General numerical description of a mass moving along a structure

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Abstract

The paper deals with vibrations of structures under a moving inertial load. The space-time finite element approach has been used for a general description of the moving mass particle. Problems occur when we perform computer simulations. In the case of wave problem numerical description of the moving inertial loads requires great mathematical care. Otherwise we get a wrong solution. There is no commercial computing packages that would enable us direct simulation of moving loads, both gravitational and inertial.

Keywords: space-time finite element method, moving mass, vibrations

1. Introduction

Engineering structures under moving loads is the important research topic in many diverse fields of engineering. Moving loads are widely used i. a. in transportation. A vehicle travelling along a road plate or airfield plate is one of numerous practical applications. A complete description of the problem should contain both gravitational and inertial action of moving load. Implementation of gravitational moving forces are simple in analytical and numerical approaches. Since it does not depend on solution, it requires only an ad hoc modification of the load vector. Much more complex are moving inertial loads. Inclusion of inertia of the moving load requires not only modification of the right-hand side vector, but also affects selected parts of global matrices of inertia, damping and stiffness of the system. Acceleration of the moving mass particle is described by the well known Renaudot formula

$$\frac{\mathrm{d}^2 w(\mathsf{v}t,t)}{\mathrm{d}t^2} = \left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=\mathsf{v}t} + 2\mathsf{v}\left.\frac{\partial^2 w(x,t)}{\partial x \partial t}\right|_{x=\mathsf{v}t} + \mathsf{v}^2 \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=\mathsf{v}t} . \tag{1}$$

Adequate terms correspond to the transverse, Coriolis and centrifugal acceleration. Several papers [1, 2, 3, 4] discuss a numerical description of the moving mass in the finite element formulation, applied to the Euler beam and Kirchhoff plate. Interpolation of displacements by 3rd order polynomials is simple. It facilitates the derivation of the matrices describing the traveling mass particle (1). Matrices known from the literature are not comprehensive. They are not suitable for general applications. In the case of the wave equations (string, Timoshenko beam, Mindlin plate) we take into account a linear relationship between displacements and angles of rotation in neighboring nodes. In the paper [5] classical finite element formulation of the moving mass travelling along the Timoshenko beam was proposed.

This paper presents a space-time approach to the moving mass problem. Characteristic matrices in the case of thin and thick plates were derived.

2. Finite element carrying a moving mass

Numerical formulation of a moving mass is performed by space-time finite elements method [6, 7, 8]. This method consists of discretization of equations of motion both in space and time. In this case velocity variant and stationary mesh was used. Finally we obtain the system of algebraic equations $\mathbf{K}^* \mathbf{v} + \mathbf{e} = \mathbf{0}$ with velocities as unknowns. In order to calculate nodal displacements vector we use the following formula

$$\mathbf{w}_{i+1} = \mathbf{w}_i + h[\beta \mathbf{v}_i + (1-\beta)\mathbf{v}_{i+1}] \ . \tag{2}$$

Let us consider space-time finite element carrying mass m moving at a constant speed v. Virtual energy in the domain $\Omega = \{(x,t)\colon 0 \le x \le b,\ \ 0 \le t \le h\}$ is written by the equation

$$\Pi_m = \int_0^h \int_0^b v^* \cdot \delta(x - x_0 - \mathsf{v}t)\, m \, \frac{\mathrm{d}^2 w(\mathsf{v}t, t)}{\mathrm{d}t^2} \, \mathrm{d}x \, \mathrm{d}t \ . \tag{3}$$

Dirac delta δ defines the position of the moving mass. v^* is the virtual velocity. However, the acceleration of the moving mass is given by (1). We apply linear interpolation of the nodal velocity

$$v(x,t) = \sum_{i=1}^{4} N_i(x,t)\, v_i \ , \tag{4}$$

where the shape function takes the following form

$$\mathbf{N} = \left[\frac{1}{bh}(x-b)(t-h) \ , \ \ -\frac{1}{bh}\, x(t-h) \ , \ \ -\frac{1}{bh}(x-b)t \ , \ \ \frac{1}{bh}\, x\, t \right] \ . \tag{5}$$

Displacement function is a result of the integration of (4)

$$w(x,t) = w(x,0) + \int_0^t \mathbf{N}\, \mathbf{v} \, \mathrm{d}t \ . \tag{6}$$

The virtual velocity is described with Dirac function

$$v^* = \delta(t - \alpha h) \left[\left(1 - \frac{x}{b} \right) v_3 + \frac{x}{b} v_4 \right] \ . \tag{7}$$

Linear interpolation of nodal physical parameters with shape functions unables determination of a centrifugal acceleration of the moving mass particle. We rewrite (1) in the equivalent form

$$\frac{\mathrm{d}^2 w(\mathrm{v}t, t)}{\mathrm{d}t^2} = \left. \frac{\partial v(x,t)}{\partial t} \right|_{x=\mathrm{v}t} + \mathrm{v} \left. \frac{\partial v(x,t)}{\partial x} \right|_{x=\mathrm{v}t} + \mathrm{v} \frac{\mathrm{d}}{\mathrm{d}t} \left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right] \ . \tag{8}$$

We assume the backward difference formula to the third term of (8). We have then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right] = \frac{1}{h} \left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right]^{t+h} - \frac{1}{h} \left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right]^{t} \ . \tag{9}$$

The upper indices indicate time at which the respective terms are defined. At time

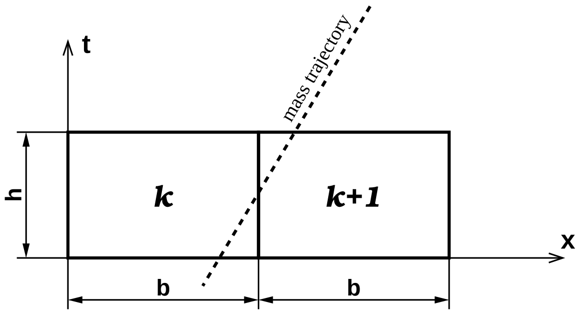

Figure 1: The transition mass between elements.

of transition of the moving load between the elements k and $k+1$ (Fig. 1), the current displacements are computed based on displacements the neighbouring element $k+1$

$$\left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right]^{t+h} = \frac{1}{b} \left(w_4^{k+1} - w_3^{k+1} \right) \ , \tag{10}$$

however, the initial displacement in the element k equals

$$\left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right]^{t} = \frac{1}{b} \left(w_2^{k} - w_1^{k} \right) \ . \tag{11}$$

The lower indices indicate the number of nodes. According to (2), (10), and (11) the finite difference scheme (9) is written as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left[\left. \frac{\partial w(x,t)}{\partial x} \right|_{x=\mathrm{v}t} \right] &= \frac{1}{bh} \left(w_2^{k+1} - w_2^{k} - w_1^{k+1} + w_1^{k} \right) + \\ &+ \frac{1}{b} \left[-\beta v_1^{k+1} + \beta v_2^{k+1} - (1-\beta) v_3^{k+1} + (1-\beta) v_4^{k+1} \right] \ . \end{aligned} \tag{12}$$

The accurate solution is obtained with $\beta = 1 - \alpha$ [9]. Therefore, we can write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\mathsf{v}t}\right] = \frac{1}{bh}\left(w_2^{k+1} - w_2^k - w_1^{k+1} + w_1^k\right) + \\ + \frac{1}{b}\left[-(1-\alpha)v_1^{k+1} + (1-\alpha)v_2^{k+1} - \alpha v_3^{k+1} + \alpha v_4^{k+1}\right] \ .\end{aligned} \tag{13}$$

Classical minimization of the energy (3) results in the following matrices

$$\mathbf{M}_m = \frac{m}{h}\left[\begin{array}{cc|cc} -(1-\kappa)^2 & -\kappa(1-\kappa) & (1-\kappa)^2 & \kappa(1-\kappa) \\ -\kappa(1-\kappa) & -\kappa^2 & \kappa(1-\kappa) & \kappa^2 \end{array}\right] , \tag{14}$$

$$\mathbf{C}_m = \frac{2m\mathsf{v}}{b}\left[\begin{array}{cc|cc} (\kappa-1)(1-\alpha) & (1-\kappa)(1-\alpha) & (\kappa-1)\alpha & (1-\kappa)\alpha \\ -\kappa(1-\alpha) & \kappa(1-\alpha) & -\kappa\alpha & \kappa\alpha \end{array}\right] , \tag{15}$$

and vector of nodal forces at the beginning of the time interval

$$\mathbf{e}_m = \frac{m\mathsf{v}}{bh}\left[\begin{array}{c} (1-\kappa)\left(w_2^{k+1} - w_2^k - w_1^{k+1} + w_1^k\right) \\ \kappa\left(w_2^{k+1} - w_2^k - w_1^{k+1} + w_1^k\right) \end{array}\right] . \tag{16}$$

The coefficient κ describes the instantaneous position of the mass in the spatial element

$$\kappa = \frac{x_0 + \mathsf{v}\alpha h}{b} \ , \quad 0 < \kappa \leq 1 \, . \tag{17}$$

We must emphasize here that the centrigugal forces are contributed in the vector $\mathbf{e}_m$ and is not described by a separate term. In the case of direct differentiation of (1) we lose information of nodal forces represented by (16). Vector $\mathbf{e}_m$ has nonzero values during the transition of the mass between neighbouring space-time elements. It can not be omitted since it contributes vital mathematical quantity, even if it mostly equals zero. The matrices (14)–(15) and the vector (16) contribute only the moving inertial particle effect. The matrices of the mass influence in a finite element of a structure must be added to the global system of equations.

3. Numerical examples

First we will consider a thin plate. Respective finite element formulation related to the plate can be found for example in [10]. We use thin plate elements in the simulation of a plate vibrations under a mass moving along the symmetry axis of the plate. The data assumed: thickness t=0.4 m, dimensions l_x=l_y=12 m, Young modulus E=30 MPa, Poisson coefficient ν=0.2, mass density ρ=2400 kg/m^3, the moving load composed of the mass m=10^4 kg and related force P=9.81·10^4 N.

In the case of thick plate we consider the Mindlin model of the plate. We use the formulation given for example in [11, 12]. We assume linear distributions of both displacements and rotations along the element, according to the interpolation functions. Comparisons of numerical and semi-analytical solution were done. The excellent coincidence is exhibited. Influence of inertia of the moving load solutions are depicted in Figures 2 and 3. Displacements of the contact point and the center of the plates are depicted. w_0 denotes the static displacement of the center of the Kirchhoff plate.

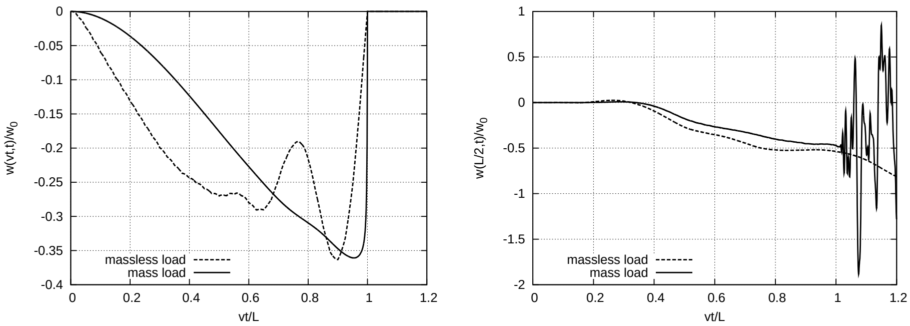

Figure 2: Vertical displacements at the contact point and at the middle of the Kirchhoff plate (thickness=0.1 m, v=360 km/h).

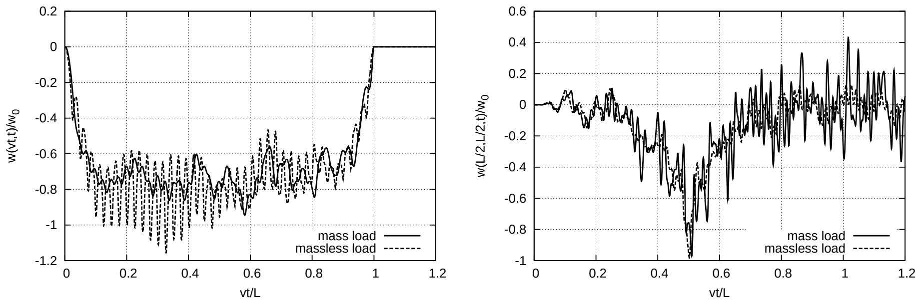

Figure 3: Vertical displacements at the contact point and at the middle of the Mindlin plate (thickness=1 m, v=360 km/h).

4. Conclusions

Original finite elements carrying a moving mass particle were elaborated. The presented approach is general and allows the accurate modeling of the point mass traveling with a constant velocity in numerical computations by using the space-time finite element method. The results confirm the significant influence of the inertia of the moving load on the solutions.

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Ogólny numeryczny opis ruchomej masy

Praca omawia problem drgań konstrukcji pod ruchomym obciążeniem bezwładnościowym. Przedstawiono ogólne macierze opisujące chwilowe położenie masy w czasoprzestrzennym elemencie skończonym. Zaprezentowane wyniki symulacji komputerowych potwierdzają znaczny wpływ bezwładności ruchomego obciążenia na otrzymane rozwiązania.