad - bc \neq 0, \qquad i, B, D = \overline{1,n}. \tag{2}$$

Then at any fixed time $t \in \mathbb{R}$ the configuration space Q of our problem is identical with the projective group $\mathrm{Pr}\,(n,\mathbb{R}) \supset \mathrm{GAf}\,(n,\mathbb{R})$ (for $N = M = \mathbb{R}^n$) and such a system of material points is called the projectively-rigid body. Due to the isomorphism $\mathrm{Pr}\,(n,\mathbb{R}) \simeq \mathrm{SL}\,(n+1,\mathbb{R})$ we may rewrite those constraints (2) in n dimensions as the constraints defining an incompressible affinely-rigid body in $n+1$ dimensions, i.e.

$$x^\mu(t,a) = \varphi^\mu_\nu(t)a^\nu, \qquad \varphi = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \qquad \det\varphi = 1, \qquad \mu,\nu = \overline{0,n}.$$

The additional variable is called 0-th only for convenience reasons. The new configuration space $\widetilde{Q}$ is identical with the unimodular linear group $\mathrm{SL}\,(n+1,\mathbb{R})$. We may introduce also non-holonomic velocities instead of the very $\dot\varphi$, i.e. $\Omega = \dot\varphi\varphi^{-1}$ and $\hat\Omega = \varphi^{-1}\dot\varphi$. They are related to each other by the configuration-dependent similarity transformation: $\Omega = \varphi\hat\Omega\varphi^{-1}$.

2 Left and right invariant geodetic problems

Let us consider the left-invariant geodetic problem on the projective group $\mathrm{Pr}\,(n,\mathbb{R})$ (or equivalently on the unimodular linear group $\mathrm{SL}\,(n+1,\mathbb{R})$), which is also right-invariant under the orthogonal subgroup $\mathrm{SO}\,(n+1,\mathbb{R})$:

$$T_{\mathrm{left}} = \frac{J}{2}\,\mathrm{Tr}\left(\hat\Omega^T\hat\Omega\right) + \frac{\alpha}{2}\,\mathrm{Tr}\left(\hat\Omega^2\right) + \frac{\beta}{2}\left(\mathrm{Tr}\,\hat\Omega\right)^2, \tag{3}$$

and the right-invariant geodetic problem on the projective group, which is left-invariant under the orthogonal subgroup:

$$T_{\mathrm{right}} = \frac{J}{2}\,\mathrm{Tr}\left(\Omega^T\Omega\right) + \frac{\alpha}{2}\,\mathrm{Tr}\left(\Omega^2\right) + \frac{\beta}{2}\left(\mathrm{Tr}\,\Omega\right)^2, \tag{4}$$

where J, α, β are generalized inertial constants, the second and third terms for both kinetic energies are identical and are the Casimir invariants.

The most adequate description of internal degrees of freedom is that based on the two-polar decomposition, i.e. we split the system of degrees of freedom into three subsystems: $\varphi = LDR^T$, where $L, R \in \mathrm{SO}\,(n+1,\mathbb{R})$ are special orthogonal matrices ($L^TL = R^TR = \hat I$, $\det L = \det R = 1$) and D is diagonal, positive and $\det D = 1$. If we take $D^{\mu\mu} = \exp(q^\mu)$, then $\sum_\mu q^\mu = 0$. In this way our system is formally represented as a composition of two $(n+1)$-dimensional fictitious rigid bodies (systems of principal axes of the Cauchy and Green deformation tensors) and n independent material points oscillating along the straight line $\mathbb{R}$. Orthogonal transformations L and R diagonalize the Cauchy and Green deformation tensors: $C = LD^{-2}L^T$ and $G = RD^2R^T$. If there is no coincidence of diagonal elements of D, i.e. if the spectra of C and G are simple, then the two-polar decomposition is finitely non-unique: $\varphi = LDR^T = \widetilde L\widetilde D\widetilde R^T$, where $\widetilde L = LO_\pi$, $\widetilde D = O_\pi^{-1}DO_\pi$, $\widetilde R^T = RO_\pi$ and O_π is an orthogonal representation of the permutation group with the restriction $O_\pi\ \mathrm{Diag}\left(q^1,\ldots,q^{n+1}\right)O_\pi^{-1} = \mathrm{Diag}\left(q^{\pi(1)},\ldots,q^{\pi(n+1)}\right)$, i.e. $S_{n+1} \ni \pi \mapsto$

$O_\pi \in \mathrm{SO}\,(n+1, \mathbb{R})$. The components of matrices O_π are only 0 or ± 1, i.e. in any row and column there is only one non-vanishing element. If the spectrum of C and G is non-simple, then the non-uniqueness of decomposition becomes continuous. An extreme situation occurs when D is completely degenerate (proportional to the unit matrix), then no meaning may be assigned separately to L and R, only LR^T is well-defined.

It is convenient to introduce non-holonomic variables adapted to the two-polar decomposition, i.e. velocity $\dot{q}$ and non-holonomic angular velocities of the L and R rigid bodies in the co-moving representation: $\lambda = L^T \dot{L} = -\lambda^T$ and $\rho = R^T \dot{R} = -\rho^T$. Then the kinetic energies (3) and (4) may be rewritten in the combined form (the upper expression is related to the left-invariant and the lower one to the right-invariant problems) as follows:

$$
\begin{aligned}
T_{\text{left/right}} = {} & \frac{J+\alpha}{2} \operatorname{Tr}\left(\dot{D}^2 D^{-2}\right) + (J-\alpha) \operatorname{Tr}\left(\lambda D \rho D^{-1}\right) \\
& + \frac{\alpha}{2} \operatorname{Tr}\left(\lambda^2 + \rho^2\right) - \frac{J}{2} \operatorname{Tr}\left(\begin{array}{c} \lambda D^2 \lambda D^{-2} + \rho^2 \\ \lambda^2 + \rho D^2 \rho D^{-2} \end{array} \right).
\end{aligned}
$$

We see that the constant β is absent in $T_{\text{left/right}}$ because of the condition $\det D = 1 \sim \sum_\mu q^\mu = 0$. If we explicitly substitute this condition into the expression for the kinetic energy, i.e. $q^0 = -\sum\limits_{i=1}^{n} q^i$, then we can rewrite the previous formula in the following form:

$$
\begin{aligned}
T_{\text{left/right}} = {} & \frac{J+\alpha}{2} \left[\sum_{i=1}^{n} \left(\dot{q}^i\right)^2 + \langle \dot{q} \rangle^2 \right] \\
& + \sum_{i=1}^{n} V_{\text{left/right}}(|q^i + \langle q \rangle|, \lambda_{0i}, \rho_{0i}) + \frac{1}{2} \sum_{i,j=1}^{n} U_{\text{left/right}}\left(|q^i - q^j|, \lambda_{ij}, \rho_{ij}\right),
\end{aligned}
\tag{5}
$$

where $\langle \bullet \rangle = \sum\limits_{j=1}^{n} \bullet^j$, one-point and binary effective interaction potentials are as follows:

$$
\begin{aligned}
V_{\text{left/right}} &= (J-\alpha) \left[(\lambda_{0i})^2 + (\rho_{0i})^2 - 2\lambda_{0i}\rho_{0i} \operatorname{ch}\left(q^i + \langle q \rangle\right) \right] + 2J \left\{ \begin{array}{c} \lambda_{0i} \\ \rho_{0i} \end{array} \right\}^2 \operatorname{sh}^2\left(q^i + \langle q \rangle\right), \\
U_{\text{left/right}} &= (J-\alpha) \left[(\lambda_{ij})^2 + (\rho_{ij})^2 - 2\lambda_{ij}\rho_{ij} \operatorname{ch}\left(q^i - q^j\right) \right] + 2J \left\{ \begin{array}{c} \lambda_{ij} \\ \rho_{ij} \end{array} \right\}^2 \operatorname{sh}^2\left(q^i - q^j\right).
\end{aligned}
$$

Let us define the canonical affine momenta (p, j, k), which are conjugate to $(\dot{q}, \lambda, \rho)$, respectively. Here j and k are skew-symmetric matrices expressing the co-moving representation of the angular momenta of L and R rigid bodies. The Legendre transformation reads:

$$
\begin{aligned}
p_i &= \frac{\partial T_{\text{left/right}}}{\partial \dot{q}^i} = (J+\alpha)\left[\dot{q}^i + \langle \dot{q} \rangle\right] \Longleftrightarrow \dot{q}^i = \frac{1}{J+\alpha}\left[p_i - \frac{1}{n+1}\langle p \rangle\right], \\
j_{0i} &= \frac{\partial T_{\text{left/right}}}{\partial \lambda_{0i}} = 2(J-\alpha)\left[\lambda_{0i} - \rho_{0i} \operatorname{ch}\left(q^i + \langle q \rangle\right)\right] + 4J \left\{ \begin{array}{c} \lambda_{0i} \\ 0 \end{array} \right\} \operatorname{sh}^2\left(q^i + \langle q \rangle\right), \\
k_{0i} &= \frac{\partial T_{\text{left/right}}}{\partial \rho_{0i}} = 2(J-\alpha)\left[\rho_{0i} - \lambda_{0i} \operatorname{ch}\left(q^i + \langle q \rangle\right)\right] + 4J \left\{ \begin{array}{c} 0 \\ \rho_{0i} \end{array} \right\} \operatorname{sh}^2\left(q^i + \langle q \rangle\right), \\
j_{ij} &= \frac{\partial T_{\text{left/right}}}{\partial \lambda_{ij}} = 2(J-\alpha)\left[\lambda_{ij} - \rho_{ij} \operatorname{ch}\left(q^i - q^j\right)\right] + 4J \left\{ \begin{array}{c} \lambda_{ij} \\ 0 \end{array} \right\} \operatorname{sh}^2\left(q^i - q^j\right), \qquad i < j, \\
k_{ij} &= \frac{\partial T_{\text{left/right}}}{\partial \rho_{ij}} = 2(J-\alpha)\left[\rho_{ij} - \lambda_{ij} \operatorname{ch}\left(q^i - q^j\right)\right] + 4J \left\{ \begin{array}{c} 0 \\ \rho_{ij} \end{array} \right\} \operatorname{sh}^2\left(q^i - q^j\right), \qquad i < j.
\end{aligned}
$$

The non-vanishing basic Poisson brackets have the following form: $\{q^i, p_j\} = \delta^i_j$, $\{j_{\mu\nu}, j_{\kappa\sigma}\} = j_{\mu\sigma}\delta_{\kappa\nu} - j_{\kappa\nu}\delta_{\mu\sigma} + j_{\sigma\nu}\delta_{\mu\kappa} - j_{\mu\kappa}\delta_{\sigma\nu}$ and $\{k_{\mu\nu}, k_{\kappa\sigma}\} = k_{\mu\sigma}\delta_{\kappa\nu} - k_{\kappa\nu}\delta_{\mu\sigma} + k_{\sigma\nu}\delta_{\mu\kappa} - k_{\mu\kappa}\delta_{\sigma\nu}$. It is more convenient later on to use the auxiliary variables $R := -j - k = -R^T$ and $A := j - k = -A^T$ instead of the very j and k:

$$R_{0i} = 4\left[(J-\alpha)(\lambda_{0i}+\rho_{0i})\,\mathrm{sh}^2\left(\frac{q^i+\langle q\rangle}{2}\right) - J\left\{\begin{array}{c}\lambda_{0i}\\ \rho_{0i}\end{array}\right\}\mathrm{sh}^2\left(q^i+\langle q\rangle\right)\right],$$

$$A_{0i} = 4\left[(J-\alpha)(\lambda_{0i}-\rho_{0i})\,\mathrm{ch}^2\left(\frac{q^i+\langle q\rangle}{2}\right) + J\left\{\begin{array}{c}\lambda_{0i}\\ -\rho_{0i}\end{array}\right\}\mathrm{sh}^2\left(q^i+\langle q\rangle\right)\right],$$

$$R_{ij} = 4\left[(J-\alpha)(\lambda_{ij}+\rho_{ij})\,\mathrm{sh}^2\left(\frac{q^i-q^j}{2}\right) - J\left\{\begin{array}{c}\lambda_{ij}\\ \rho_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right], \qquad i<j,$$

$$A_{ij} = 4\left[(J-\alpha)(\lambda_{ij}-\rho_{ij})\,\mathrm{ch}^2\left(\frac{q^i-q^j}{2}\right) + J\left\{\begin{array}{c}\lambda_{ij}\\ -\rho_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right], \qquad i<j.$$

They satisfy the following Poisson bracket rules: $\{R_{\mu\nu}, R_{\kappa\sigma}\} = \{A_{\mu\nu}, A_{\kappa\sigma}\} = -R_{\mu\sigma}\delta_{\kappa\nu} + R_{\kappa\nu}\delta_{\mu\sigma} - R_{\sigma\nu}\delta_{\mu\kappa} + R_{\mu\kappa}\delta_{\sigma\nu}$ and $\{R_{\mu\nu}, A_{\kappa\sigma}\} = -A_{\mu\sigma}\delta_{\kappa\nu} + A_{\kappa\nu}\delta_{\mu\sigma} - A_{\sigma\nu}\delta_{\mu\kappa} + A_{\mu\kappa}\delta_{\sigma\nu}$.

Geodetic Hamiltonians corresponding to our kinetic energies (5) may be written as follows:

$$\Im_{\text{left/right}} = \frac{1}{2(J+\alpha)}\left[\sum_{i=1}^{n}(p_i)^2 - \frac{1}{n+1}\langle p\rangle^2\right] + \frac{1}{32(J-\alpha)}\sum_{\mu,\nu=0}^{n}\left[R^2_{\mu\nu}+A^2_{\mu\nu}\right] \tag{6}$$

$$\pm\frac{J}{8(J^2-\alpha^2)}\sum_{\mu,\nu=0}^{n}R_{\mu\nu}A_{\mu\nu} + \sum_{i=1}^{n}V_{\text{eff}}(|q^i+\langle q\rangle|, R_{0i}, A_{0i}) + \frac{1}{2}\sum_{i,j=1}^{n}U_{\text{eff}}(|q^i-q^j|, R_{ij}, A_{ij}),$$

where even in the purely geodetic problems we have the "internal" effective interaction potentials:

$$V_{\text{eff}} = \frac{1}{16(J+\alpha)}\left\{R^2_{0i}\,\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right) + A^2_{0i}\,\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\},$$

$$U_{\text{eff}} = \frac{1}{16(J+\alpha)}\left\{R^2_{ij}\,\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right) + A^2_{ij}\,\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\}.$$

The Hamilton equations of motion may be expressed in terms of Poisson brackets as follows:

$$\frac{dp_i}{dt} = \{p_i, H\} = \frac{1}{4(J+\alpha)\,\mathrm{sh}\left(q^i+\langle q\rangle\right)}\left\{R^2_{0i}\,\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right) - A^2_{0i}\,\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\}$$
$$+\sum_{j=1}^{n}\frac{1}{8(J+\alpha)\,\mathrm{sh}\left(q^i-q^j\right)}\left\{R^2_{ij}\,\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right) - A^2_{ij}\,\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\},$$

$$\frac{dR_{\mu\nu}}{dt} = \{R_{\mu\nu}, R_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{\partial R_{\kappa\sigma}} + \{R_{\mu\nu}, A_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{A_{\kappa\sigma}},$$

$$\frac{dA_{\mu\nu}}{dt} = \{A_{\mu\nu}, R_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{\partial R_{\kappa\sigma}} + \{A_{\mu\nu}, A_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{A_{\kappa\sigma}},$$

where the corresponding partial derivatives of the kinetic energies (6) have the following form:

$$\frac{\partial\Im_{\text{left/right}}}{\partial R_{0i}} = \frac{\left[J\,\mathrm{ch}\left(q^i+\langle q\rangle\right)-\alpha\right]R_{0i} \pm 2JA_{0i}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i+\langle q\rangle\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial A_{0i}} = \frac{\left[J\,\mathrm{ch}\left(q^i+\langle q\rangle\right)-\alpha\right]A_{0i} \pm 2JR_{0i}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i+\langle q\rangle\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial R_{ij}} = \frac{\left[J\,\mathrm{ch}\left(q^i-q^j\right)-\alpha\right]R_{ij} \pm 2JA_{ij}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i-q^j\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial A_{ij}} = \frac{\left[J\,\mathrm{ch}\left(q^i-q^j\right)-\alpha\right]A_{ij} \pm 2JR_{ij}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i-q^j\right)/2\right]}.$$

3 Geodetic problems on the projective line

In n dimensions a projectively-rigid body is such a body all projective relations between constituents of which during any admissible motion are invariant, i.e. material straight lines remain straight lines and all cross-ratios of any four points placed on the same straight lines are constant. It is interesting that the cross-ratio of four points on the line, i.e.

$$(x_1, x_2, x_3, x_4) = \frac{x_4 - x_1}{x_4 - x_2} : \frac{x_3 - x_1}{x_3 - x_2},$$

plays here the same role as the usual mutual ratio of segments for the affine and the distance for the metrical geometries (see e.g. [11]). After choosing the appropriate homogeneous coordinates and adding to the consideration a set of non-proper points in the infinity, the cross-ratio is constant under the action of the whole projective group Pr $(n, \mathbb{R})$.

Let us consider now the very simple and in some sense trivial but nevertheless very illustrative example of the one-dimensional left and right invariant geodetic problems on the projective group Pr $(1, \mathbb{R}) \simeq$ SL $(2, \mathbb{R})$. Hence:

$$D = \begin{bmatrix} e^{-q} & 0 \\ 0 & e^{q} \end{bmatrix}, \qquad L = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix},$$

$$\lambda = L^{-1}\dot{L} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}, \qquad \rho = R^{-1}\dot{R} = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix}, \qquad \lambda = \dot{\gamma}, \qquad \rho = \dot{\delta}.$$

The kinetic energy (5) now has the following form:

$$T^{n=1}_{\text{left/right}} = (J+\alpha)\dot{q}^2 + (J-\alpha)\begin{bmatrix} \rho - \mathrm{ch}\,(2q)\lambda \\ \lambda - \mathrm{ch}\,(2q)\rho \end{bmatrix}^2 + (J+\alpha)\,\mathrm{sh}^2(2q)\begin{bmatrix} \lambda^2 \\ \rho^2 \end{bmatrix}. \tag{7}$$

The canonical momenta (p, j, k) (equivalently, the auxiliary valuables R, A) and corresponding velocities $(\dot{q}, \lambda, \rho)$ are connected by the Legendre transformation in the following way:

$$p = 2(J+\alpha)\dot{q}, \qquad j_{\text{left/right}} = 2(J-\alpha)\left[\lambda - \rho\,\mathrm{ch}(2q)\right] + 4J\begin{Bmatrix} \lambda \\ 0 \end{Bmatrix}\mathrm{sh}^2(2q),$$

$$k_{\text{left/right}} = 2(J-\alpha)\left[\rho - \lambda\,\mathrm{ch}\,(2q)\right] + 4J\begin{Bmatrix} 0 \\ \rho \end{Bmatrix}\mathrm{sh}^2(2q),$$

$$R_{\text{left/right}} = 2(J-\alpha)(\lambda+\rho)\,\mathrm{sh}^2 q - 4J\begin{Bmatrix} \lambda \\ \rho \end{Bmatrix}\mathrm{sh}^2(2q),$$

$$A_{\text{left/right}} = 2(J-\alpha)(\lambda-\rho)\,\mathrm{ch}^2 q + 4J\begin{Bmatrix} \lambda \\ -\rho \end{Bmatrix}\mathrm{sh}^2(2q).$$

Geodetic Hamiltonians corresponding to our kinetic energies (7) are as follows:

$$\begin{aligned} \Im^{n=1}_{\text{left/right}} &= \frac{p^2}{4(J+\alpha)} + \frac{1}{4(J-\alpha)}\begin{Bmatrix} k^2 \\ j^2 \end{Bmatrix} + \frac{1}{4(J+\alpha)\,\mathrm{sh}^2(2q)}\begin{bmatrix} j + \mathrm{ch}\,(2q)k \\ k + \mathrm{ch}\,(2q)j \end{bmatrix}^2 \\ &= \frac{p^2}{4(J+\alpha)} + \frac{R^2 + A^2}{16(J-\alpha)} \pm \frac{JRA}{4(J^2-\alpha^2)} + V^{n=1}_{\text{eff}}(q, R, A), \end{aligned}$$

where the one-dimensional effective potential and corresponding effective force have the following form:

$$V^{n=1}_{\text{eff}}(q, R, A) = \frac{R^2\,\mathrm{cth}^2 q + A^2\,\mathrm{th}^2 q}{16(J+\alpha)}, \qquad F^{n=1}_{\text{eff}} := -\frac{\partial V^{n=1}_{\text{eff}}}{\partial q} = \frac{R^2\,\mathrm{cth}^2 q - A^2\,\mathrm{th}^2 q}{4(J+\alpha)\,\mathrm{sh}(2q)}.$$

The canonical momenta j and k (equivalently R and A) in the one-dimensional case are the constants of motion, so the Newton equation of motion of the fictitious particle on the line $\mathbb{R}$ in the "internal" effective potential $V_{\mathrm{eff}}^{n=1}$ is as follows:

$$\ddot{q} = \frac{1}{2(J+\alpha)} F_{\mathrm{eff}}^{n=1} = \frac{R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q}{8(J+\alpha)^2 \operatorname{sh}(2q)}, \tag{8}$$

Assigning some special values to such constants of motion as energy E and canonical momenta R, A we can write the first-order differential equation on the q variable as follows:

$$\dot{q}^2 = \frac{1}{J+\alpha} E_{\mathrm{eff}}^{\pm} - \frac{R^2 \operatorname{cth}^2 q + A^2 \operatorname{th}^2 q}{16(J+\alpha)^2}, \qquad E_{\mathrm{eff}}^{\pm} = E - \frac{R^2 + A^2}{16(J-\alpha)} \mp \frac{JRA}{4(J^2-\alpha^2)},$$

and finally we have the following solution of the (8):

$$\begin{aligned} t(q) &= \int dq \frac{4(J+\alpha)}{\sqrt{16(J+\alpha)E_{\mathrm{eff}}^{\pm} - R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q}} \\ &= \frac{2(J+\alpha)}{\sqrt{\Theta}} \ln\left[R^2 - A^2 - \Theta \operatorname{ch}^2(2q) - 2\sqrt{\Theta} \operatorname{sh} q \sqrt{16(J+\alpha)E_{\mathrm{eff}}^{\pm} - R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q} \right], \end{aligned}$$

where $\Theta = 16(J+\alpha)E - 2J(R \pm A)^2/(J-\alpha)$.

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[1] Arnold V.I., Mathematical methods of classical mechanics, *Springer Graduate Texts in Mathematics*, Vol. 60, New York, Springer, 1978.

[2] Sławianowski J.J., Analytical mechanics of finite homogeneous strains, *Archive of Mech.*, 1974, V.24, N 4, 569–587.

[3] Sławianowski J.J., The mechanics of an affinely-rigid body, *Intern. J. Theor. Phys.*, 1975, V.12, N 4, 271–296.

[4] Sławianowski J.J., Affinely-rigid body and Hamiltonian systems on $\mathrm{GL}(n,\mathbb{R})$, *Rep. Math. Phys.*, 1988, V.26, N 1, 73–119.

[5] Sławianowska A., On certain nonlinear many-body problems on lines and circles, *Archive of Mech.*, 1989, V.41, N 5, 619–640.

[6] Sławianowski J.J. and Sławianowska A., Hamiltonian systems on linear groups and one-dimensional lattices with internal parameters, in Proceedings of 5th Polish-German Workshop on Dynamical Problems in Mechanical Systems (31 August – 6 September, 1997, Zakopane, Poland), Warsaw University of Technology, Institute of Machine Design Fundamentals, *Machine Dynamics Problems*, 1998, V.20, 263–273.

[7] Sławianowski J.J., One-dimensional chains, matrix groups and quantization of Poisson structures, *Journal of Tech. Phys.*, 1998, V.39, N 2, 163–187.

[8] Sławianowski J.J., Quantum and classical models based on $\mathrm{GL}\,(n)$-symmetry, in Proceedings of the Second International Symposium on Quantum Theory and Symmetries (18–21 July, 2001, Kraków, Poland), Editors E. Kapuścik and A. Horzela, New Jersey – London – Singapore – Hong Kong, World Scientific, 2002, 582–588.

[9] Sławianowski J.J., Group theoretic approach to internal and collective degrees of freedom in mechanics and field theory, *Tech. Mech.*, 2002, V.22, N 1, 8–13.

[10] Sławianowski J.J. and Kovalchuk V., Invariant geodetic problems on the affine group and related Hamiltonian systems, *Rep. Math. Phys.*, 2003, V.51, N 2/3, 371–379.

[11] Rudnicki J., Analitycal geometry, Part 1, Toruń, T. Szczęsny's Scientific Publishing House, 1949.