Ergodic and chaotic behaviour of partial differential equations and applications to biological models

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IPPT Seminar on Mechanics
6.2.2023

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When can we say that a system is chaotic?

Answer: A system is chaotic if it has a simple and **deteministic** description, but it behavies in a complicated and **”random”** way.

R.R. Math. Meth. Appl. Sci. **27** (2004), 723–738.
R.R. Discrete and Continuous Dynamical Systems **35** (2015), 757–770.

1. **Macroscopic approach:** The existence of global attractors with complicated structure (*strange attractors*).

2. **Microscopic approach:** The existence of trajectories which are unstable, turbulent or dense in the phase space; topological mixing.

3. **Stochastic approach:** The existence of invariant measures having strong ergodic and analytic properties.

X - metric space

$\{S_t\}_{t \geq 0}$ - semiflow on X

a) $S_t : X \to X$, for $t \geq 0$,

b) $S_0 = \mathrm{Id}$, $S_{t+s} = S_t \circ S_s$, $t, s \geq 0$,

c) $S_t(x)$ is a continuous function of (t, x) .

Example:

$$x'(t) = f(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n$$

$$S_t(x_0) = x(t).$$

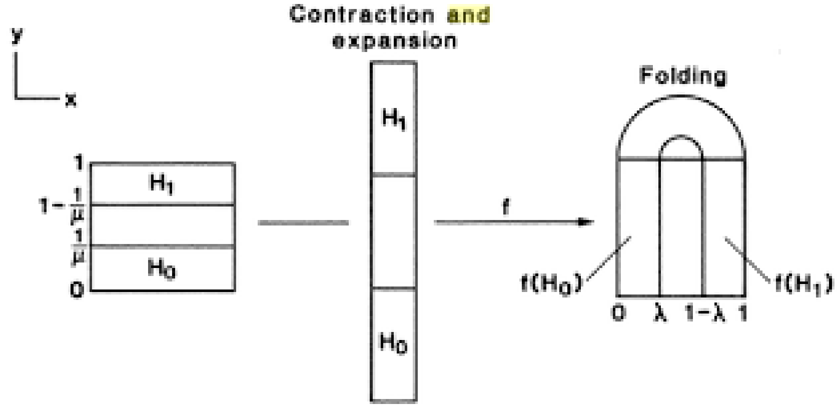
Iterates of a transformation $S \colon X \to X$ (discrete time semiflow).

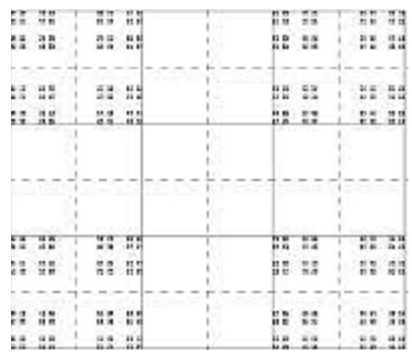
Macroscopic approach – strange properties of attractors of a semiflow.

Attractor – a compact set A for which there is an open set U such that:
$A \subset U$,
$S_t(\mathrm{cl}\, U) \subset U$ for $t > 0$,
$A = \bigcap_{t>0} S_t(U)$.

An attractor is called a *strange attractor* if it is a fractal set, i.e. if it has different topological and Hausdorff dimensions.

Smale horseshoe





Dynamics (on the vertical part of Sh) similar to the shift transformation on Cantor set:

$$C = \prod_{n\in\mathbb{N}} \{0,2\}_n, \quad (Tx)_n = x_{n+1}.$$

$$C = \{a \in [0,1] : a = \sum_{n=1}^{\infty} a_n 3^{-n}, a_n \in \{0,2\}\}$$

C is a strange set and trajectories expands: $|T^n(x)-T^n(y)| = 3^n|x-y|$ for $n = 1,\ldots,n(x,y)$, $n(x,y)$ is large if $|x-y|$ is small.

Examples: the logistic map $T(x) = 4x(1-x)$, the Smale's horseshoe, the Lorenz' flow and $T : H(\mathbb{C}) \to H(\mathbb{C})$, $Tf = f'$.

Microscopic approach:

Chaos in the sense of Auslander-Yorke:

(a) each trajectory is unstable,

(b) there exists a dense trajectory.

Chaos in the sense of Devaney: **(b)** + the set of periodic points is dense in X

Topological mixing: for any two open subsets U, V of X there exists $t_0 > 0$ such that

$$S_t(U) \cap V \neq \emptyset \quad \text{for } t \geq t_0.$$

Turbulent trajectory (Lasota-Yorke): no periodic points in the closure of the trajectory.

Turbulent trajectory (Bass):

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T S_t(x)\,dt = x_0$$

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T [S_t(x) - x_0][S_{t+\tau}(x) - x_0]\,dt = \gamma(\tau)$$

$$\gamma(0) \not\equiv 0, \qquad \lim_{\tau\to\infty} \gamma(\tau) = 0.$$

Stochastic approach: probabilistic properties of dynamical systems

μ - probability measure on the σ-algebra $\mathcal{B}(X)$ of Borel subsets of X.

μ invariant w.r. $\{S_t\}_{t\geq 0}$ if for $A \in \mathcal{B}(X)$, $t > 0$

$$\mu(S_t^{-1}(A)) = \mu(A).$$

μ is ergodic if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(S_t(x))\,dt = \int_X f(x)\,\mu(dx) \quad \mu-\text{a.e.}$$

$f = 1_A \Rightarrow$ (mean time of visiting A) $= \mu(A)$

μ is mixing if $\lim_{t\to\infty} \mu(S_t^{-1}(A)\cap B) = \mu(A)\mu(B)$.

$$\lim_{t\to\infty} P(S_t(x) \in A \,|\, x \in B) = \mu(A).$$

μ is exact if $\lim_{t\to\infty} \mu(S_t(A)) = 1$ for $\mu(A) > 0$.

exactness $\Rightarrow$ mixing $\Rightarrow$ ergodicity.

(P) supp $\mu = X$ (positivity on open sets)

Mixing + (P) $\Rightarrow$ chaos (A-Y)

Ergodicity + (P) $\Rightarrow$ a.a. traj. are dense in X

Mixing + (P) $\Rightarrow$ unstability of all trajectories

Mixing + (P) $\Rightarrow$ topological mixing

Mixing + (P) + exist. of the 2-moment of μ
$\Rightarrow$ almost all trajectories are turbulent (Bass)

If X is a finite dimesional space, then ergodic properties of transformations and semiflows on X can be successfully investigated by means of Frobenius–Perron operators:

A. Lasota and M.C. Mackey, *Chaos, Fractals and Noise. Stochastic Aspects of Dynamics*, 1994.

(X, Σ, m) a σ-finite measure space, $S : X \to X$ a measurable transformation s.t.
if $m(A) = 0$, then $m(S^{-1}(A)) = 0$.
The operator $P : L^1(X) \to L^1(X)$ s.t.

$$\int_A Pf(x)\, m(dx) = \int_{S^{-1}(A)} f(x)\, m(dx)$$

for all $f \in L^1$ and $A \in \Sigma$ is called *Frobenius–Perron operator* for S.

μ- a probability measure $\mu < m$,
Let $f_* = \frac{d\mu}{dm}$ be a density of μ.

μ is invariant under $S \Leftrightarrow Pf_* = f_*$ for $t > 0$.

P F–P operator to the system (X, Σ, μ, S):

S	P
ergodic	$\mathbf{1}_X$ is a unique invariant density of P
mixing	w-$\lim_{t\to\infty} P^t f = \mathbf{1}_X$ for each $f \in D$
exact	$\lim_{t\to\infty} P^t f = \mathbf{1}_X$ for each $f \in D$

Invariant measure for p.d.e.

A.Lasota, Rend. Sem. Math. Univ. Padova **61** (1979), 40-48.

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u$$

$$S_t v(x) = u(t,x); \quad S_t v(x) = e^{\lambda t} v(e^{-t}x).$$

$$X = \{v \in C[0,1] : \quad v(0) = 0\}.$$

Theorem 1 *If $\lambda \geq 2$ then there is a continuous ergodic measure μ on X invariant w.r. $\{S_t\}$.*

(continuous $\mu(Per) = 0$)

Lemma 1 *Let* $S : X \to X$ *be a continuous map. If for some nonempty compact disjoint sets* A *and* B *we have*

$$A \cup B \subset S(A) \cap S(B),$$

then there exists a turbulent trajectory (L-Y).

Lemma 2 *(Bogoluboff-Kriloff). Let* $S : X \to X$ *be a continuous map of a compact metric space. Then there exists a probability Borel measure* μ *invariant and ergodic w.r.* S.

Invariant measure for p.d.e.

R.R. (1985), (1988).

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(g(x)u) = f(x,u)$$

$$u(0,x) = v(x), \text{ for } \ x \in [0,1].$$

$g(0) = 0$, $g(x) > 0$ for $x \in (0,1]$,

$f(0,u_0) = 0$, $\frac{\partial f}{\partial u}(0,u_*) > 0$.

$$\{V_t\}_{t\geq 0}, \qquad V_t v(x) = u(t,x)$$

$$X = \{v \in C[0,1] : \quad v(0) = u_*\}.$$

Theorem 2 *There exists a probability measure μ which satisfies:*

(a) *μ is invariant w.r. to $\{V_t\}$,*

(b) *μ is exact,*

(c) $\operatorname{supp}\mu = X$,

(d) $\int_X \|v^2\|\,\mu(dv) < \infty$.

Moreover, we proved that the set of periodic points of $\{V_t\}$ is dense in X.

Draft of the proof:

$\{T_t\}$ left-side shift on

$$Y = \{\varphi : [0, \infty) \to \mathbb{R}\}$$

$$(T_t\varphi)(s) = \varphi(s+t) \text{ for } t, s \geq 0.$$

1. Semiflows (V_t, X) and (T_t, Y) are conjugated (isomorphic), a.e. the map $Q : X \to Y$, given by $Qv_0(t) = v(t, 1)$ is a homeomorphism from X onto $Q(X) \subset Y$ and

$$Q \circ S_t = T_t \circ Q, \quad \text{for } t \geq 0.$$

2. Let $\xi_t = e^t w_{e^{-2t}}$, where w_t, $t \geq 0$ is the Wiener process. Then ξ_t is a stationary Gaussian process with continuous trajectories. Let

$$m(A) = P\{\omega : \xi.(\omega) \in A\} \quad A \in \mathcal{B}(Y).$$

The measure m is invariant under $\{T_t\}$ and $m(Q(X)) = 1$. The measure $\nu(A) = m(Q(A))$ is invariant under $\{S_t\}$.

3. Exactness. (T_t, Y) is exact iff the σ-algebra $\mathcal{A}_\infty = \bigcap_{t>0} T_t^{-1}(\mathcal{B}(Y))$ contains only sets of measure zero or one.

Let $\mathcal{F}_{\leq t}$ be the σ-algebra generated by $w_s, s \leq t$. Then σ-algebra $T_t^{-1}(\mathcal{B}(Y))$ is generated by ξ_s, $s \geq t$, therefore, $T_t^{-1}(\mathcal{B}(Y)) = \mathcal{F}_{[0,e^{-2t}]}$. Thus $\mathcal{A}_\infty = \bigcap_{r>0} \mathcal{F}_{[0,r]}$ and according to Blumenthal's zero-one law $\mathcal{A}_\infty$ contains only sets of measure zero or one.

4. Positivity of ν on open sets can be obtained from the following property of Wiener process:

$$\text{Prob}\{\omega : f(t) < w_t(\omega) < g(t) \text{ for } t \in [a,b]\} > 0.$$

for continuous functions $f < g$ and $0 < a < b$.

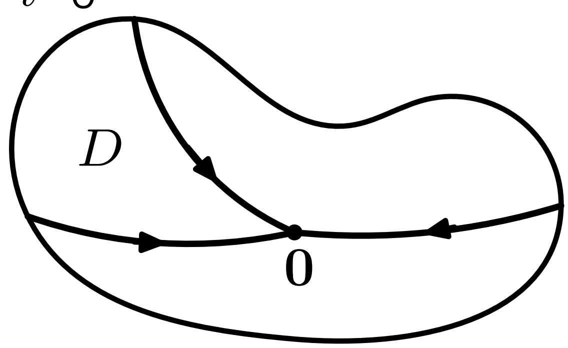
Invariant measure for p.d.e.

$$\frac{\partial u}{\partial t} + a_1(x)\frac{\partial u}{\partial x_1} + \cdots + a_d(x)\frac{\partial u}{\partial x_d} = f(x,u) \quad (\star)$$

$x \in D$, D diffeomorphic with $B(\mathbf{0},1)$, $\mathbf{0} \in \operatorname{Int} D$.
$a\colon D \to \mathbb{R}^d$ is C^1 function, $a(\mathbf{0}) = \mathbf{0}$.

$$x'(t) = -a(x(t)), \quad x(0) = x_0 \in D, \quad \pi_t x_0 = x(t).$$

Assume that if $x_0 \in D$ then $\pi_t x_0 \in D$ for $t \geq 0$ and $\lim_{t\to\infty} \pi_t x_0 = \mathbf{0}$.



There exists $u_0^{\mathbf{0}} \in \mathbb{R}$ such that $f(\mathbf{0}, u_0^{\mathbf{0}}) = 0$ and $\dfrac{\partial f}{\partial u}(\mathbf{0}, u_0^{\mathbf{0}}) > 0$;

there exist $u_-^{\mathbf{0}} \in [-\infty, u_0^{\mathbf{0}})$ and $u_+^{\mathbf{0}} \in (u_0^{\mathbf{0}}, \infty]$ such that $f(\mathbf{0}, u) < 0$ for $u \in (u_-^{\mathbf{0}}, u_0^{\mathbf{0}})$ and $f(\mathbf{0}, u) > 0$ for $u \in (u_0^{\mathbf{0}}, u_+^{\mathbf{0}})$;

if $u_-^{\mathbf{0}} > -\infty$, then $f(\mathbf{0}, u_-^{\mathbf{0}}) = 0$, $\dfrac{\partial f}{\partial u}(\mathbf{0}, u_-^{\mathbf{0}}) < 0$;
if $u_+^{\mathbf{0}} < \infty$, then $f(\mathbf{0}, u_+^{\mathbf{0}}) = 0$, $\dfrac{\partial f}{\partial u}(\mathbf{0}, u_+^{\mathbf{0}}) < 0$;

Lemma 3 *If* $u^{\mathbf{0}}_{-} > -\infty$*, then there exists a unique stationary solution* $u_{-}\colon D \to \mathbb{R}$ *of* $(\star)$ *such that* $u_{-}(\mathbf{0}) = u^{\mathbf{0}}_{-}$*. Analogously if* $u^{\mathbf{0}}_{+} < \infty$*, then ...* $u_{+}(\mathbf{0}) = u^{\mathbf{0}}_{+}$*.*

We set $u_{-} \equiv -\infty$ if $u^{\mathbf{0}}_{-} = -\infty$ and $u_{+} \equiv \infty$ if $u^{\mathbf{0}}_{+} = \infty$. Let

$$V_0 = \{v \in C(D)\colon\ u_{-}(x) < v(x) < u_{+}(x) \text{ for } x \in D \text{ and } v(\mathbf{0}) = u^{\mathbf{0}}_{0}\}.$$

If $v(x) = u(0, x)$, $v \in V_0$, then $S_t v = u(t, \cdot) \in V_0$.

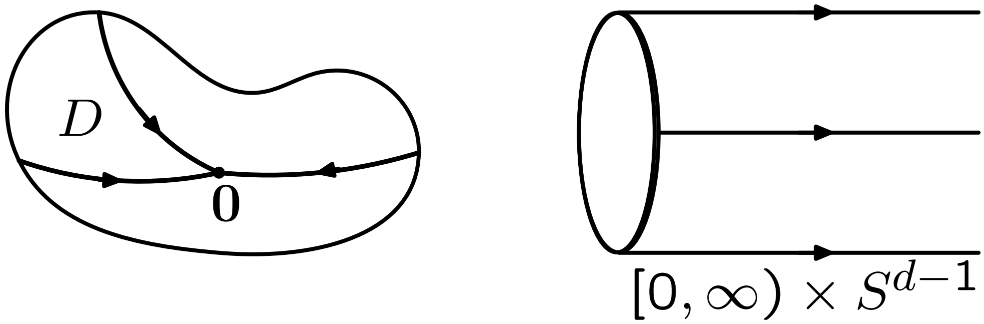
Theorem 3 *There exists a measure* m *supported on* V_0 *s.t.* $(V_0, \mathcal{B}(V_0), m; S_t)$ *is exact.*

Idea of the proof.

1. We replace the Wiener process by *Lévy d-parameter Brownian motion*, which is a Gaussian random field $(\xi(x))$ on $\mathbb{R}^d$ with zero mean and covariance function

$$c(x,y) = \mathsf{E}\xi(x)\xi(y) = \frac{1}{2}(|x| + |y| - |x-y|).$$

2. We set $W = C([0,\infty) \times S^{d-1})$ and define a semiflow $(T_t)_{t\geq 0}$ on the space W by $T_t w(s,y) = w(s+t,y)$, $s,t \geq 0$ and $y \in S^{d-1}$.



3. Starting from the random field (ξ_x) we construct an invariant measure μ on the space W invariant w.r. to $(T_t)_{t\geq 0}$ supported on W.
4. We show that systems $(V_0, \mathcal{B}(V_0), m; S_t)$ and $(W, \mathcal{B}(W), \mu; T_t)$ are isomorphic.

If $f(x, u_0^{\mathbf{0}}) \equiv 0$, then we can consider a semiflow (S_t) restricted to the space

$$V_0^+ = \{v \in V_0 \colon\ u_0^{\mathbf{0}} \leq v(x) < u_+(x) \text{ for } x \in D\}.$$

Theorem 4 *There exists a measure m supported on V_0^+ s.t. $(V_0^+, \mathcal{B}(V_0^+), m; S_t)$ is exact.*

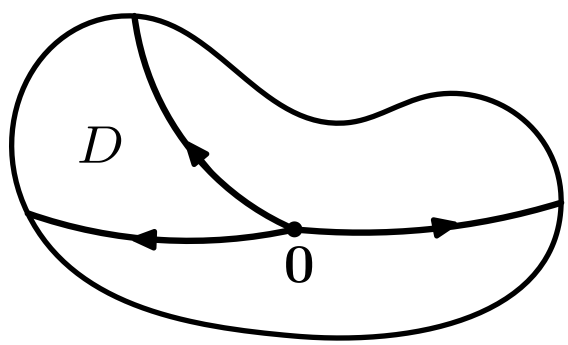
Equation in a divergence form

$$\frac{\partial u}{\partial t}(t,x) + \mathrm{div}(a(x)u(t,x)) = g(x,u(t,x)), \quad (\star\star)$$

where $\mathrm{div}(a(x)u(t,x)) = \sum_{i=1}^{d} \frac{\partial(a_i(x)u(t,x))}{\partial x_i}$.

Eq. $(\star\star)$ describes the growth of a population. Any individual is characterized by a vector x which changes according to Eq. $x' = a(x)$.

$g(x,u)$ – is a growth rate, $u(t,x)$ is the population distribution w.r. to x.



Eq. ($\star\star$) can be written in the form ($\star$) with

$$f(x,u) = g(x,u) - u \,\mathsf{div}\, a(x).$$

If $g(\mathbf{0}, u_0^{\mathbf{0}}) = u_0^{\mathbf{0}} \,\mathsf{div}\, a(\mathbf{0})$, $\dfrac{\partial g}{\partial u}(\mathbf{0}, u_0^{\mathbf{0}}) > \mathsf{div}\, a(\mathbf{0})$

then $f(\mathbf{0}, u_0^{\mathbf{0}}) = 0$, $\dfrac{\partial f}{\partial u}(\mathbf{0}, u_0^{\mathbf{0}}) > 0$.

Space structure population with logistic growth

We consider a population in which individuals disperse according to equation $x'(t) = a(x)$ and then leave the set D.
Let $g(x,u) = \lambda(1 - u/K(x))u$ be the growth rate. Then the solution of Eq. $(\star\star)$ is the space distribution of the number of individuals in D.

Here $u_0 \equiv 0$. If $\lambda > \operatorname{div} a(\mathbf{0})$ and if

$$u_+^{\mathbf{0}} = K(0)\Big(1 - \lambda^{-1} \operatorname{div} a(\mathbf{0})\Big),$$

then there is a stationary solution u_+ of Eq. $(\star\star)$ such that $u_+(\mathbf{0}) = u_+^{\mathbf{0}}$.

According to Theorem 4 there exists a measure m supported on V_0^+ s.t. $(V_0^+, \mathcal{B}(V_0^+), m; S_t)$ is exact, where

$$V_0^+ = \{v \in C(D)\colon\ 0 \leq v(x) < u_+(x) \text{ for } x \in D,\ v(\mathbf{0}) = 0\}.$$

Flow with jumps

We consider a movement of particles with velocity $a(x)$ in the domain D. When a particle reaches the boundary ∂D it jumps to the set D and chooses its new position according to the distribution $v(t,x)$ of other particles.

$$\frac{\partial v}{\partial t}(t,x) + \mathrm{div}(a(x)v(t,x)) = \left(\int_{\partial D} a(y) \cdot n(y) v(t,y)\, \sigma(dy) \right) v(t,x). \qquad (\clubsuit)$$

Here $n(y)$ is the outward pointing unit normal to ∂D at y; $\sigma(dy)$ is the surface measure on ∂D; and the term between large brackets is the total flow across the boundary ∂D.

If V_0^d the subset of V_0^+ consisting of probability densities, then there exists a measure m supported on V_0^d s.t. the semiflow $(V_0^d, \mathcal{B}(V_0^d), m_d; P_t)$ is exact.

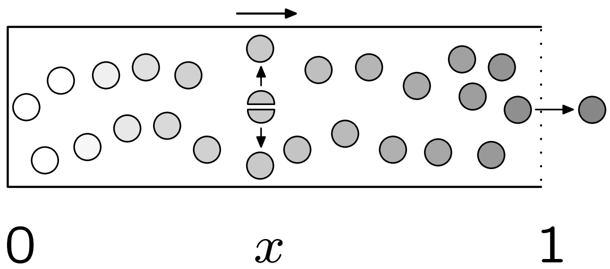
Proof. Let $u(t,x)$ is a positive solution of Eq. $(\star)$ with

$$f(x,u) = (\lambda - \operatorname{div} a(x))u.$$

If $U(t) = \int_D u(t,x)\,dx$, then $v(t,x) = u(t,x)/U(t)$ is a solution of $(\clubsuit)$.

Blood cell production system

R.R. Chaos: An Interdisciplinary Journal of Nonlinear Science, **19** (2009), 043112, 1–6.



The evolution of maturity of blood cells in the bone marrow (precursors of any blood cells). $x' = g(x)$, x-maturity of a cell.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(g(x)u) = g(1)u(t,1)u(t,x) \qquad (1)$$

Corollary 1 *The semiflow* $\{U_t\}_{t\geq 0}$ *generated by (1) is topologically mixing, chaotic in the sense of Devaney and turbulent in the sense of Bass.*

Size-structured cell population model

R.R. J. Math. Anal. Appl. **393** (2012), 151–165.

x - cell size, $x' = g(x)$

$b(x)$, $d(x)$ - birth i death coefficients,

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}(g(x)u(t,x)) = -\mu(x)u(t,x) + 4b(2x)u(t,2x),$$

where $\mu(x) = d(x) + b(x)$.

Theorem on stability. If $g(2x) \neq 2g(x)$ at least for one x, then there exist $\lambda \in \mathbb{R}$ and a density v^* s.t.

$$\lim_{t \to \infty} e^{-\lambda t} u(t,x) = C(u(0,x)) v^*(x).$$

Question: What can happen when $g(x) = x$?

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = au(t,x) + bu(t,2x),$$

El Mourchid, G. Metafune, A. Rhandi and J. Voigt, *J. Math. Anal. Appl.* **339** (2008), 918–924. : $u(t,2x)\mathbf{1}_{[0,1/2]}(x)$

Theorem 5 *If* $2^a b \log 2 < e^{-1}$ *and if we choose the space* X *in a "proper way" then there exists a probability measure* μ *which satisfies:*

(a) μ *is invariant w.r. to* $\{U_t\}$*,*

(b) μ *is mixing,*

(c) $\operatorname{supp} \mu = X$*,*

(d) $\int_X \|v^2\| \, \mu(dv) < \infty$.

The set of periodic points of $\{U_t\}$ is dense.

Corollary 2 *The semiflow* $\{U_t\}_{t \geq 0}$ *is topologically mixing, chaotic in the sense of Devaney and turbulent in the sense of Bass.*

Thank you!