Sedimentation of a polydisperse non-Brownian suspension

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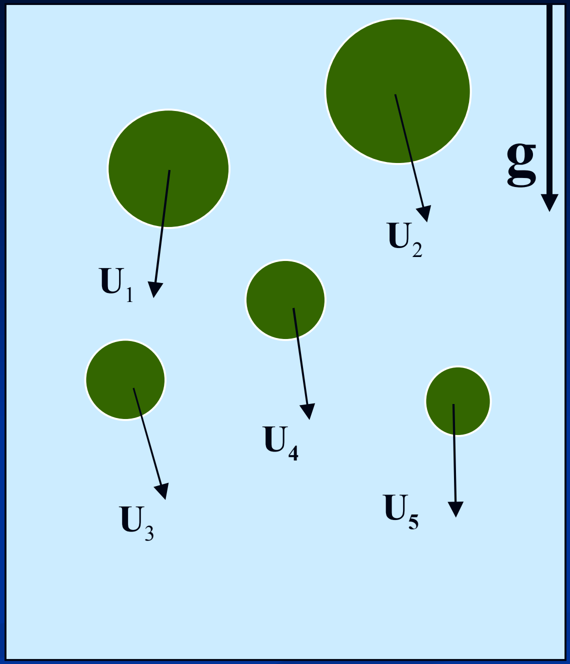
Overview

- Introduction and formulation of the problem
- Discussion of Batchelors theory
- Towards a correct and self-consistent solution
- Results
- Experimental data and discussion

Introduction

- Slow sedimentation of hard spheres (radius app. 5-100 μm) in a viscous (η), non-compressible fluid.
- No Brownian motion, Reynolds numbers small
- Stokesian dynamics, stick boundary conditions on the particles.



Particle velocities are linearly proportional to the external forces acting on them:

$$U_i = \sum_j \mu_{ij}(X) F_j$$

Configuration of all the particles

The mobility matrix- a function of the particle positions.

Scattering expansion in terms of one- and two-particle operators

Formulation of the problem

Stationary sedimentation at low particle concentrations

Two-particle correlaction function

Mean sedimentation velocity

Sedimentation coefficient

Mean sedimentation velocity

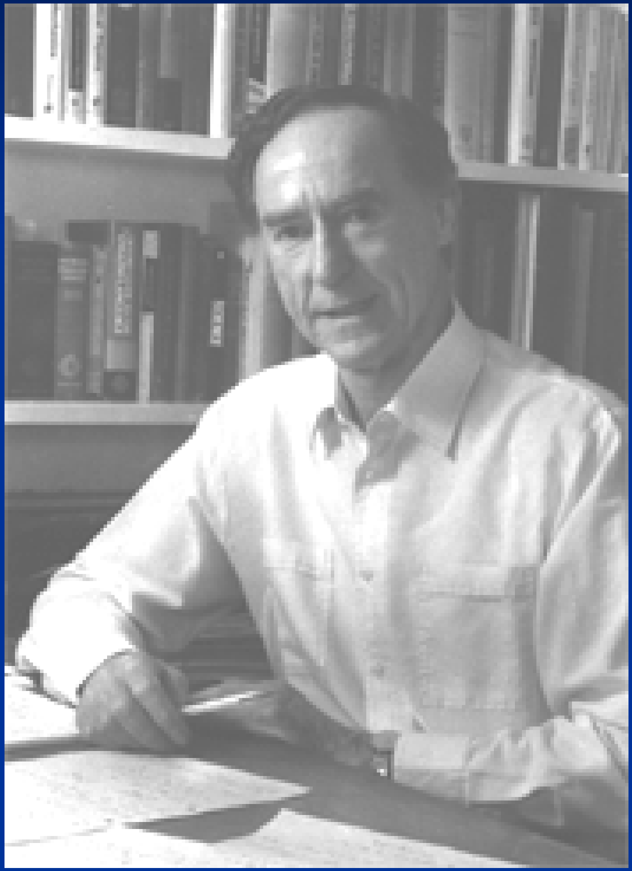
Sedimentation coefficient

$$\frac{\langle U_i \rangle}{U_{0i}} = 1 + \sum_j S_{ij} \phi_j$$

Stokes velocity

Volume fraction of particles with radius a_j and density ρ_j

Discussion of Batchelors theory

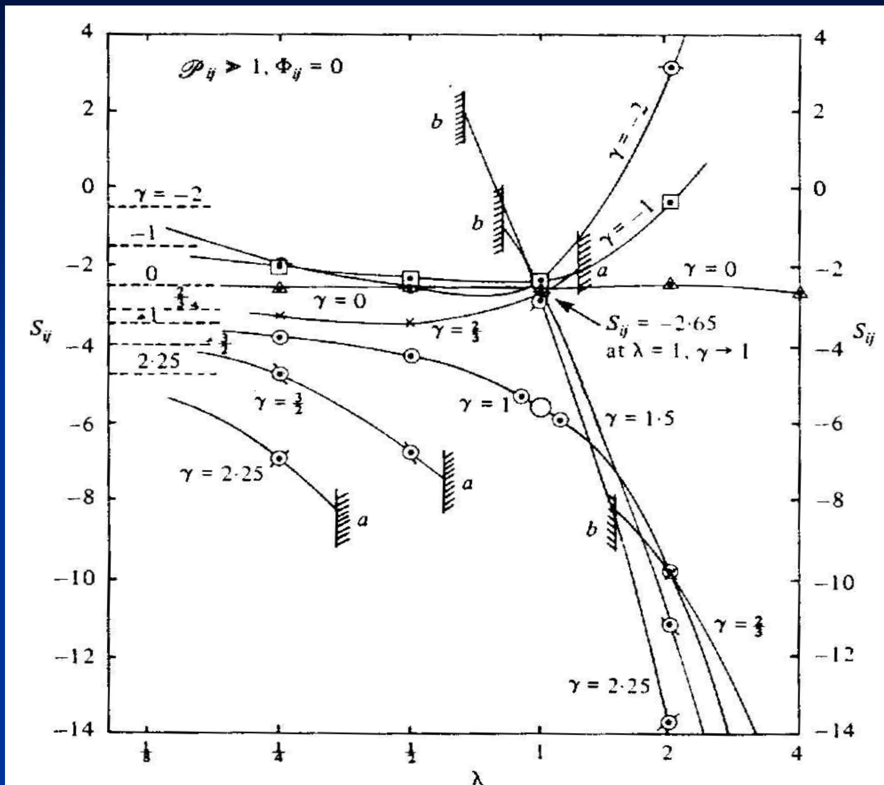


George Keith Batchelor (March 8, 1920 - March 30, 2000)

- Monodisperse suspension (1972)
  - Random distribution of particles
  - S = -6.55
- Polydisperse suspension (1982)
  - Consideration of only two-particle dynamics

$$\frac{\partial g(12)}{\partial t} + \sum_{i=1}^{2} \nabla_i \cdot \{\boldsymbol{U}_i g(12)\} = 0$$

Batchelors results for non-Brownian suspensions

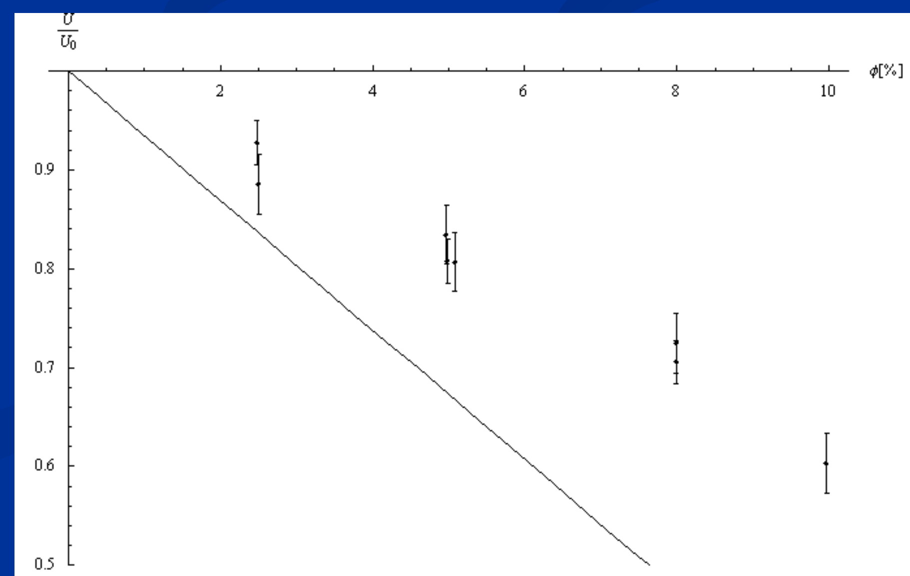


- discontinuities in the form of the distribution function and the value of the sedimentation coefficient when calculating the limit of identical particles,
- due to the existence of closed trajectories the solution of the problem does not exist for all particle sizes and densities.

$$\lambda = \frac{a_j}{a_i}, \quad \gamma = \frac{\rho - \rho_j}{\rho - \rho_i}.$$

Monodisperse suspension:

- S = -6.55
- Experimental results S= -3.9 (Ham&Homsy 1988)



Towards a correct and self-consistent solution

- Liouville Equation:

$$\frac{\partial \rho(\boldsymbol{X};t)}{\partial t} + \sum_{i,j} \nabla_i \cdot \{(\boldsymbol{\mu}_{ij}(\boldsymbol{X})\boldsymbol{F}_j \rho(\boldsymbol{X};t)\} = 0.$$

- Reduced distribution functions

$$n(1,2,\ldots,s;t) = \frac{N!}{(N-s)!} \int d\boldsymbol{R}_{(s+1)} \cdots d\boldsymbol{R}_N \rho(\boldsymbol{X};t).$$

- Cluster expansion of the mobility matrix

$$\mu_{11}(X) = \mu_{11}^{(1)}(1) + \sum_{i=2}^{N} \mu_{11}^{(2)}(1i) + \frac{1}{2!} \sum_{i,j=2}^{N} \mu_{11}^{(3)}(1ij) + \ldots,$$

- BBGKY hierarchy

$$\left(\frac{\partial}{\partial t} + \mathfrak{L}(s)\right) n(s;t) = -\sum_{l=1}^{\infty} \int d(s+l)\mathfrak{L}(s|s+l)n(s+l;t),$$

- Correlation functions

$$
\begin{aligned}
n(1) &= h(1) \\
n(12) &= h(1)h(2) + h(12) \\
n(123) &= h(1)h(2)h(3) + h(12)h(3) + h(13)h(2) + h(23)h(1) + h(123) \\
&\vdots \\
n(\boldsymbol{s}) &= \sum_{\bigsqcup_i \boldsymbol{m}_i = \boldsymbol{s}} \prod_i h(\boldsymbol{m}_i),
\end{aligned}
$$

⇒ Hierarchy equations for h(s)

Cluster expansion of mobility matrix

$$
\begin{aligned}
\frac{\partial h(12;t)}{\partial t} = & -\sum_{i=1,2} \nabla_i \cdot \{\boldsymbol{\mu}^{(1)}_{ii}(i)\boldsymbol{F}_i h(12;t)\} \\
& -\sum_{i,j=1,2} \nabla_i \cdot \{\boldsymbol{\mu}^{(2)}_{ij}(12)\boldsymbol{F}_j (h(12;t) + h(1;t)h(2;t))\} + \ldots .
\end{aligned}
$$

- **Hierarchy contains infinite-range terms and divergent integrals!!**

Solution

- Low concentration limit – truncation of the hierarchy
- Correlations in steady state must be integrable (group property)
- Finite velocity fluctuations (Koch&Shaqfeh 1992)

$$\int d\boldsymbol{r} h(\boldsymbol{r}) = -n.$$

$$h(\boldsymbol{r}) = h^{(s)}(\boldsymbol{r}) + \bar{h}(\boldsymbol{r}\phi^{\beta}),$$

- Describes correlation at the particle size length-scale.
- Equation derived based on the analysis of multi-particle hydrodynamic interactions and the assumption of integrability of correlations.
- Formula for this function and its asymptotic form may be found analytically. Explicit values for arbitrary particle separations and particle sizes/densities are calculated using multipole expansion numerical codes with lubrication corrections.

- The long-range structure scales with the particle volume fraction ($\beta>0$)
- Satisfies the Koch-Shaqfeh criterion for finite particle velocity fluctuations
- Once isotropy is assumed, the long-range structure function does not contribute to the value of the sedimentation coefficient.

Screening on two different length scales

Explicit solution

$$g^s(s,\lambda) = \frac{1}{1-A(s,\lambda)} \exp\left[\int_s^\infty \frac{3(B(s',\lambda)-A(s',\lambda))}{s'(1-A(s',\lambda))} ds'\right].$$

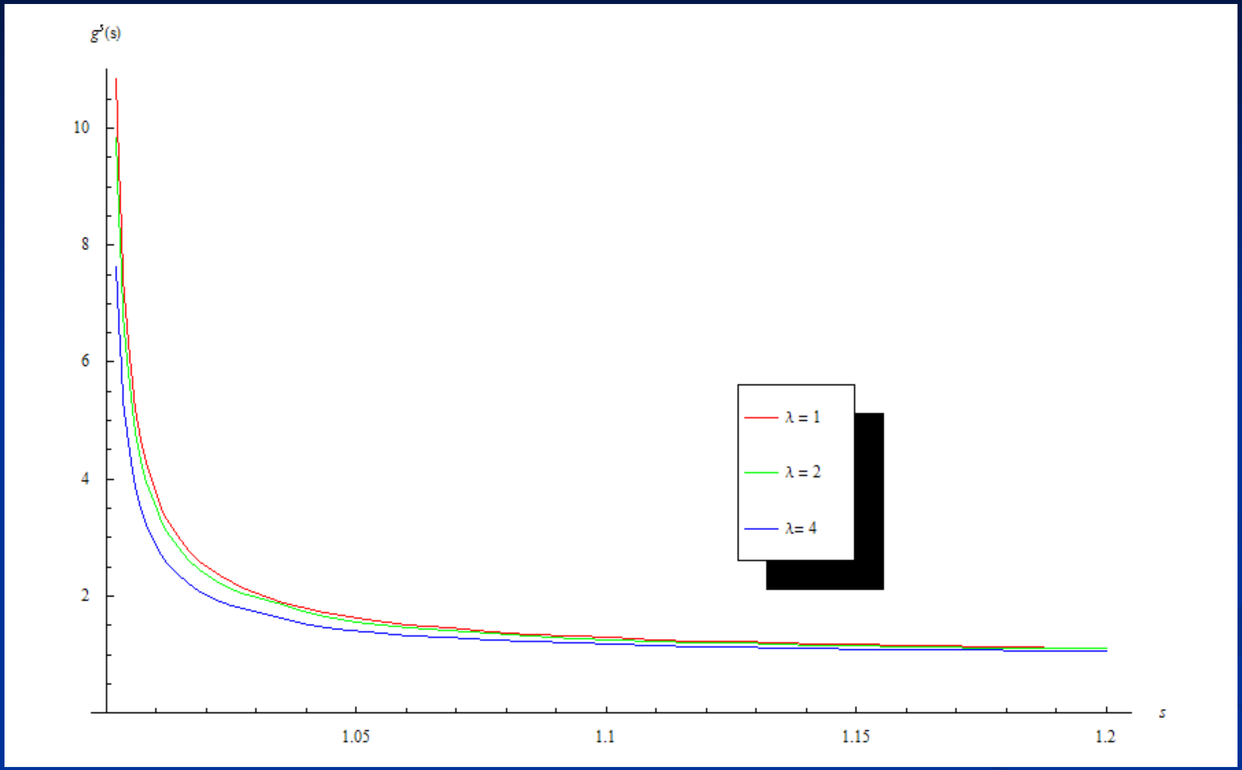
Functions describing two-particle hydrodynamic interactions

$$s = \frac{r}{a_1+a_2}, \qquad \lambda = \frac{a_2}{a_1},$$

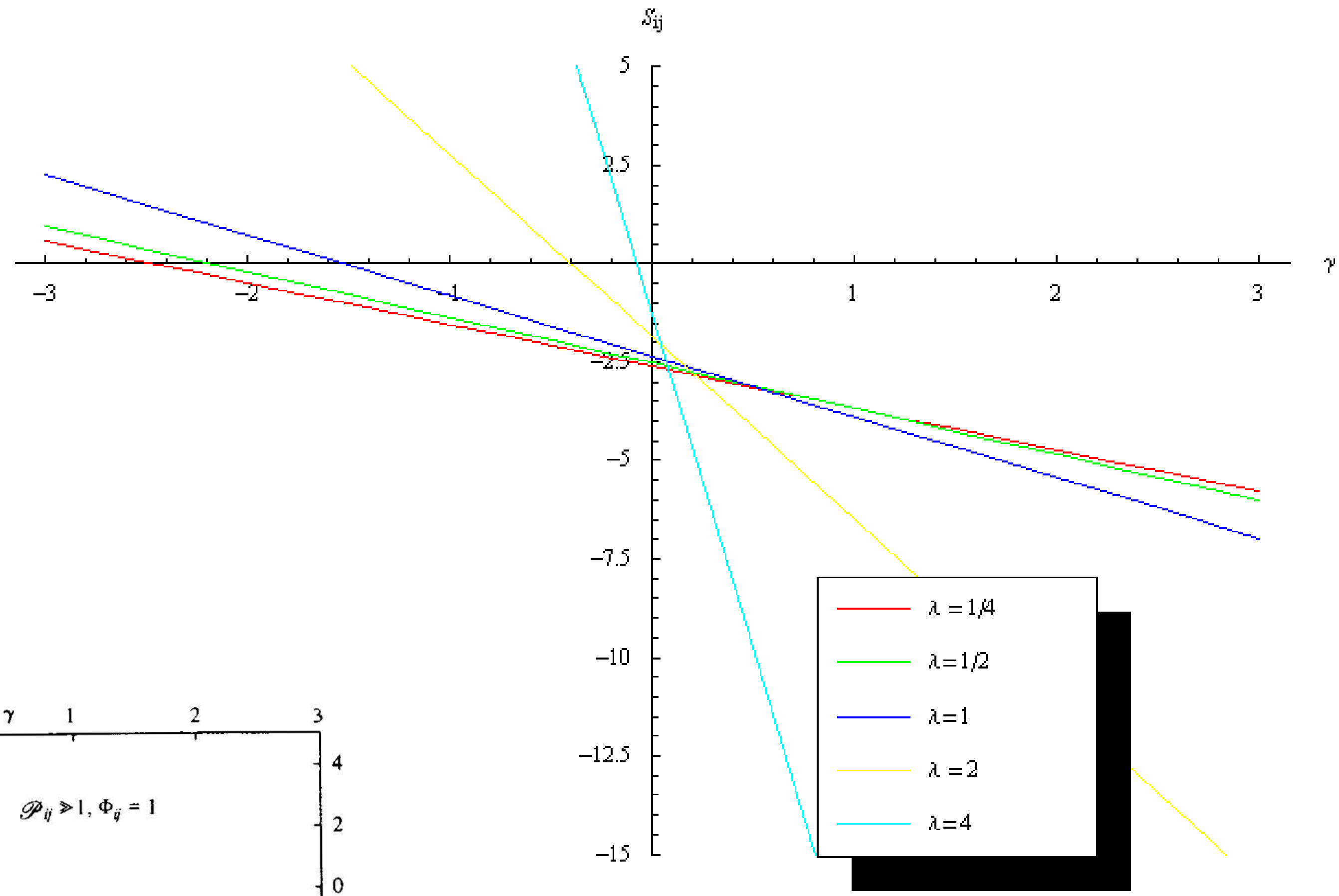
$$n(1)n(2)g^s(r) = n^s(r).$$

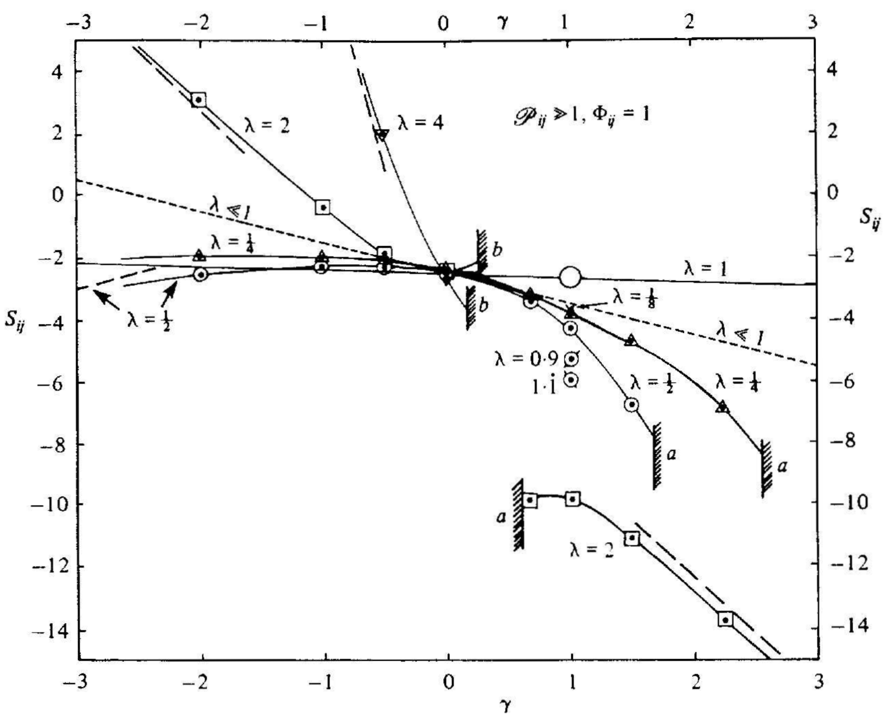
- Assymptotic form for large s: $$g^s(s,\lambda) - 1 \approx 0.1953/s^6.$$

Results



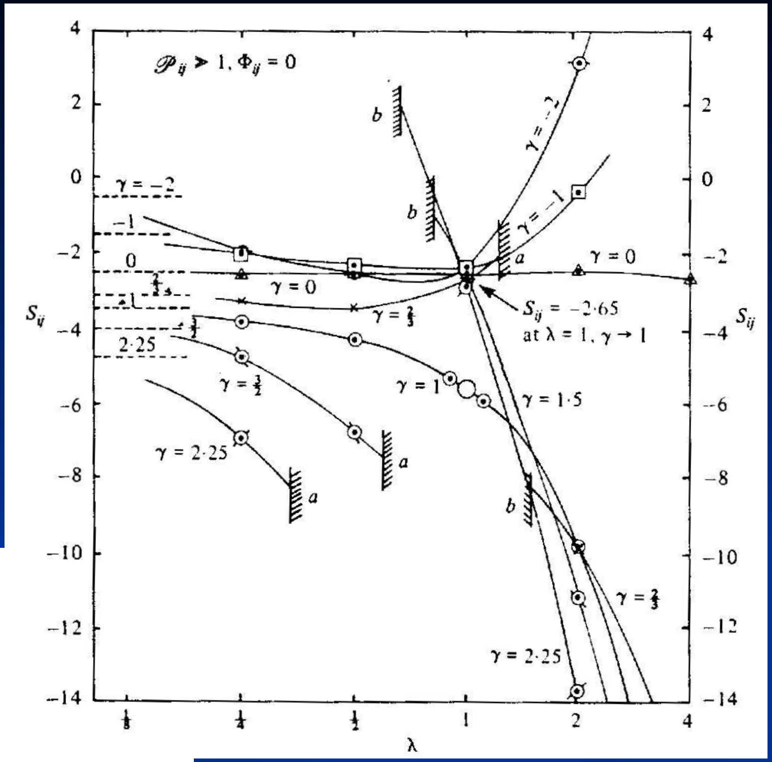
- Excess amount of close pairs of particles
- Function does not depend on the densities of particles
- Isotropic
- Well defined for all particle sizes and densities. The limit of identical particles is continuous.



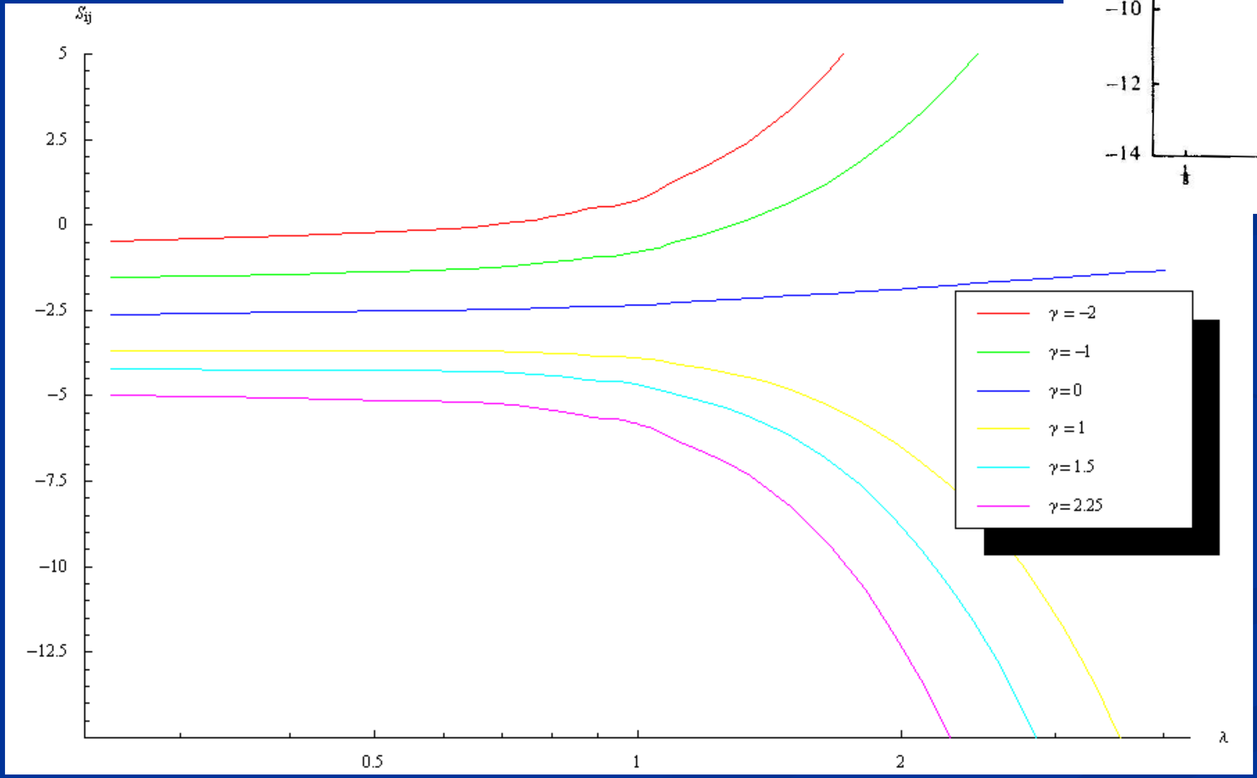


$$S_{ij} = S_{ij}^{(0)}(\lambda) + \gamma S_{ij}^{(1)}(\lambda),$$

$$\gamma = \frac{\rho - \rho_j}{\rho - \rho_i}.$$

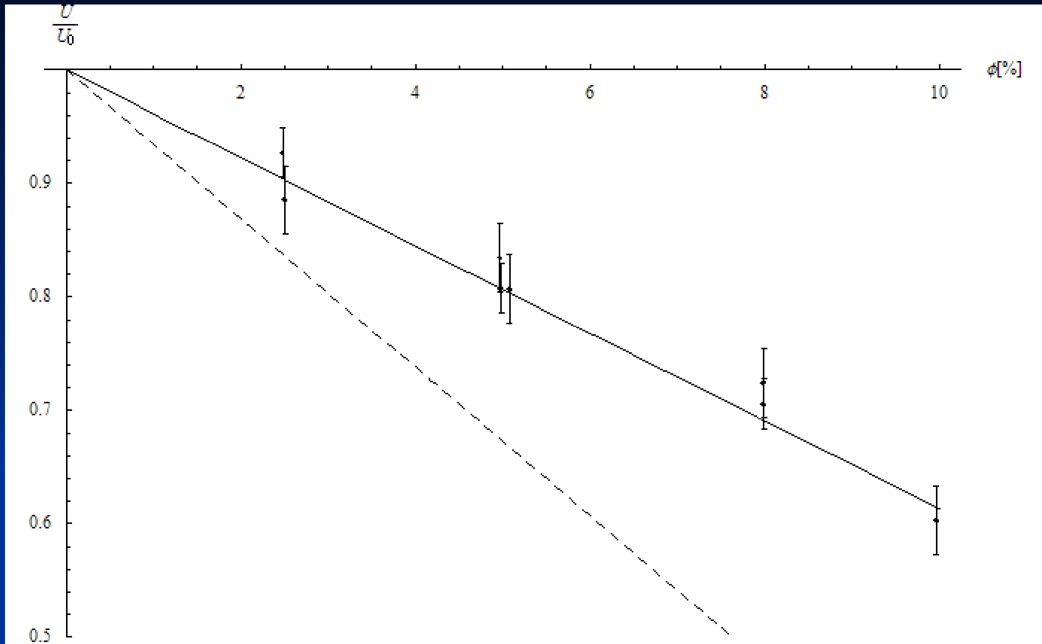


$\mathscr{P}_{ij} \gg 1, \Phi_{ij} = 0$
S_{ij}
$\gamma = -2$
$\gamma = -1$
$\gamma = 0$
$\gamma = \frac{2}{3}$
$\gamma = 1$
$\gamma = 1 \cdot 5$
$\gamma = \frac{3}{2}$
$\gamma = 2 \cdot 25$
$S_{ij} = -2 \cdot 65$ at $\lambda = 1, \gamma \to 1$
a
b
λ



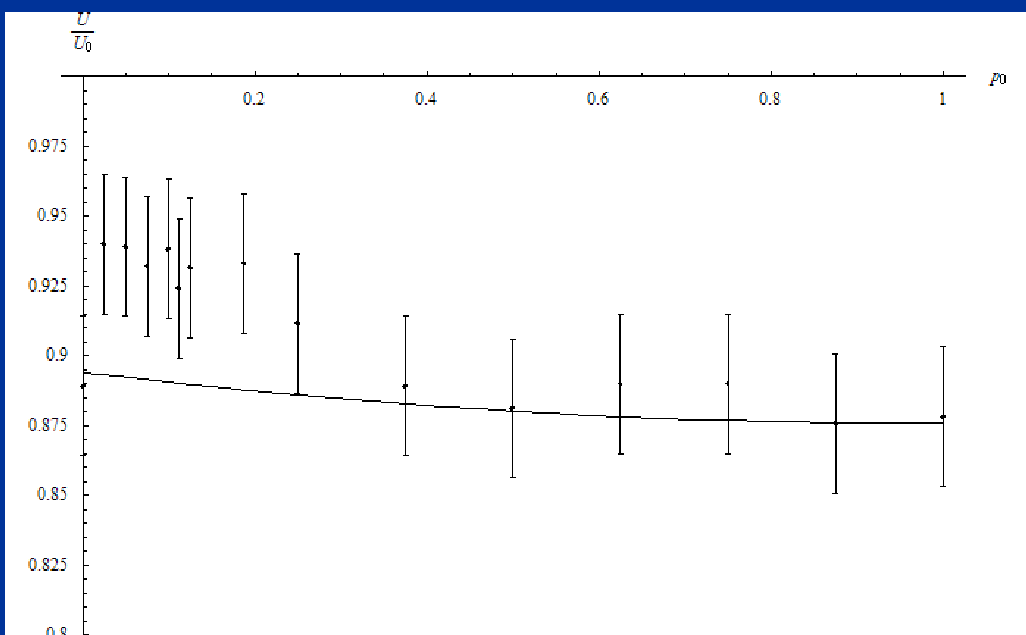
S_{ij}
$\gamma = -2$
$\gamma = -1$
$\gamma = 0$
$\gamma = 1$
$\gamma = 1.5$
$\gamma = 2.25$
λ

Comparison do experiment



Monodisperse suspension:

- S = -3.87
- Batchelor: S = -6.55
- Experimental results S= -3.9 (Ham&Homsy 1988)



Polydisperse suspension

- Suspension of partcles with different radii and densities (D.Bruneau et al. 1990)
- Batchelors theory not valid.

$$p_0 = \frac{\phi_m}{\phi} = \frac{\phi - \phi_s}{\phi},$$

$$\phi = \phi_s + \phi_m,$$

Discussion

- Local formulation of the problem – well defined in the thermodynamic limit
- Multi-particle dynamics
- Self-consistent
- Comparison to experimental data very promising.
- Practical