Lie point symmetries and invariant solutions of equations for turbulence statistics

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Outline

- Randomness vs. symmetries in turbulence
- Complete statistical description of turbulence in terms of multipoint correlations
- Group analysis and invariant solutions
- Invariant turbulence modelling
- Conclusions and perspectives

Introduction

- Randomness of turbulence follows from its dependence on small variations in the initial and boundary conditions

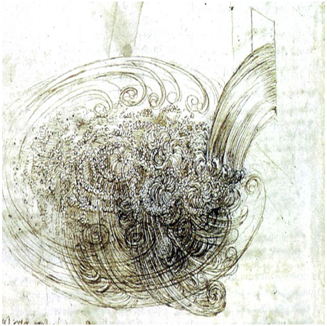


Figure: Whirlpools of water. Image: Leonardo da Vinci, RL 12660, Windsor, Royal Library.

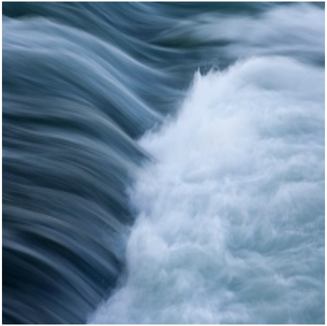


Figure: Laminar flow turns turbulent. Creative Commons credit, Image: James Marvin Phelps.

Introduction

Symmetries summarize regularities of the laws of nature, provide a route between abstract theory and experimental observations

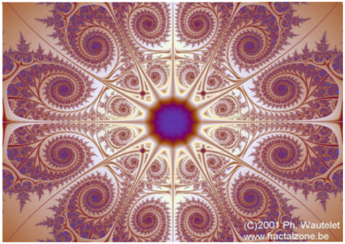


Figure: Fractal structure from www.fractalzone.be

Figure: Top: wake behind rotating cylinders, S. Kumar, B. Gonzalez [Phys. Fluids 23, 014102 (2011)].
Bottom: Santa Cristina della Fondazza, University of Bologna.

Introduction

Symmetry breaking in the wake behind two rotating cylinders

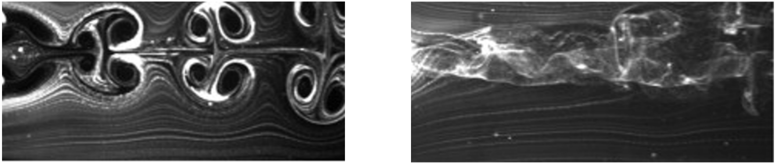
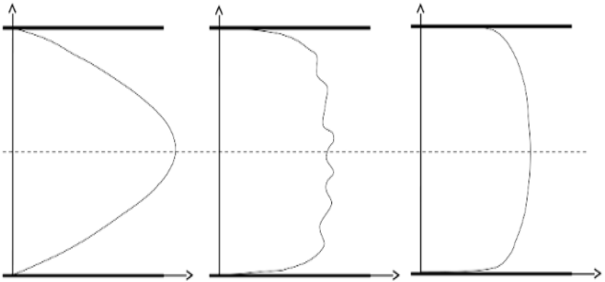


Figure: Wake behind rotating cylinders, S. Kumar, B. Gonzalez [Phys. Fluids 23, 014102 (2011)]
www.aps.org - image gallery, http://dx.doi.org/10.1063/1.3528260

Introduction



Breaking of symmetry after the transition to turbulence

Mean solution - reflection symmetry is recovered

Statistical description of turbulence

- After the transition to turbulence we usually observe breaking of symmetries, however, the symmetries may be recovered in the statistical sense.
- "To identify underlying symmetries is a central problem of the statistical physics of infinite-dimensional strongly fluctuating systems."

  Bernard et al. Inverse Turbulent Cascades and Conformally Invariant Curves, PRL 024501 (2007)
- Aim: Investigate symmetries of a system of equations describing turbulence statistics

Probability density functions (pdf's) of velocity

The probability that a random variable $\boldsymbol{U}$ is contained within $\boldsymbol{v}$ and $\boldsymbol{v} + \mathrm{d}\boldsymbol{v}$: $P(\boldsymbol{v} \leq \boldsymbol{U} \leq \boldsymbol{v} + \mathrm{d}\boldsymbol{v}) = f(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$

Pdf of velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ at point $\boldsymbol{x}$ and at time t: $f(\boldsymbol{v}; \boldsymbol{x}, t)$

Ensemble average:

$$\langle G(\boldsymbol{U}(\boldsymbol{x}, t))\rangle = \int G(\boldsymbol{v}) f(\boldsymbol{v}; \boldsymbol{x}, t)\mathrm{d}\boldsymbol{v}$$

Multipoint correlations (MPC)

Reynolds-averaged Navier-Stokes equations

$$\frac{\partial \langle U^i \rangle}{\partial x^i} = 0, \quad \frac{\partial \langle U^i \rangle}{\partial t} + \frac{\partial \langle U^i U^j \rangle}{\partial x^j} = -\frac{1}{\varrho}\frac{\partial \langle P \rangle}{\partial x^i} + \nu \frac{\partial^2 \langle U^i \rangle}{\partial x^j \partial x^j}.$$

Equations for the two-point correlations: $\langle U^i(\boldsymbol{x}_1, t) U^j(\boldsymbol{x}_2, t)\rangle$

$$\frac{\partial \langle U^i(\boldsymbol{x}_1,t)U^j(\boldsymbol{x}_2,t)\rangle}{\partial t} + \sum_{n=1}^{2} \frac{\partial \langle U^i(\boldsymbol{x}_1,t)U^j(\boldsymbol{x}_2,t)U^k(\boldsymbol{x}_n,t)\rangle}{\partial x_n^k} =$$

$$-\frac{1}{\rho}\left(\frac{\partial \langle U^i(\boldsymbol{x}_1,t)P(\boldsymbol{x}_2,t)\rangle}{\partial x_2^j} + \frac{\partial \langle P(\boldsymbol{x}_1,t)U^j(\boldsymbol{x}_2,t)\rangle}{\partial x_1^i}\right) + \nu \sum_{n=1}^{2} \frac{\partial \langle U^i(\boldsymbol{x}_1,t)U^j(\boldsymbol{x}_2,t)\rangle}{\partial x_n^k \partial x_n^k}$$

Multipoint correlations (MPC)

- Equation for the n-th point velocity correlation contains an unclosed correlation of the order $n+1$
- This forms an infinite hierarchy of MPC equations - the Friedman-Keller hierarchy

  [L. Keller and A. Friedmann, Proc. First. Int. Congr. Appl. Mech. (Technische Boekhandel en Drukkerij, Delft, 1924]
- Analogously, an infinite hierarchy for the multipoint pdf's can be derived where the n-th equation contains an unknown $n+1$-point pdf.

  [T. S. Lundgren, Phys. Fluids 10, 1967]

LMN hierarchy

Infinite Lundgren-Monin-Novikov hierarchy for n-point pdf:

$$\frac{\partial f_1}{\partial t} + v_1^i \frac{\partial f_n}{\partial x_1^i} = \frac{\partial}{\partial v_1^i}\left[F_1^i\left([f_2]\right)\right]$$

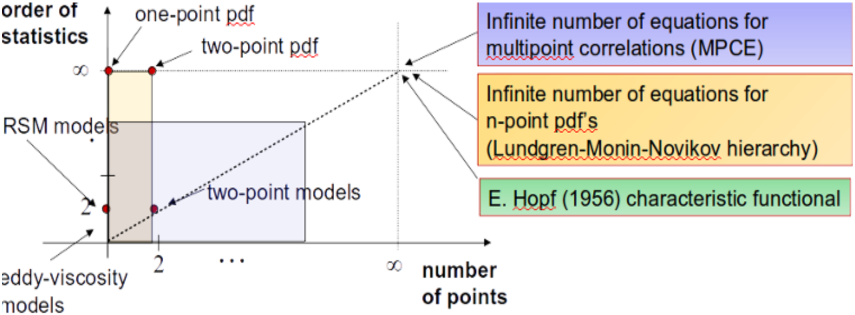
$$\frac{\partial f_n}{\partial t} + \sum_{k=1}^{2} v_k^i \frac{\partial f_n}{\partial x_k^i} = \sum_{k=1}^{2} \frac{\partial}{\partial v_k^i}\left[F_k^i\left([f_3]\right)\right]$$

$$\ldots$$

$$\frac{\partial f_n}{\partial t} + \sum_{k=1}^{n} v_k^i \frac{\partial f_n}{\partial x_k^i} = \sum_{k=1}^{n} \frac{\partial}{\partial v_k^i}\left[F_k^i\left([f_{n+1}]\right)\right]$$

$$\ldots$$

Statistical description of turbulence



Different levels of statistical description of turbulence

Lie symmetries

Transformations of variables which do not change the form of a considered equation

$$\boldsymbol{x}^* = \phi_{\boldsymbol{x}}(\boldsymbol{x}, t, \boldsymbol{U}, \epsilon), \quad t^* = \phi_t(\boldsymbol{x}, t, \boldsymbol{U}, \epsilon), \quad \boldsymbol{U}^* = \phi_{\boldsymbol{U}}(\boldsymbol{x}, t, \boldsymbol{U}, \epsilon)$$

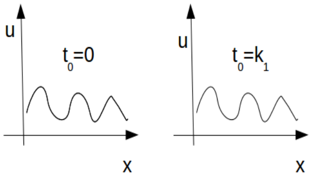
Example: scaling group of Euler equation: $\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho} \nabla P,$

$$t^* = t,\ \boldsymbol{x}^* = \mathrm{e}^{\epsilon} \boldsymbol{x},\ \boldsymbol{U}^* = \mathrm{e}^{\epsilon} \boldsymbol{U}, \quad P^* = \mathrm{e}^{2\epsilon} P$$

Transformations satisfy group properties:

1. closure	$\boldsymbol{x}^* = \mathrm{e}^{\epsilon_1 + \epsilon_2} \boldsymbol{x} = \mathrm{e}^{\epsilon_3} \boldsymbol{x}$
2. associativity	$\boldsymbol{x}^* = \mathrm{e}^{(\epsilon_1 + \epsilon_2) + \epsilon_3} \boldsymbol{x} = \mathrm{e}^{\epsilon_1 + (\epsilon_2 + \epsilon_3)} \boldsymbol{x}$
3. identity element	$\boldsymbol{x}^* = \mathrm{e}^{0} \boldsymbol{x} = \boldsymbol{x}$
4. inverse element	$\boldsymbol{x}^* = \mathrm{e}^{\epsilon} \mathrm{e}^{-\epsilon} \boldsymbol{x} = \boldsymbol{x}$

Lie symmetries - time translation



Time translation

$$T_1: \quad t^* = t + k_1, \quad \boldsymbol{x}^* = \boldsymbol{x},$$
$$\boldsymbol{U}^* = \boldsymbol{U}, \quad P^* = P,$$

Lie symmetries - scaling

$$T_2: \quad \boldsymbol{x}^* = e^{k_2}\boldsymbol{x}, \quad \boldsymbol{U}^* = e^{k_2}\boldsymbol{U}, \quad P^* = e^{2k_2}P, \quad \Longrightarrow \quad x^{i*}/x^i = v^{i*}/v^i$$

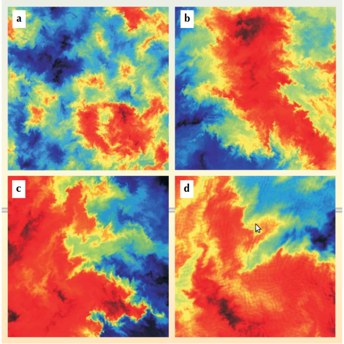
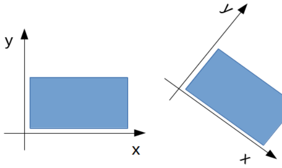


Figure: Scale invariance. From: G. Falkovich and K. R. Sreenivasan Lessons from Hydrodynamic Turbulence, Phys. Today **59**, 43-49, 2006

Two-dimensional passive scalar field. "The four panels in the sequence a-d show increasingly magnified views of a fixed location at a specific time. Clearly the four images are not identical. But they are similar in a statistical sense" (statistical self-similarity).

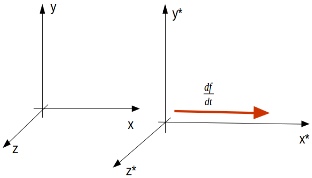
Lie symmetries - rotational invariance



Rotational invariance

$$T_4 - T_6: \; t^* = t, \;\; \boldsymbol{x}^* = \mathbf{a} \cdot \boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathbf{a} \cdot \boldsymbol{U}, \;\; P^* = P,$$

Lie symmetries - generalised Galilean invariance



Galilei invariance - laws of motion do not change in a moving frame.

$$T_7 - T_9 : \quad t^* = t, \quad \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t),$$

$$\boldsymbol{U}^* = \boldsymbol{U} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t}, \quad P^* = P - \boldsymbol{x} \cdot \frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2},$$

Lie symmetries

Symmetries of the Euler equations with $\rho = const$

$$T_1: \; t^* = t + k_1, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \boldsymbol{U}^* = \boldsymbol{U}, \;\; P^* = P,$$

$$T_2: \; t^* = t, \;\; \boldsymbol{x}^* = \mathrm{e}^{k_2}\boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathrm{e}^{k_2}\boldsymbol{U}, \;\; P^* = \mathrm{e}^{2k_2}P,$$

$$T_3: \; t^* = \mathrm{e}^{k_3}t, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathrm{e}^{-k_3}\boldsymbol{U}, \;\; P^* = \mathrm{e}^{-2k_3}P,$$

$$T_4 - T_6: \; t^* = t, \;\; \boldsymbol{x}^* = \mathbf{a} \cdot \boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathbf{a} \cdot \boldsymbol{U}, \;\; P^* = P,$$

$$T_7 - T_9: \; t^* = t, \;\; \boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{f}(t), \;\; \boldsymbol{U}^* = \boldsymbol{U} + \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t}, \quad P^* = P - \boldsymbol{x} \cdot \frac{\mathrm{d}^2\boldsymbol{f}}{\mathrm{d}t^2},$$

$$T_{10}: \; t^* = t, \;\; \boldsymbol{x}^* = \boldsymbol{x}, \;\; \boldsymbol{U}^* = \boldsymbol{U}, \;\; P^* = P + f_4(t) \;\; ,$$

For the Navier-Stokes equations instead of T_2 and T_3 we have T_{NaSt}

$$T_{NaSt}: \;\; t^* = \mathrm{e}^{2k_4}t, \;\; \boldsymbol{x}^* = \mathrm{e}^{k_4}\boldsymbol{x}, \;\; \boldsymbol{U}^* = \mathrm{e}^{-k_4}\boldsymbol{U}, \;\; P^* = \mathrm{e}^{-2k_4}P,$$

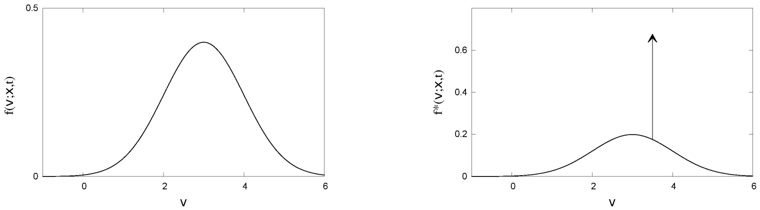
Symmetries of the MPC equations

MPC equations have symmetries which follow from the symmetries of the Navier-Stokes equations

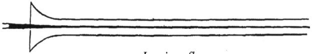
+ additional scaling and translational invariance.

[Rosteck & Oberlack, J. Nonlinear Math. Phys. 18, 2011]

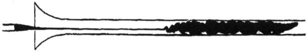
New symmetries transform a turbulent signal into a laminar or intermittent laminar-turbulent signal [Wacławczyk et al. 2014. PRE, 90, 013022]



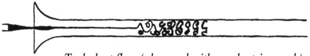
Symmetries of LMN hierarchy



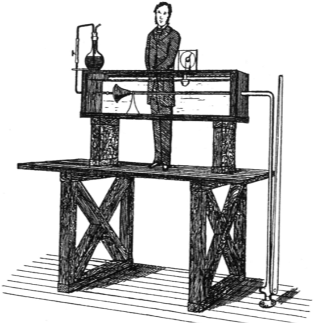
Laminar flow



Turbulent flow



Turbulent flow (observed with an electric spark)



Lie symmetries

Invariant function $f(\boldsymbol{x}, t, \boldsymbol{U}) = f(\boldsymbol{x}^*, t^*, \boldsymbol{U}^*)$

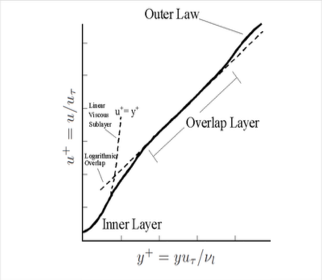
- Invariant functions describe scaling laws - as attractors of the instantaneous, fluctuating solutions of the Navier-Stokes equations.

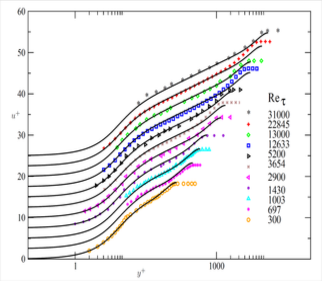
Invariant modelling

- To properly describe a physical phenomenon (e.g. turbulence), a model should be invariant under the same set of symmetries as the original equations (here: MPC equations).
- Additional symmetries of a model, not seen in the original equations lead to unphysical results (example: $k - \epsilon$ model which is additionally frame-rotation invariant).

Invariant functions

Logarithmic profile





Rona A. et al., Aeronautical Journal -New Series- 116(1180) · June 2012

Invariant functions

New scaling and translation symmetries:

$$\langle U \rangle^* = \mathrm{e}^{k_s} \langle U \rangle, \quad \langle U \rangle^* = \langle U \rangle + C_1 \left(1 - \frac{y^2}{H^2} \right)$$

+

Classical scaling symmetry: $\langle U \rangle^* = \mathrm{e}^{-k_{NS}} \langle U \rangle, \quad y^* = \mathrm{e}^{k_{NS}} y$

Lead to the characteristic system: $\frac{\mathrm{d}\langle U \rangle}{(k_s - k_{NS})\langle U \rangle + C_1 \left(1 - \frac{y^2}{H^2}\right)} = \frac{\mathrm{d}y}{k_{NS} y}$

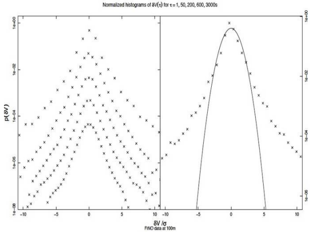
which for $k_{NS} = k_s$ has the solution

$$\langle U \rangle = \frac{C_1}{k_s} \ln(y) + \frac{C_1}{2k_s} \left(1 - \frac{y^2}{H^2} \right) + \mathcal{C}$$

Invariant functions

Exponential tails:

- Based on the new symmetries it is possible to derive solutions for pdf's, e.g. exponential tails



Pictures from http://www.uni-oldenburg.de/en/physics/research/twist/research/turbulent-flows/atmospheric-turbulence

Symmetry analysis - current work

Symmetry analysis of infinite systems - cannot be performed with computer algebra systems. New symmetries - guessed (not calculated), hence it is possible that the set of symmetries is not complete.

Lie group analysis of equations for pdf's:

- Lie group methods for integro-differential equations
- Infinite system - additional difficulty (look for some recurrence relations)
- Requires tedious algebra - but results are very promising

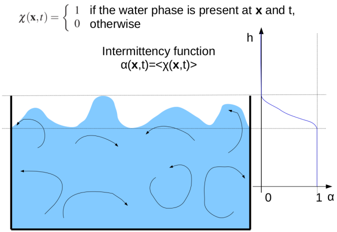
Wacławczyk M., Grebenev V. N., Oberlack M., submitted to Journal of Physics A: Mathematical and Theoretical, 2016

Invariant modelling

Modelling of external intermittency: air-water flow

Wacławczyk M. & Oberlack M., Int. J. Multiphase Flow, 37, 2011

Wacławczyk M. & Wacławczyk T., Int. J. Heat Fluid Flow, 52, 2015



Invariant modelling

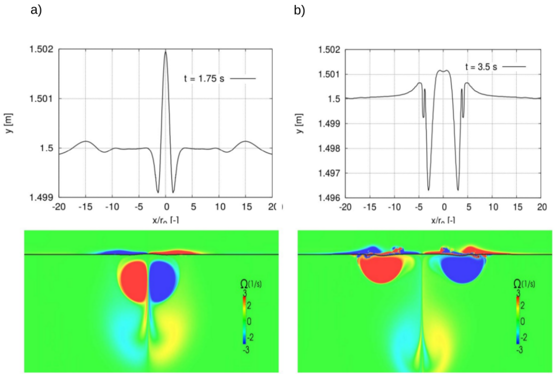
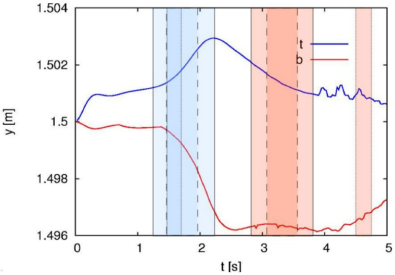


Figure: Two eddies reaching and deforming the surface. [Wacławczyk M. & Wacławczyk T., Int. J. Heat Fluid Flow, 52, 2015]

Invariant modelling



$$\frac{\partial \alpha}{\partial t} + \langle \mathbf{u} \rangle \cdot \nabla \alpha = \underline{\nabla \cdot (D_t \nabla \alpha)} + \underline{\nabla \cdot (C_t \alpha (1-\alpha) \hat{\mathbf{n}})}$$

Diffusion term — Contraction term

Figure: Evolution of the intermittency region. [Wacławczyk M. & Wacławczyk T., Int. J. Heat Fluid Flow, 52, 2015], Development of numerical methods: [Wacławczyk T., J. Comp. Phys. 299 July 2015]

Invariant modelling

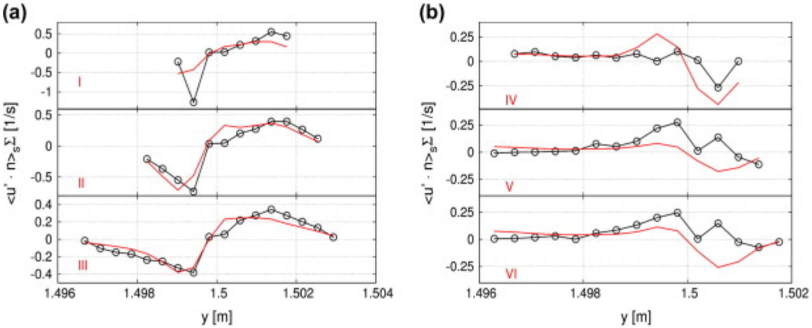


Figure: Evolution of the intermittency region. Symbols: values of $\langle \mathbf{u} \cdot \mathbf{n} \rangle_s \Sigma$ from numerical experiment, Lines: model $\nabla \cdot (D_t \nabla \alpha) + \nabla \cdot (C_t \alpha (1 - \alpha) \mathbf{n})$ [Wacławczyk M. & Wacławczyk T., Int. J. Heat Fluid Flow, 52, 2015]

Conclusions and perspectives

- Lie symmetry analysis provides connection between theory and experimental observations
- New statistical symmetries reflect the fact that a flow may have different character (laminar or turbulent)
- Lie symmetry analysis of LMN hierarchy was performed - first results obtained.

Perspectives

- Invariant solutions for pdf's
- Invariant turbulence modelling - internal intermittency in atmospheric flows, external intermittency

Acknowledgements

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