APPLIED MECHANICS SERIES

Jan Rychlewski

“C E I I I N O S S S T T U V”
Mathematical Structure of Elastic Bodies

Translated and provided with extended commentary
by Andrzej Ziółkowski

Institute of Fundamental Technological Research
Polish Academy of Sciences

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P O L I S H A C A D E M Y O F S C I E N C E S

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Mathematical Structure of Elastic Bodies

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WARSAW 2023

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Preface

The insightful essay of Professor Jan Rychlewski presented below was devised during his stay as a visiting professor at the Institute of Mechanical Problems in Moscow in 1983. It is devoted to the spectral analysis of symmetric fourth-order tensors – stiffness $\mathbf{C}$ and compliance $\mathbf{S}$, describing the properties of linear elastic anisotropic materials. The presented general analysis has been divided into individual classes of material anisotropies, such as: full (triclinic) anisotropy, monoclinic, orthotropic, tetragonal, trigonal, cubic anisotropy, as well as transversal and full isotropy.

The methodology of spectral analysis has been described in detail in linear algebra textbooks (see A. Kiełbasiński, H. Schwetlick, *Numeryczna algebra liniowa* [Numerical Linear Algebra], WTN, 1992, and T. Kaczorek, *Wektory i macierze w automatyce i elektrotechnice* [Vectors and Matrices in Automation and Electrical Engineering], WTN, 1998), especially regarding aspects of numerical calculations. The eigenvalue problem for a symmetric matrix $\mathbf{A}$ taking the form $\mathbf{A}\mathbf{u}_j = \lambda_j \mathbf{u}_j$ requires determination of two sets of mutually coupled objects: scalar eigenvalues λ_j and eigenvectors $\mathbf{u}_j$. The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ can be brought to the form of a characteristic polynomial, whose roots determine scalar eigenvalues λ_j. The second important issue of the eigenvalue problem is to express the matrix $\mathbf{A}$ as a spectral decomposition with respect to the eigenvalues: $\mathbf{A} = \mathbf{u}\boldsymbol{\Lambda}\mathbf{u}^T = \Sigma \mathbf{u}_j \lambda_j \mathbf{u}_j^T$. For non-symmetric matrices, a similar decomposition exists with respect to the right and left eigenvectors: $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^T = \Sigma \mathbf{u}_j \lambda_j \mathbf{v}_j^T$, where the following relations are valid $\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{u}_j$, $\mathbf{u}_j^T \mathbf{A} = \lambda_j \mathbf{v}_j^T$. Numerical analysis of eigenproblems for symmetric and non-symmetric matrices $\mathbf{A}$ has been discussed in detail in a monograph, see A. Kiełbasiński, H. Schwetlick.

Passing from general spectral analysis in matrix calculus to discussion of linear mechanics of elastic bodies, the states of strain and stress are expressed in terms of second-order tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, and Hooke's law $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, with the symmetric fourth-order elastic stiffness tensor $\mathbf{C}$. The parametric form of this tensor and number of linearly independent parameters characterizing it depend on the anisotropy class of the material. The spectral analysis of the tensor $\mathbf{C}$ carried out in the work of J. Rychlewski for specific classes of anisotropy showed that the tensor $\mathbf{C}$ can be represented in a spectral form as the sum of maximum six products of Kelvin moduli λ_i, $i = 1, 2, ..., 6$ and corresponding to them eigentensors $\mathbf{C} = \lambda_1\omega_1 \otimes \omega_1 + \lambda_2\omega_2 \otimes \omega_2 + \lambda_3\omega_3 \otimes \omega_3 + \lambda_4\omega_4 \otimes \omega_4 + \lambda_5\omega_5 \otimes \omega_5 + \lambda_6\omega_6 \otimes \omega_6$. In the report explicit formulas were presented for true (Kelvin) moduli of elasticity and corresponding to them forms of eigentensors for different classes of material anisotropy. The problem of determining tensor $\mathbf{C}$ symmetry groups for different anisotropy classes was also formulated.

J. Rychlewski's monograph is a ground-breaking work. The subsequent known works on the same issue, i.e. the determination of the true elastic moduli and the corresponding eigentensors by M.M. Mehrabadi, S.C. Cowin, Eigentensors of linear anisotropic materials, *The Quarterly Journal of Mechanics and Applied Mathematics*, **43**(1): 15–41, 1990, and also by S. Sutcliffe, Spectral decomposition of the elasticity tensor, *Journal of Applied Mechanics*, **59**(4), 762–773, 1992 were published much later, i.e. in 1990 and 1992, respectively.

Prof. Zenon Mróz
Chairman of the Editorial Committee
Applied Mechanics Series

Contents

Foreword

The constitutive law of the classical theory of elasticity – Hooke's law was considered. The concept of an elastic eigenstate of a particle was introduced. On this basis, the structure of the stiffness tensor was explained. In particular, it has been shown that the system of 21 constants, which describes the elastic properties in continuous manner, in fact consists of three distinct subsystems: 6 true stiffness moduli, 12 stiffness distributors and 3 angles. It has also been shown that Hooke's law for any anisotropic body can be represented as a sum $\rho \leq 6$ of the laws of simple proportionality.

Jan Rychlewski
Institute of Problems in Mechanics, USSR Academy of Sciences, 1983

§1 Introductory remarks

In all the mechanics of continuous media, it is difficult to indicate a simpler, more well-known and more frequently used relationship in engineering practice than Hooke's law. I am going to show that the limits of its understanding have not been reached, and that you can also encounter green shoots here, attracting with freshness and bringing practical useful fruits.

The law discovered by Robert Hooke [1] was refined by the greatest mathematicians, mechanicians and physicists. From the point of view that is adopted in this work, two periods were particularly significant. The first concerns the works of the brilliant French school from the 1820s. Then A. Cauchy completed creating the concept of the stress tensor, which gave him, C. Navier and S.D. Poisson the possibility of giving Hooke's law for isotropic bodies an essentially complete, almost modern form [2].

It is true that the dispute in their discussion about the number of elastic constants has been settled relatively recently [3]. The second period covers the activities of F. Neuman, his student W. Voigt and the Voigt school. They developed the foundations of the anisotropic theory of elasticity and provided a description of the elastic properties of crystals [4]. The results of this school are still the basis for the relevant chapters in crystal physics [5, 6] and gradually entered the canon of basic textbooks on the theory of elasticity and mechanics of continuous media.

The research of F. Neumann and W. Voigt was permeated with the pathos of taking into account the symmetry that governs the phenomena in crystals. "The crystal can be compared [...] with an orchestra led by a good conductor. [...] This analogy artistically explains why in the case of crystals whole areas of phenomena appear completely absent in other bodies [...] Some phenomena flourish in them with wonderful diversity and grace, in other bodies they can only be captured in the form of indistinct and monotonous mean values [...]" – as W. Voigt wrote [4].

In all natural sciences, the presence of symmetry in an object leads to fundamental conclusions. As a rule, the amount of information resulting from the consideration of symmetry is the greater the higher the symmetry is. But this also has a drawback: as the degree of symmetry decreases, the amount of this information decreases, and naturally to zero, when no non-trivial symmetry can be observed. The highly developed machine of group theory, in particular the theory of group representation, stalls with the lack of symmetry at a dead point, and with its disappearance it begins to spin in place – with nowhere to latch onto.

This is also the case in the theory of elastic properties. Isotropy, transverse isotropy, orthotropy and cubic symmetry illustrate the effectiveness of the existing description of elastic properties by symmetry methods. At the same time, bodies devoid of trivial symmetry elements are packed by the existing theory into one bag with a numb label "triclinic symmetry" [5–8]. Their properties are distinguished only by unclear and monotonous sets of 21 non-zero components of the stiffness tensor in a randomly selected basis. Beautiful examples of distinguishing special material bases in such bodies [9, 10], alleviate the situation but only partially and not in all cases. The helplessness of the theory condemns here the experimenter to tedious searches.

It is worth emphasizing that bodies showing strongly anisotropic elastic properties will obviously appear in technology more and more often. The thing is that symmetry is quickly lost as the structure of composite materials becomes complicated. For example, it is enough to reinforce an isotropic matrix with three different and non-perpendicular fiber bundles to make the composite completely unsymmetrical.

Fortunately, no matter how important symmetry is, regularities do not emerge solely as a result of it. In this work, I propose to look at Hooke's law through the concept of the body's elastic eigenstates. It turns out that even with a complete lack of symmetry in an elastic body, phenomena can be detected that can be captured in the form of completely clear and by no means monotonous quantities. In particular, it is shown for example that the aforementioned set of 21 stiffness tensor components can be replaced by a set of 6 true **stiffness moduli***, 12 **stiffness distributors** and 3 **angles** orienting the body in the laboratory.

A characteristic feature of the proposed approach is that it gives the more information about elastic properties, the lower the symmetry of the body under consideration.

The work considers the main, quite simple part of the theory. The only difficult question can be formulated as follows: why was it not done (if it was not done!) 80 years ago?

*The bolded words are keywords in this work. All the starred footnotes are from the translator. All numbered footnotes are from the author. – editorial note.

Notation. We use index-free (absolute) tensor notation; otherwise, it would be difficult to discern the simplicity of the proposed ideas in the maze of indexes.

All tensors in this work are Euclidean, i.e. they are elements of tensor spaces $T_p = E \otimes ... \otimes E$ (p-fold) starting from the "physical" 3-dimensional vector Euclidean space E.

It is:

$p = 0$ – numbers	$a, ..., \varepsilon, ..., L, ...$
$p = 1$ – vectors	$\mathbf{m}, \mathbf{n}, ...$
$p = 2$ – second-order tensors	$\boldsymbol{\omega}, \boldsymbol{\tau}, ...$
$p = 4$ – fourth-order tensors	$\mathbf{C}, \mathbf{S}, ...$

Second-order tensors $\boldsymbol{\omega}, ...$ are symmetric, if $\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathrm{T}}$, i.e. they are from a tensor subspace*

$$\mathcal{S} \equiv \mathrm{sym}\ E \otimes E \subset T_2.$$

Fourth-order tensors only come from the tensor subspace

$$\mathcal{T} \equiv \mathcal{S} \otimes \mathcal{S} \subset T_4.$$

Second-order orthogonal tensors are denoted by the symbol $\mathbf{Q}, ...$

The properties of Euclidean tensors are described in Appendix A. However, to understand the essence of this work, it is not necessary to read this Appendix and to know the index-free tensor notation. It is enough to be able to rewrite any formula in the usual Cartesian index notation. The following is a self-explanatory list of all the equivalences that are necessary and sufficient for this purpose**:

$\mathbf{n}$, $\boldsymbol{\omega}$, $\mathbf{C}$	$\leftrightarrow$	n_i, ω_{ij}, C_{ijkl}
$\mathbf{1}$	$\leftrightarrow$	δ_{ij}
$\mathbf{n} \otimes \mathbf{m}$, $\mathbf{n} \otimes \boldsymbol{\omega}$	$\leftrightarrow$	$n_i m_j$, $n_i \omega_{jk}$
$\boldsymbol{\omega} \otimes \boldsymbol{\tau}$	$\leftrightarrow$	$\omega_{ij} \tau_{kl}$
$\boldsymbol{\omega}^2$, $\boldsymbol{\omega}^3$	$\leftrightarrow$	$\omega_{ij}\omega_{jk}$, $\omega_{ij}\omega_{jk}\omega_{kl}$
$\boldsymbol{\omega}\mathbf{n}$, $\mathbf{n}\boldsymbol{\omega}\mathbf{m}$	$\leftrightarrow$	$\omega_{ij} n_j$, $\omega_{ij} n_i m_j$
$\|\|\boldsymbol{\omega}\|\|$***	$\leftrightarrow$	$(\omega_{ij}\omega_{ij})^{1/2}$
$\boldsymbol{\alpha\tau}$	$\leftrightarrow$	$\alpha_{ij}\tau_{jk}$

*Translator note: For clarity: $\mathrm{sym} E \otimes E = T_2^s$ and $\mathcal{S} \otimes \mathcal{S} = T_4^s$.

**Translator note: The denotation of tensorial operations have the following meaning: " " – no marking means contraction over two indices (second-order tensors), "$\cdot$" – dot means full contraction, "$\circ$"– hollow dot means contraction over two indices (fourth-order tensors), "$\star$" – means orthogonal transformation, "$\sigma\times$" means permutation.

***Translator note: $||\boldsymbol{\omega}|| = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})^{1/2}$.

$$
\begin{array}{lll}
\boldsymbol{\alpha}\cdot\boldsymbol{\beta} & \leftrightarrow & \alpha_{ij}\beta_{ij} \\
\mathbf{C}\cdot\boldsymbol{\omega} & \leftrightarrow & C_{ijkl}\omega_{kl} \\
\boldsymbol{\alpha}\cdot\mathbf{C}\cdot\boldsymbol{\beta} & \leftrightarrow & C_{ijkl}\alpha_{ij}\beta_{kl} \\
\mathbf{A}\cdot\mathbf{B} & \leftrightarrow & A_{ijkl}B_{ijkl} \\
\mathbf{C}\circ\mathbf{S} & \leftrightarrow & C_{ijkl}S_{klpq} \\
\mathbf{Q}*\boldsymbol{\omega} & \leftrightarrow & Q_{ij}Q_{kl}\omega_{jl}{}^{*} \\
\mathbf{Q}*\mathbf{C} & \leftrightarrow & Q_{ij}Q_{kl}Q_{pq}Q_{st}C_{jlqt} \\
\|\mathbf{C}\| & \leftrightarrow & (C_{ijkl}C_{ijkl})^{1/2}
\end{array}
$$

An important role in our considerations is played by the well-known fourth-order tensor $\mathbf{I}^{(4s)}$**, uniquely defined by the following condition:

$$\mathbf{I}^{(4s)}\cdot\boldsymbol{\omega}=\boldsymbol{\omega} \qquad \text{for any } \boldsymbol{\omega}\in\mathcal{S}.$$

We call it a **unit** tensor. From this definition it follows that

$$\mathbf{I}^{(4s)}=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj}).$$

We also note that

$$\mathbf{Q}*\mathbf{I}^{(4s)}=\mathbf{I}^{(4s)} \qquad \text{for any } \mathbf{Q}\in\mathcal{O},$$

where $\mathcal{O}$ denotes a group of orthogonal transformations of space E, which, for brevity, we will call rotations***.

*Translator note: $\mathbf{Q}*\boldsymbol{\omega}\equiv\mathbf{Q}\boldsymbol{\omega}\mathbf{Q}^T\leftrightarrow Q_{ij}Q_{kl}\omega_{jl}$ $(\mathbf{Q}\mathbf{Q}^T=\mathbf{1})$.

**Translator note: For consistency and clarity of notation, the original designation of $\mathbb{1}$ of the fourth-order tensor is replaced by $\mathbf{I}^{(4s)}$, which in the index notation takes the form $I^{(4s)}_{ijkl}$.

***Translator note: $\mathcal{O}=\{\mathbf{Q}\in T_2:\mathbf{Q}\mathbf{Q}^{\mathrm{T}}=\mathbf{1},\ \det\mathbf{Q}=\pm1\}$ – orthogonal group, $\mathcal{R}=\{\mathbf{Q}\in T_2:\mathbf{Q}\mathbf{Q}^T=\mathbf{Q}^T\mathbf{Q}=\mathbf{1},\det\mathbf{Q}=+1\}$ – proper orthogonal group.

§2 Elastic eigenstates

The subject of this work is the well-known Hooke's law. We assume that:

1) stress is a linear function of strain,

2) deformations are small,

3) deformation is calculated from a certain natural, unstressed reference state,

4) the stress tensor is symmetric,

5) the influence of temperature and other fields is negligible.

Under these assumptions Hooke's Law takes the form

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon}, \qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}. \tag{2.1}$$

The tensor $\mathbf{C}$ we call the **stiffness tensor**. In general, Hooke's law can be written inversely

$$\boldsymbol{\varepsilon} = \mathbf{S} \cdot \boldsymbol{\sigma}, \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}. \tag{2.2}$$

We call tensor $\mathbf{S}$ the **compliance tensor**. Tensors $\mathbf{C}$ and $\mathbf{S}$[*] are considered elements of the 36-dimensional tensor subspace $\mathcal{T} \subset T_4$ and are mutually reciprocal in the sense that[1]

$$\mathbf{C} \circ \mathbf{S} = \mathbf{S} \circ \mathbf{C} = \mathbf{I}^{(4s)}. \tag{2.3}$$

[1] Strictly speaking, for this purpose it is necessary to relate the stress to an arbitrarily fixed frame of reference; then $\boldsymbol{\sigma}$ loses its physical dimension. The reader may assume that this has been done, if he is so comfortable with it.

[*] Translator note: Stiffness and compliance tensors $\mathbf{C}$ and $\mathbf{S}$ apart from the fact that they have symmetries $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$ resulting from their belonging to subspace T_4, additionally have symmetries $C_{ijkl} = C_{klij}$; cf. also eqs. (3.1) and (3.5) and accompanying text.

(We still not use as a rule, in the sequel the index notation of formulae, referring the reader interested in the index notation to the "dictionary" provided on pp. 3 and 4).

According to the established tradition, we describe the elastic properties with the stiffness tensor $\mathbf{C}$.

Let us consider **$\mathcal{O}$-orbits** of the tensor $\mathbf{C}$ orbit in space $\mathcal{T}$, i.e. all the stiffness tensors $\mathbf{Q} * \mathbf{C}$ that are obtained from $\mathbf{C}$ as a result of all rotations $\mathbf{Q} \in \mathcal{O}$. We denote them by $\langle \mathbf{C} \rangle$*. When in mechanics the term "elastic material" is used, then one means (implicitly) just the $\mathcal{O}$-orbit. We will call the $\mathcal{O}$-orbit an **elastic material $\langle \mathbf{C} \rangle$**.

Invariants defined on $\mathcal{T}$, i.e. scalar functions of the form $\pi : \mathcal{T} \to T$, satisfying the condition,

$$\pi(\mathbf{Q} * \mathbf{C}) = \pi(\mathbf{C}) \quad \text{for all} \quad \mathbf{C} \in \mathcal{T}, \quad \mathbf{Q} \in \mathcal{O}, \tag{2.4}$$

are actually set not on the stiffness tensors but on the materials. It is natural to call them material invariants, and their values $\pi(\mathbf{C})$ **material elasticity constants**. (The components C_{ijkl} of the stiffness tensor $\mathbf{C}$ in an arbitrarily chosen basis **are not** material elasticity constants; this has been recognized long ago [27], but for some reason it is silent about it in textbooks.) Isotropic functions defined on $\mathcal{T}$, i.e. functions of the form: $f : \mathcal{T} \to T_p$ (of any order p), satisfying condition

$$f(\mathbf{Q} * \mathbf{C}) = \mathbf{Q} * f(\mathbf{C}) \quad \text{for all} \quad \mathbf{C} \in \mathcal{T}, \quad \mathbf{Q} \in \mathcal{O}, \tag{2.5}$$

are natural to be called material tensor functions, and their values $f(\mathbf{C})$ **material tensors**.

Let $\mathcal{B}$ be a solid with material points $X, \ldots$ We say that the body $\mathcal{B}$ is made of elastic material $\langle \mathbf{C} \rangle$, if for any point $X \in \mathcal{B}$ Hooke's law is valid with the stiffness tensor $\mathbf{C}(X)$, whereas $\mathbf{C}(X) \in \langle \mathbf{C} \rangle$ but for any point $X \in \mathcal{B}$. We call a body $\langle \mathbf{C} \rangle$ made of an elastic material homogeneous, if

$$\mathbf{C}(X) = \mathbf{C} = \text{const} \quad \text{for all} \quad X \in \mathcal{B}. \tag{2.6}$$

*Translator note: "Two tensors lie in one orbit if and only if the disordered systems of their principal values are the same". J. Rychlewski, *Symmetry of Causes and Results*, PWN, Warsaw, 1991, p. 33.

(Colloquially speaking, this means that all particles have not only identical elastic properties, but are also oriented equally.) A homogeneous elastic body can simply be equated with a stiffness tensor. And in this sense, we continue to use the expression: homogeneous elastic body $\mathbf{C}$ or, briefly, **elastic body C**.

We now move from these boring, albeit obligatory, definitions to the essence of this study.

Let us take any elastic body $\mathbf{C}$. In the general case, the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors, related by Hooke's law (2.1) and (2.2) are not only disproportionate, i.e. $\boldsymbol{\sigma} \neq \lambda\boldsymbol{\varepsilon}$ for any λ, but also non-coaxial*. The isotropic body is no exception. However,

> it may happen that the strain $\boldsymbol{\varepsilon}$ and stresses $\boldsymbol{\sigma}$ are selected for a given elastic body $\mathbf{C}$ such that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are strictly proportional, i.e.
>
> $$\boldsymbol{\sigma} = \lambda\boldsymbol{\varepsilon} \qquad \sigma_{ij} = \lambda\varepsilon_{ij} \tag{2.7}$$
>
> for some constant λ.

This is of course for $\boldsymbol{\varepsilon} = \boldsymbol{\omega}$, where λ and $\boldsymbol{\omega}$ are defined by the condition

$$\boxed{\mathbf{C} \cdot \boldsymbol{\omega} = \lambda\boldsymbol{\omega}}\,. \tag{2.8}$$

The words that arise here – eigenvalue and eigenstate – are encrypted (or, if the reader wishes, deciphered) using a mechanical terminology and expressed with due diligence.

> **Definition 0.** A parameter λ will be called the stiffness modulus of an elastic body $\mathbf{C}$, if there is a symmetric second-order tensor $\boldsymbol{\omega}$, that satisfies the condition (2.8). The $\boldsymbol{\omega}$ tensor itself will be called the **elastic eigenstate** of body $\mathbf{C}$, corresponding to the stiffness modulus λ.

The elastic eigenstate $\boldsymbol{\omega}$ can be interpreted as either a strain state or a stress state if necessary (see footnote 1 on page 5).

*Translator note: Two tensors $\mathbf{A}$ and $\mathbf{B} \in \mathcal{S}$, we call **consistent**, if their **principal directions** coincide, i.e., $\mathbf{A} = A_1\boldsymbol{\mu}_{\mathrm{I}} \otimes \boldsymbol{\mu}_{\mathrm{I}} + A_2\boldsymbol{\mu}_{\mathrm{II}} \otimes \boldsymbol{\mu}_{\mathrm{II}} + A_3\boldsymbol{\mu}_{\mathrm{III}} \otimes \boldsymbol{\mu}_{\mathrm{III}}$, $\mathbf{A} = B_1\boldsymbol{\mu}_{\mathrm{I}} \otimes \boldsymbol{\mu}_{\mathrm{I}} + B_2\boldsymbol{\mu}_{\mathrm{II}} \otimes \boldsymbol{\mu}_{\mathrm{II}} + B_3\boldsymbol{\mu}_{\mathrm{III}} \otimes \boldsymbol{\mu}_{\mathrm{III}}$. Some of A_k may be equal, and similarly some of B_k ($\boldsymbol{\mu}_k \cdot \boldsymbol{\mu}_l = \delta_{kl}$). Two tensors $\mathbf{A}$, $\mathbf{B} \in \mathcal{S}$, we call **coaxial**, when their eigen subspaces coincide, i.e. they are consistent, and when some of their principal values are multiple, then when e.g. $A_1 = A_2$ then it must be $B_1 = B_2$, cf. also p. 68 in [P10].

The simplest elastic eigenstate is shown in Figure 1, where **a**, **b**, **c** we will treat as the orthonormal triplet of the material fibers of the body. **Pure shear deformation**

$$\boldsymbol{\varepsilon} = \frac{1}{2}\gamma(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}) \quad \sim \quad \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in which the length of the fibers **a**, **b**, **c** does not change (within the theory of small deformations!), and the pairs of fibers (**a**, **c**), (**b**, **c**) remain perpendicular, here it corresponds to the **pure shear stress**

$$\boldsymbol{\sigma} = \tau(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}) \quad \sim \quad \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

wherein the stiffness modulus is a shear modulus $\lambda = G(\mathbf{a},\mathbf{b})$

$$\boldsymbol{\sigma} = 2G(\mathbf{a},\mathbf{b})\boldsymbol{\varepsilon}, \quad \tau = G\gamma.$$

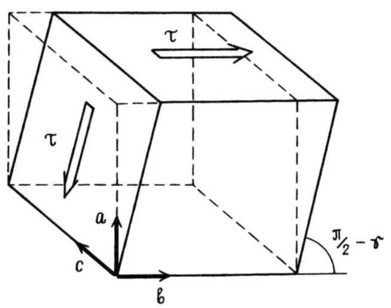

Fig. 1. Pure shear. The simplest elastic eigenstate.[*]

In the case of an isotropic body, this will be the case for any pair of orthogonal fibers **a**, **b**. In §10 we show that this property defines an isotropic body. On the other hand, there are bodies for which not a single pair of orthogonal fibers can be found **a**, **b**, having the property shown in Figure 1.

The second simplest elastic eigenstate is $\boldsymbol{\omega} = \mathbf{1}$ (cf. Figure 2). **Purely volumetric deformation**

[*]Editorial note: Captions to all drawings were elaborated by the translator.

$$\boldsymbol{\varepsilon} = \frac{1}{3}\varepsilon\mathbf{1} \quad \sim \quad \begin{pmatrix} \frac{1}{3}\varepsilon & 0 & 0 \\ 0 & \frac{1}{3}\varepsilon & 0 \\ 0 & 0 & \frac{1}{3}\varepsilon \end{pmatrix}$$

corresponds here to the **state of hydrostatic stress**

$$\boldsymbol{\sigma} = \sigma\mathbf{1} \quad \sim \quad \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}.$$

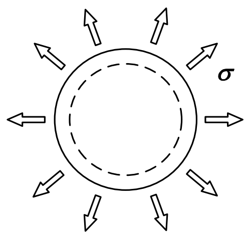

Fig. 2. Purely volumetric (spherical) deformation.
The second simplest elastic eigenstate.

The modulus of stiffness is the modulus of the volumetric stiffness K

$$\boldsymbol{\sigma} = 3K\,\boldsymbol{\varepsilon}, \qquad \sigma = K\,\varepsilon.$$

This is the case for an isotropic body, but there are many bodies, which do not have such a property. A sphere made of such a material under the influence of hydrostatic pressure deforms into an ellipsoid.

We immediately prove the theorem establishing the physical content of the introduced concepts.

> **Theorem 1.** If a tensor $\boldsymbol{\omega}$ is an elastic eigenstate of the body $\mathbf{C}$, corresponding to the stiffness modulus λ, then for any rotation $\mathbf{Q}$ the tensor $\mathbf{Q} * \boldsymbol{\omega}$ is an elastic eigenstate of the body $\mathbf{Q} * \mathbf{C} \in \langle\mathbf{C}\rangle$ corresponding to **the same** stiffness modulus λ.

Proof. If $\mathbf{C} \cdot \boldsymbol{\omega} = \lambda\boldsymbol{\omega}$, then

$$(\mathbf{Q} * \mathbf{C}) \cdot (\mathbf{Q} * \boldsymbol{\omega}) \equiv \mathbf{Q} * (\mathbf{C} \cdot \boldsymbol{\omega}) = \lambda\mathbf{Q} * \boldsymbol{\omega}. \quad \blacklozenge \tag{2.9}$$

Hence, the most important conclusion:

> moduli of stiffness and invariants of elastic eigenstates are the same for any two bodies of the same material.

The proposed method of describing the elastic properties is based on the following belief:

> the eigenstates of elasticity and moduli of stiffness contain all the information on the macrostructure of a body that are necessary to describe its elastic behavior.

This is proven in the next section.

Note 1. The idea described here and the first results were presented during lectures on the mechanics of continuous media, which I gave from the late 1960's at many research centers in Poland and the Soviet Union. As for my actions in matters far from mechanics, I did not fulfill my promise to my students to print the work then or later, except for a very brief overview of the essence in my handwritten script [11], p. 54 (see Appendix B). As A. Blinowski kindly informed me, recently this idea appeared many times in [12] for one specific case, with reference to the author of the script [11].

§3 Structural formula

We consider only those elastic bodies for which there is an **elastic potential**

$$2\Phi(\boldsymbol{\varepsilon}) \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \mathbf{C} \cdot \boldsymbol{\varepsilon}, \tag{3.1}$$

$$\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}} \Phi. \tag{3.2}$$

All this work is based on the pleasant circumstance that the bilinear form

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} \equiv \boldsymbol{\alpha} \cdot \mathbf{I}^{(4s)} \cdot \boldsymbol{\beta} \equiv \mathrm{tr}(\boldsymbol{\alpha}\boldsymbol{\beta}) \tag{3.3}$$

turns out to be a correctly defined **dot product** in $\mathcal{S}$, and it is consistent with the structure of the tensor product $\mathcal{S} = E \otimes E$. A space $\mathcal{S}$ with this dot product, while continuing to be a tensor product, is also a 6-dimensional abstract Euclidean space (see Appendix A).

We consider tensors $\mathbf{C} \in \mathcal{T}$ as linear operators $l : \mathcal{S} \to \mathcal{S}$, transforming $\mathcal{S}$ into each other:

$$l(\boldsymbol{\omega}) \equiv \mathbf{C} \cdot \boldsymbol{\omega} \quad \text{for any} \quad \boldsymbol{\omega} \in \mathcal{S}. \tag{3.4}$$

The existence of the elastic potential (3.1) is equivalent to the following condition:

$$\boldsymbol{\alpha} \cdot \mathbf{C} \cdot \boldsymbol{\beta} = \boldsymbol{\beta} \cdot \mathbf{C} \cdot \boldsymbol{\alpha} \quad \text{for any} \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{S}, \tag{3.5}$$

i.e. $\boldsymbol{\alpha} \cdot l(\boldsymbol{\beta}) = \boldsymbol{\beta} \cdot l(\boldsymbol{\alpha})$. Such linear transformations of Euclidean spaces are called symmetric. So,

> the stiffness tensor $\mathbf{C}$ can be considered a symmetric linear operator that transforms the 6-dimensional Euclidean space of symmetric tensors of the second-order $\mathcal{S}$ into itself.

Still, for the sake of brevity, we do not distinguish the tensor from the linear transformation carried out by it according to formula (3.4).

The shown completely natural view of the stiffness tensors is the core of the work. Follow up – this is just a matter of technique which is well known. Indeed, the operation of symmetric operators in Euclidean spaces has been described by mathematicians long time ago and comprehensively (see, for example, [13–15]). All we have to do is translate the available information into the language of Euclidean tensors in an extremely concise form.

Each linear operator operating in a finite dimensional space is uniquely defined by its values on the elements of an arbitrarily determined basis. We take an orthonormal basis for convenience $\mathcal{S}$

$$\boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}} \qquad \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL} \equiv \begin{cases} 0 & K \neq L, \\ 1 & K = L, \end{cases} \tag{3.6}$$

where $K, L = \mathrm{I}, \ldots, \mathrm{VI}$. (We continue to label the bases in $\mathcal{S}$ with large Latin indices; **no summation is assumed** over repeating indices denoted by large Latin indices.) Writing any tensor $\boldsymbol{\alpha} \in \mathcal{S}$ in this base, we have

$$\boldsymbol{\alpha} = \alpha_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \alpha_{\mathrm{VI}} \boldsymbol{\omega}_{\mathrm{VI}}, \qquad \alpha_K \equiv \boldsymbol{\alpha} \cdot \boldsymbol{\omega}_K. \tag{3.7}$$

Now,

$$\begin{aligned} \mathbf{C} \cdot \boldsymbol{\alpha} &= \alpha_{\mathrm{I}} \mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \alpha_{\mathrm{VI}} \mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{VI}} \\ &= \left[(\mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{I}}) \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + (\mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{VI}}) \otimes \boldsymbol{\omega}_{\mathrm{VI}} \right] \cdot \boldsymbol{\alpha} \end{aligned} \tag{3.8}$$

for any $\boldsymbol{\alpha} \in \mathcal{S}$. This implies a **fundamental identity**: for any tensor $\mathbf{C} \in \mathcal{T}$ and for any orthonormal basis $\boldsymbol{\omega}_K$, $K = \mathrm{I}, \ldots, \mathrm{VI}$

$$\mathbf{C} = (\mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{I}}) \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + (\mathbf{C} \cdot \boldsymbol{\omega}_{\mathrm{VI}}) \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{3.9}$$

This is a special case of the formula (A.20), cf. Appendix A.

Note 2. Because of the importance of the deduced identity, we give it in Cartesian index notation without using geometric terminology. We claim that for any tensor C_{ijkl} with an internal symmetry

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}, \tag{3.10}$$

and any six symmetrical tensors

$$\omega_{\mathrm{I}\,ij}, \ldots, \omega_{\mathrm{VI}\,ij}, \qquad \omega_{K\,ij} = \omega_{K\,ji}, \tag{3.11}$$

meeting the conditions

$$\omega_{K\,ij} \omega_{L\,ij} = \delta_{KL}, \qquad K, L = \mathrm{I}, \ldots, \mathrm{VI}, \tag{3.12}$$

the following identity is true

$$C_{ijkl} = C_{ijpq}\omega_{\mathrm{I}\,pq}\omega_{\mathrm{I}\,kl} + \ldots + C_{ijpq}\omega_{\mathrm{VI}\,pq}\omega_{\mathrm{VI}\,kl}, \tag{3.13}$$

or, equivalently, for any six-tuple (3.11) and (3.12), the identity is satisfied

$$\omega_{\mathrm{I}\,ij}\omega_{\mathrm{I}\,kl} + \ldots + \omega_{\mathrm{VI}\,ij}\omega_{\mathrm{VI}\,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}).^{*} \quad \blacklozenge \tag{3.14}$$

The identity (3.9) is equivalent to the following statement: for any orthonormal basis (3.6)

$$\mathbf{I}^{(4s)} = \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{3.15}$$

Indeed, from (3.9) it follows (3.15) according to the definition of $\mathbf{I}^{(4s)}$ itself. Conversely, from (3.15) it follows (3.9), because $\mathbf{C} \circ \mathbf{I}^{(4s)} = \mathbf{C}$ for any $\mathbf{C} \in \mathcal{T}$.

Projectors play a major role in the theory of symmetric mappings of Euclidean spaces. We find their tensor image. Consider a subspace $\mathcal{P} \subset \mathcal{S}$ and its orthogonal complement $\mathcal{P}^{\perp}$. The formula

$$\mathcal{S} = \mathcal{P} \oplus \mathcal{P}^{\perp} \tag{3.16}$$

means that for each tensor $\boldsymbol{\alpha} \in \mathcal{P}$ there are exactly two tensors $\boldsymbol{\alpha}_{\mathcal{P}}$ and $\boldsymbol{\alpha}_{\mathcal{P}}^{\perp}$, and such that

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathcal{P}} + \boldsymbol{\alpha}_{\mathcal{P}}^{\perp}, \qquad \boldsymbol{\alpha}_{\mathcal{P}} \cdot \boldsymbol{\alpha}_{\mathcal{P}}^{\perp} = 0, \qquad \boldsymbol{\alpha}_{\mathcal{P}} \in \mathcal{P}. \tag{3.17}$$

We call tensor $\boldsymbol{\alpha}_{\mathcal{P}}$, as usual, **an orthogonal projection $\boldsymbol{\alpha}$ onto a subspace** $\mathcal{P}$. We consider a tensor $\mathbf{P} \in \mathcal{T}$, uniquely defined by the formula

$$\mathbf{P} \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathcal{P}} \quad \text{for any} \quad \boldsymbol{\alpha} \in \mathcal{S}. \tag{3.18}$$

We call this tensor **an orthogonal projector on a subspace** $\mathcal{P}$ (cf. Figure 3). It is easy to get an explicite form of $\mathbf{P}$.

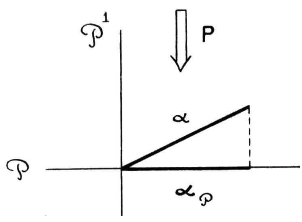

Fig. 3. Projection on subspace $\mathcal{P}$ using the orthogonal projector $\mathbf{P}$.

*Translator note: $\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) = \mathbf{I}^{(4s)}$.

Let $L \equiv \dim \mathcal{P} \leq 6$. Let us take an orthonormal basis $\boldsymbol{\omega}_I, \dots, \boldsymbol{\omega}_{\mathrm{VI}}$, such that L tensors

$$\boldsymbol{\omega}_{K+\mathrm{I}}, \boldsymbol{\omega}_{K+\mathrm{II}}, \dots, \boldsymbol{\omega}_{K+L} \tag{3.19}$$

were located in $\mathcal{P}$. Then $\mathbf{P} \cdot \boldsymbol{\omega}_{K+\mathrm{I}} = \boldsymbol{\omega}_{K+\mathrm{I}}, \dots, \mathbf{P} \cdot \boldsymbol{\omega}_{K+L} = \boldsymbol{\omega}_{K+L}$ and simultaneously $\mathbf{P} \cdot \boldsymbol{\omega}_T = 0$ for any $\boldsymbol{\omega}_T \notin \mathcal{P}$. According to the identity (3.9) we get

$$\mathbf{P} = \boldsymbol{\omega}_{K+\mathrm{I}} \otimes \boldsymbol{\omega}_{K+\mathrm{I}} + \dots + \boldsymbol{\omega}_{K+L} \otimes \boldsymbol{\omega}_{K+L}. \tag{3.20}$$

In $\mathcal{P}$ itself, the projector acts as a unit operator: $\mathbf{P} \cdot \boldsymbol{\alpha} = \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in \mathcal{P}$. Therefore,

$$\overset{s}{\mathbf{P}} \equiv \mathbf{P} \circ \ \dots \ \circ \mathbf{P} = \mathbf{P} \quad \text{for any} \quad s \geq 1. \tag{3.21}$$

The dimension of $\mathcal{P}$ is equal to the number of terms in the representation (3.20)

$$\dim \mathcal{P} = P_{ijij}. \tag{3.22}$$

Two orthogonal projectors: $\mathbf{P}_1$ on $\mathcal{P}_1$ and $\mathbf{P}_2$ on $\mathcal{P}_2$ we call **mutually orthogonal**, if the subspaces $\mathcal{P}_1$, $\mathcal{P}_2$ are orthogonal, $\mathcal{P}_1 \perp \mathcal{P}_2$. Of course, this is equivalent to the equality

$$\mathbf{P}_1 \circ \mathbf{P}_2 = \mathbf{P}_2 \circ \mathbf{P}_1 = \mathbf{0}. \tag{3.23}$$

A system of pairs of mutually orthogonal projectors $\mathbf{P}_1, \dots, \mathbf{P}_\rho$, is called the **unity distribution**, if

$$\mathbf{I}^{(4s)} = \mathbf{P}_1 + \dots + \mathbf{P}_\rho. \tag{3.24}$$

Any unity distribution can be obtained by selecting the appropriate base according to the formula (3.15) and grouping the component expressions accordingly. The distribution of unity (3.24) corresponds to the distribution of space into simple sum subspaces, in pairs mutually orthogonal

$$\mathcal{S} = \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_\rho, \tag{3.25}$$

where $\mathcal{P}_\alpha \equiv \operatorname{Im} \mathbf{P}_\alpha$, $\alpha = 1, \dots, \rho$.

Conversely, the distribution (3.25), where $\mathcal{P}_\alpha \perp \mathcal{P}_\beta$ for $\alpha \neq \beta$, is the distribution of unity (3.24), where $\mathbf{P}_\alpha$ is an orthogonal projector on $\mathcal{P}_\alpha$.

Now we are able to present the main theorem – the **Spectral Theorem** – of the theory of symmetric mappings of Euclidean spaces, in our case. (Here and later, we use Greek indices for elements of the distribution (3.24), (3.25) and all corresponding to them elements; **no summation over repeated indices is assumed**.)

Theorem 2. For any elastic body $\mathbf{C}$ there exists exactly one orthogonal distribution of the space of symmetric tensors of the second-order

$$\mathcal{S} = \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_\rho, \qquad \mathcal{P}_\alpha \perp \mathcal{P}_\beta, \qquad \alpha \neq \beta, \qquad \rho \leq 6, \tag{3.26}$$

and exactly one set of in pairs different parameters,

$$\lambda_1, ..., \lambda_\rho, \qquad \lambda_\alpha \neq \lambda_\beta \quad \text{for} \quad \alpha \neq \beta,$$

such that

$$\boxed{\mathbf{C} = \lambda_1 \mathbf{P}_1 + ... + \lambda_\rho \mathbf{P}_\rho}, \tag{3.27}$$

where $\mathbf{P}_v$ are orthogonal projectors on $\mathcal{P}_v$.

Proof. May be found in slightly different terminology, e.g., in [13, 14, 16]. ♦

The components $\mathcal{P}_\alpha$ of a simple sum (3.26), which continue to play a major role, have a completely clear physical interpretation:

> a subspace $\mathcal{P}_\alpha$ consists of all elastic eigenstates with modulus of stiffness λ_α, $\alpha = 1, ..., \rho$.

Indeed, for any $\boldsymbol{\omega} \in \mathcal{P}_\alpha$ we have

$$\mathbf{C} \cdot \boldsymbol{\omega} = (\lambda_1 \mathbf{P}_1 + ... + \lambda_\rho \mathbf{P}_\rho) \cdot \boldsymbol{\omega} = \lambda_\alpha \boldsymbol{\omega}. \tag{3.28}$$

Projectors $\mathbf{P}_\alpha$ in formula (3.27) are called **material projectors** of subspaces, $\mathcal{P}_\alpha$ – **material subspaces**, and distributions (3.26) and (3.27) **material distributions**, corresponding to the body $\mathbf{C}$.

After writing the system of equations:

$$\begin{aligned} \mathbf{I}^{(4s)} &= \mathbf{P}_1 + ... + \mathbf{P}_\rho, \\ \mathbf{C} &= \lambda_1 \mathbf{P}_1 + ... + \lambda_\rho \mathbf{P}_\rho, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{(\rho-1)}{\mathbf{C}} &= \lambda_1^{\rho-1} \mathbf{P}_1 + ... + \lambda_\rho^{\rho-1} \mathbf{P}_\rho \end{aligned} \tag{3.29}$$

with the help of the Vandermonde determinant

$$\Delta = \begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_\rho \\ \vdots & \ldots & \vdots \\ \lambda_1^{\rho-1} & \ldots & \lambda_\rho^{\rho-1} \end{vmatrix} = \prod_{\rho \geq \alpha > \beta \geq 1} (\lambda_\alpha - \lambda_\beta) \tag{3.30}$$

we get explicit expressions [13]

$$\mathbf{P}_\alpha = \frac{(\mathbf{C} - \lambda_1 \mathbf{I}^{(4s)}) \circ ... \circ (\mathbf{C} - \lambda_{\alpha-1} \mathbf{I}^{(4s)}) \circ (\mathbf{C} - \lambda_{\alpha+1} \mathbf{I}^{(4s)}) \circ ... \circ (\mathbf{C} - \lambda_\rho \mathbf{I}^{(4s)})}{(\lambda_\alpha - \lambda_1) \ldots (\lambda_\alpha - \lambda_{\alpha-1})(\lambda_\alpha - \lambda_{\alpha+1}) \ldots (\lambda_\alpha - \lambda_\rho)}. \tag{3.31}$$

We also pay attention to the formula

$$\lambda_\alpha = \mathbf{C} \cdot \mathbf{P}_\alpha. \tag{3.32}$$

The stiffness moduli λ_α are, as usual, the roots of the characteristic equation, which takes the form here

$$\det(\mathbf{C} - \lambda\,\mathbf{I}^{(4s)}) = \lambda^6 + a_1(\mathbf{C})\,\lambda^5 + \ldots + a_5(\mathbf{C})\,\lambda + a_6(\mathbf{C}) = 0, \tag{3.33}$$

where

$$\det(\mathbf{A}) \equiv \det(A_{KL}), \tag{3.34}$$

wherein

$$A_{KL} \equiv \boldsymbol{\nu}_K \cdot \mathbf{A} \cdot \boldsymbol{\nu}_L = A_{LK} \tag{3.35}$$

is a matrix (6×6) related to the tensor $\mathbf{A}$ and the freely fixed orthonormal basis $\boldsymbol{\nu}_{\mathrm{I}}, \ldots, \boldsymbol{\nu}_{\mathrm{VI}}$ in $\mathcal{S}$. The choice of this base does not affect the coefficients a_i.

The number

$$q_\alpha \equiv \dim \mathcal{P}_\alpha{}^* = P_{(\alpha)\,ijij} \tag{3.36}$$

is the multiplicity λ_α as the root of equation (3.33).

We took the spectral theorem in the strongest possible way, perhaps a little extraordinary. Other equivalent wording may be given. Taking into account the fact that this theorem plays, in our opinion, a central role in describing the properties of elastic bodies, we give another three more **statements, equivalent to Theorem 2**:

1) For any elastic body $\mathbf{C}$ there is at least one orthonormal basis in $\mathcal{S}$,

$$\boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}}, \qquad \boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL} \tag{3.37}$$

consisting of its elastic eigenstates,

$$\mathbf{C} \cdot \boldsymbol{\omega}_K = \lambda_K\, \boldsymbol{\omega}_K. \tag{3.38}$$

2) For any elastic body $\mathbf{C}$ there is at least one orthonormal basis (3.37) and six parameters $\lambda_{\mathrm{I}}, \ldots, \lambda_{\mathrm{VI}}$, such that

$$\boxed{\mathbf{C} = \lambda_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \lambda_{\mathrm{VI}} \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}}\,. \tag{3.39}$$

3) For any elastic body $\mathbf{C}$ there exists at least one orthonormal basis (3.37), such that matrix

$$C_{KL} \equiv \boldsymbol{\omega}_K \cdot \mathbf{C} \cdot \boldsymbol{\omega}_L \tag{3.40}$$

is diagonal.

*Translator note: $\dim \mathcal{P}_\alpha = \mathrm{tr}\,(\mathbf{P}_{(\alpha)}) \equiv \mathbf{P}_{(\alpha)} \cdot \mathbf{I}^{(4s)}$.

We discuss the matrix (3.40). If $\boldsymbol{\omega}_K$, $K = \mathrm{I}, ..., \mathrm{VI}$ is the basis in $\mathcal{S}$, then by the very definition of the dot product of linear spaces, 36 tensors,

$$\boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_L, \qquad K, L = \mathrm{I}, ..., \mathrm{VI} \tag{3.41}$$

is the basis in $\mathcal{T} \equiv \mathcal{S} \otimes \mathcal{S}$. The matrix (3.40) is the matrix of the components $\mathbf{C}$ in this base, i.e. the following identity is satisfied

$$\mathbf{C} = \sum_{K,L=\mathrm{I}}^{\mathrm{VI}} C_{KL} \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_L. \tag{3.42}$$

This is a special case of the formula (A.14), cf. Appendix A [p. 84].

It is useful to trace the **equivalence of formulae (3.27) and (3.39)**. The formula (3.39) results from (3.27) due to (3.20), and conversely, if **all coinciding** λ_K are taken into account in (3.39), then on the basis of (3.20) we come to (3.27) with the uniqueness of all terms of the sum. We note that if λ_K are all pairwise different (i.e. the roots (3.33) are singular), then (3.39) is just the formula (3.27) with

$$\mathbf{P}_1 = \boldsymbol{\omega}_\mathrm{I} \otimes \boldsymbol{\omega}_\mathrm{I}, ..., \mathbf{P}_6 = \boldsymbol{\omega}_\mathrm{VI} \otimes \boldsymbol{\omega}_\mathrm{VI}. \tag{3.43}$$

The formula (3.39) is a very special case of the formula (B.1).

The formulae (3.27) (or (3.39)) are called **the basic structural formula of an elastic body**. We call any orthonormal set of eigenstates $\boldsymbol{\omega}_K$, $K = \mathrm{I}, ..., \mathrm{VI}$ of the tensor $\mathbf{C}$ its **tensorial material basis**.

We choose some order of terms in (3.27), for example: the term $\lambda_\alpha \mathbf{P}_\alpha$ is written before the term $\lambda_\beta \mathbf{P}_\beta$ (no summation!) if $q_\alpha < q_\beta$, and with $q_\alpha = q_\beta$, if $\lambda_\alpha > \lambda_\beta$. Then all λ_α and $\mathbf{P}_\alpha$ (as well as, of course, the set of components ρ) are values of some functions on $\mathcal{T}$ of the argument $\mathbf{C}$. This is what we mean when formulating the next theorem.

Theorem 3. The parameters $\lambda_1, ..., \lambda_\rho$ corresponding to an elastic body $\mathbf{C}$ according to the structural formula (3.27) are **material elastic constants**, and material projectors $\mathbf{P}_1, ..., \mathbf{P}_\rho$ are **material tensors** corresponding to $\mathbf{C}$.

Proof. Invariance of the stiffness moduli

$$\lambda_\alpha(\mathbf{Q} * \mathbf{C}) = \lambda_\alpha(\mathbf{C}), \qquad \alpha = 1, ..., \rho, \tag{3.44}$$

follows from Theorem 1. Isotropy of $\mathbf{P}_\alpha$

$$\mathbf{P}_\alpha(\mathbf{Q} * \mathbf{C}) = \mathbf{Q} * \mathbf{P}_\alpha(\mathbf{C}) \tag{3.45}$$

results immediately from explicit formulae (3.31). ♦

The spaces of eigenstates for the body $\mathbf{Q} * \mathbf{C}$ will be, according to Theorem 1, images of the spaces of eigenstates of body $\mathbf{C}$, upon rotation $\mathbf{Q}$.

Finally, we introduce one more useful concept. The material distribution of space $\mathcal{S}$ (3.26) corresponds to the decomposition of a number $6 \equiv \dim \mathcal{S}$ into positive integer components $q_1, \ldots, q_\rho$ (3.36) (or, if you wish, Young's diagram [17]).

We write this distribution as

$$\langle q_1 + \ldots + q_\rho \rangle, \qquad q_1 \leq \ldots \leq q_\rho, \tag{3.46}$$

and call it, **the first structural index** of the body $\mathbf{C}$ under consideration. The first structural index, of course, is the same for bodies of the same material, i.e. it is a material characteristic.

Note 3. We note that in fact $\lambda_\alpha(\mathbf{C})$, $\mathbf{P}_\alpha(\mathbf{C})$ are even "more invariant than necessary"[2].

Indeed, let us consider a group $\mathcal{A}$, consisting of all linear transformations $\mathcal{S}$ on itself, preserving the dot product, i.e. the group of automorphisms of $\mathcal{S}$ treated as the 6-dimensional Euclidean space [18]. The defining property of the transformation from $\mathcal{A}$ is as follows: the image of any orthonormal basis $\boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}}$ is another orthonormal basis, say $\boldsymbol{\nu}_{\mathrm{I}}, \ldots, \boldsymbol{\nu}_{\mathrm{VI}}$. Identifying, as usual, in accordance with the formula (3.4) transformations from $\mathcal{A}$ with tensors $\mathbf{K} \in \mathcal{T}$, we obtain according to (3.9)

$$\mathbf{K} = \boldsymbol{\nu}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \boldsymbol{\nu}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{3.47}$$

This is a general expression of the automorphism $\mathcal{S}$ as the 6-dimensional Euclidean space. The rotation group $\mathcal{O}$ of basic space E, operating in $\mathcal{S}$ is a 3-parameter subgroup in the 15-parameter group $\mathcal{A}$. The rotation $\mathbf{Q} \in \mathcal{O}$ corresponds to the tensor

$$\mathbf{K}(\mathbf{Q}) = \mathbf{Q} * \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \mathbf{Q} * \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{3.48}$$

[2]Yes, as we say, $\operatorname{tr}(\boldsymbol{\omega})$ "is more invariant than necessary"; because it is invariant with respect to the entire linear group of transformations of $\mathcal{S}$, and not just its orthogonal part.

We transfer the operation $\mathcal{A}$ from $\mathcal{S}$ to the tensorial square $\mathcal{T} \equiv \mathcal{S} \otimes \mathcal{S}$, in usual manner, i.e.

$$\mathbf{C} \to \mathbf{K} \circ \mathbf{C} \circ \mathbf{K}^{-1}, \tag{3.49}$$

where

$$\mathbf{K}^{-1} \equiv (\boldsymbol{\nu}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots)^{-1} \equiv (\boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\nu}_{\mathrm{I}} + \ldots). \tag{3.50}$$

It is easy to show that Theorem 3 can be generalized as follows: for any $\mathbf{K} \in \mathcal{A}$

$$\lambda_\alpha(\mathbf{K} \circ \mathbf{C} \circ \mathbf{K}^{-1}) = \lambda_\alpha(\mathbf{C}), \tag{3.51}$$

$$\mathbf{P}_\alpha(\mathbf{K} \circ \mathbf{C} \circ \mathbf{K}^{-1}) = \mathbf{K} \circ \mathbf{P}_\alpha(\mathbf{C}) \circ \mathbf{K}^{-1}. \tag{3.52}$$

This invariance seems to matter less because **the body $\mathbf{K} \circ \mathbf{C} \circ \mathbf{K}^{-1}$ is made of a different material than the body $\mathbf{C}$**, in general.

§4 On the mathematical and physical content of a structural formula

When discussing the structural formula in highly qualified teams, the following doubts arose, of a non-trivial nature.

4.1. Is the structural formula not the result of *a priori* information introduced to the description of Hooke's law from the theory of n-dimensional transformations of Euclidean spaces?

No, it is not. Operations:

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta}) \to \boldsymbol{\alpha} \cdot \boldsymbol{\beta}, \qquad (\alpha_{ij},\ \beta_{kl}) \to \alpha_{ij}\beta_{ij}, \tag{4.1}$$

$$\boldsymbol{\alpha} \to \mathbf{C} \cdot \boldsymbol{\alpha}, \qquad \alpha_{ij} \to C_{ijkl}\alpha_{kl}, \tag{4.2}$$

are tensorial operations. At the same time, thanks to a fortunate coincidence, the operation (4.1) is a correct definition of a dot product, and the operation (4.2) is a linear projection of $\mathcal{S}$ into itself. **We did not use any other assumptions, either explicit or implicit**. The linear space $\mathcal{S}$ with the operation (4.1), consistent with the tensor structure in $\mathcal{S}$, **is** (and **is not artificially replaced!**) a 6-dimensional Euclidean space. All the conclusions resulting from this fact are true, but they are certainly not **the whole truth** about $\mathcal{S}$, as $\mathcal{S}$ is not only a 6-dimensional Euclidean space. (Those, who do not like the language of modern algebra, I refer to **Appendix C** [p. 87], where the derivation of the structural formula is framed by full "arithmetization" of the structures under consideration.)

Thus, the answer to the question about the mathematical status of the structural formula has been exhausted.

4.2. The whole structure is based on the definition of the dot product (4.1). A scalar product can be introduced into a linear space in an infinite number of ways. Does this deprive the structural formula of physical content?

It is not like that, but to explain this circumstance we have to toil a little. We formulate the problem.

Consider any dot product in $\mathcal{S}$, i.e. a bilinear form

$$(\boldsymbol{\alpha},\ \boldsymbol{\beta}) \to \boldsymbol{\alpha} \odot \boldsymbol{\beta} \tag{4.3}$$

symmetric ($\boldsymbol{\alpha} \odot \boldsymbol{\beta} = \boldsymbol{\beta} \odot \boldsymbol{\alpha}$), positive definite ($\boldsymbol{\alpha} \odot \boldsymbol{\alpha} > 0$ for $\boldsymbol{\alpha} \neq 0$).*

We need that two conditions are satisfied for it:

1) invariance with respect to the rotation group $\mathcal{O}$ of generating space E

$$(\mathbf{Q} * \boldsymbol{\alpha}) \odot (\mathbf{Q} * \boldsymbol{\beta}) = \boldsymbol{\alpha} \odot \boldsymbol{\beta} \tag{4.4}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{S}$ and for any $\mathbf{Q} \in \mathcal{O}$ (this is the requirement of compliance of the form (4.3) with the tensor structure already available in $\mathcal{S}$);

2) symmetry of any stiffness tensor $\mathbf{C}$, satisfying the condition of existence of elastic potential (3.1), and with reference to (4.3):

$$\boldsymbol{\alpha} \odot (\mathbf{C} \cdot \boldsymbol{\beta}) = \boldsymbol{\beta} \odot (\mathbf{C} \cdot \boldsymbol{\alpha}) \tag{4.5}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{S}$.

Now the problem can be formulated as follows: are there other dot products (4.1) besides (4.3) that satisfy the requirements (4.4) and (4.5)? **The answer is positive, but banal**. We start with a lemma that has autonomous value.

> **Lemma 1.** Any scalar product in the space of symmetric tensors of the second-order $\mathcal{S} \equiv \operatorname{sym} E \otimes E$ invariant with respect to the group of rotations $\mathcal{O}$ of the primal 3-dimensional Euclidean space E has the form
>
> $$\begin{aligned} \boldsymbol{\alpha} \odot \boldsymbol{\beta} &= k_1 \operatorname{tr}(\boldsymbol{\alpha}) \operatorname{tr}(\boldsymbol{\beta}) + k_2 \boldsymbol{\alpha} \cdot \boldsymbol{\beta} \\ &= k_1 \alpha_{ii} \beta_{jj} + k_2 \alpha_{ij} \beta_{ij}, \end{aligned} \tag{4.6}$$
>
> where
>
> $$3k_1 + k_2 \geq 0, \qquad k_2 \geq 0. \tag{4.7}$$

*Translator note: Denotation of dot product “×” used in the original text of this discourse has been replaced in the translation with the designation “⊙”, so that this operation is not confused with the vector product, for which the designation “×” is commonly used.

Proof. The sufficiency is obvious: (4.6) is bilinear form, symmetric and positive definite. Let us consider a necessity, since the representation (4.6) must be invariant, then there exists such a real function f of nine variables that

$$\begin{aligned} \boldsymbol{\alpha} \odot \boldsymbol{\beta} = f(&\mathrm{tr}(\boldsymbol{\alpha}),\ \mathrm{tr}(\boldsymbol{\beta}),\ \mathrm{tr}(\boldsymbol{\alpha}\boldsymbol{\beta}), \\ &\mathrm{tr}(\boldsymbol{\alpha}^2),\ \mathrm{tr}(\boldsymbol{\beta}^2),\ \mathrm{tr}(\boldsymbol{\alpha}^2\boldsymbol{\beta}), \\ &\mathrm{tr}(\boldsymbol{\alpha}\boldsymbol{\beta}^2),\ \mathrm{tr}(\boldsymbol{\alpha}^3),\ \mathrm{tr}(\boldsymbol{\beta}^3)), \end{aligned} \tag{4.8}$$

where in parentheses it is listed the well-known functionally complete set of invariants on $\mathcal{S} \odot \mathcal{S}$. By superimposing on f the conditions of bilinearity, symmetry and positivity, we get (4.6) and (4.7) $\mathrm{tr}(\boldsymbol{\alpha}\boldsymbol{\beta}) \equiv \boldsymbol{\alpha} \cdot \boldsymbol{\beta}$. ♦

Now we can give an answer.

Theorem 4. The stiffness tensor $\mathbf{C}$ is symmetric with respect to the invariant dot product (4.6) if and only if $k_1 = 0$, i.e., when

$$\boldsymbol{\alpha} \odot \boldsymbol{\beta} = k(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}), \qquad k > 0. \tag{4.9}$$

Proof. The sufficiency is obvious. We consider the necessity. Let (4.5) be satisfied for (4.6). We take $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{S}$ non-zero and orthogonal in the old sense

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0, \qquad \boldsymbol{\alpha} \neq 0, \qquad \boldsymbol{\beta} \neq 0. \tag{4.10}$$

According to the structural formula (it is proved for (4.1); there is only doubt about its physical uniqueness!) of course, there exists such a tensor of elasticity for which $\mathbf{P}_1 = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha}$, $\mathbf{P}_2 = \boldsymbol{\beta} \otimes \boldsymbol{\beta}$, i.e.,

$$\mathbf{C} \cdot \boldsymbol{\alpha} = \lambda\, \boldsymbol{\alpha}, \qquad \mathbf{C} \cdot \boldsymbol{\beta} = \mu\, \boldsymbol{\beta}, \qquad \lambda \neq \mu. \tag{4.11}$$

Now

$$\mu(\boldsymbol{\alpha} \odot \boldsymbol{\beta}) = \boldsymbol{\alpha} \odot (\mathbf{C} \cdot \boldsymbol{\beta}) = \boldsymbol{\beta} \odot (\mathbf{C} \cdot \boldsymbol{\alpha}) = \lambda(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}), \tag{4.12}$$

hence

$$\boldsymbol{\alpha} \odot \boldsymbol{\beta} = 0. \tag{4.13}$$

Thus, each pair $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ orthonormal in the old sense (4.1), should be orthonormal also in the new sense (4.3). Taking into consideration (4.6), we get $k_1 = 0$. ♦

Thus, the dot product for which the construction of the structural formula can be made is determined, as would be expected, to the nearest positive multiplier. In our case, this result means the following:

> **The structural formula (3.27) is defined for any body with the accuracy of selecting the system of dimensional units for stress**.

This exhausts the answer to the question about the physical content of the proposed method of describing elastic bodies.

§5 Representation of Hooke's law in the form of an orthogonal decomposition

We take the elastic body $\mathbf{C}$. Its structural formula allows any tensor $\boldsymbol{\tau} \in \mathcal{S}$ to be transformed as follows. We introduce orthogonal projections $\boldsymbol{\tau}$ on material subspaces $\mathcal{P}_\alpha$ of the body $\mathbf{C}$

$$\boldsymbol{\tau}_\alpha \equiv \mathbf{P}_\alpha \cdot \boldsymbol{\tau}, \qquad \alpha = 1, ..., \rho. \tag{5.1}$$

Let us recall that

$$\mathbf{P}_\beta \cdot \boldsymbol{\tau}_\alpha = 0, \quad \text{for any} \quad \alpha \neq \beta, \tag{5.2}$$

or, equivalently

$$\boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta = 0, \quad \text{for any} \quad \alpha \neq \beta. \tag{5.3}$$

We denote the projection norms as follows,

$$\tau_\alpha \equiv \ |\boldsymbol{\tau}_\alpha| = (\boldsymbol{\tau} \cdot \mathbf{P}_\alpha \cdot \boldsymbol{\tau}). \tag{5.4}$$

Tensor $\boldsymbol{\tau}$ can be unambiguously written as the sum of its projections into material subspaces

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + ... + \boldsymbol{\tau}_\rho. \tag{5.5}$$

A special case of this representation is the representation of the tensor in the form of the sum of the spherical and deviatoric parts, cf. (10.7).

We use the representation (5.5) for the stress tensor and the strain tensor

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + ... + \boldsymbol{\sigma}_\rho, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + ... + \boldsymbol{\varepsilon}_\rho. \tag{5.6}$$

Substituting this into (2.1), we get the general form of Hooke's law for any elastic body

$$\boxed{\boldsymbol{\sigma}_1 = \lambda_1\, \boldsymbol{\varepsilon}_1, \ \dots\ , \ \boldsymbol{\sigma}_\rho = \lambda_\rho\, \boldsymbol{\varepsilon}_\rho}\,. \tag{5.7}$$

This new form of Hooke's law generalizes the textbook form of Hooke's law for an isotropic body presented in the form of two tensor equations: the law of proportionality of spherical parts and the law of proportionality of deviatoric parts of tensors $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ (10.9).

Each of the $\rho \le 6$ tensor equations (5.7) is linearly independent of the others. An equality with number α corresponds to ρ_α scalar equations.

From (5.7) it follows, in particular, the proportionality of the norms

$$\sigma_\alpha = \lambda_\alpha \, \varepsilon_\alpha, \qquad \alpha = 1, ..., \rho. \tag{5.8}$$

If some material tensor basis $\boldsymbol{\omega}_K$ is used (for $\rho = 6$ one defined with precision to signs), then it is

$$\boldsymbol{\sigma} = \sigma_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} + ... + \sigma_{\mathrm{VI}}\boldsymbol{\omega}_{\mathrm{VI}}, \qquad \sigma_K \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_K, \tag{5.9}$$

$$\boldsymbol{\varepsilon} = \varepsilon_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} + ... + \varepsilon_{\mathrm{VI}}\boldsymbol{\omega}_{\mathrm{VI}}, \qquad \varepsilon_K \equiv \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega}_K, \tag{5.10}$$

and Hooke's Law boils down to six scalar equations

$$\boxed{\sigma_{\mathrm{I}} = \lambda_{\mathrm{I}}\, \varepsilon_{\mathrm{I}}, ..., \sigma_{\mathrm{VI}} = \lambda_{\mathrm{VI}}\, \varepsilon_{\mathrm{VI}}}\,. \tag{5.11}$$

Hooke's law in the form (5.7), and even more so in the form (5.11), almost directly reflects the formulation of the very author of the anagram *ut tensio sic vis* [1, 2], which is in the title of this work. It can be assumed that Robert Hooke would be pleased.

We consider the inverse of Hooke's law (2.2). You can enter the eigenstates $\boldsymbol{\tau}$ of compliance tensor $\mathbf{S}$ and the **compliance moduli** μ according to the formula

$$\mathbf{S} \cdot \boldsymbol{\tau} = \mu\, \boldsymbol{\tau}. \tag{5.12}$$

But it is unnecessary, as shows the simple theorem:

> **Theorem 5.** The eigenstates of the compliance tensor coincide with the corresponding elastic eigenstates of the stifness tensor, and the compliance moduli are the reciprocals of the stiffness moduli.

Proof. If $\mathbf{C} \cdot \boldsymbol{\omega} = \lambda \boldsymbol{\omega}$, then

$$\mathbf{S} \cdot \boldsymbol{\omega} = \frac{1}{\lambda}\mathbf{S} \cdot (\mathbf{C} \cdot \boldsymbol{\omega}) = \frac{1}{\lambda}\mathbf{I}^{(4s)} \cdot \boldsymbol{\omega} = \frac{1}{\lambda}\, \boldsymbol{\omega}. \quad \blacklozenge \tag{5.13}$$

Therefore, we immediately get the reverse of $\mathbf{C}(X)$

$$\boxed{\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + ... + \frac{1}{\lambda_\rho}\mathbf{P}_\rho}\,. \tag{5.14}$$

Here $\lambda_1, ..., \lambda_\rho$ are stiffness moduli, and $\mathbf{P}_1, ..., \mathbf{P}_\rho$ are the material projectors, mentioned in Theorem 2. If we apply the material tensorial base $\boldsymbol{\omega}_K$, then

$$\mathbf{S} = \frac{1}{\lambda_{\mathrm{I}}} \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + ... + \frac{1}{\lambda_{\mathrm{VI}}} \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}. \tag{5.15}$$

The inverse of Hooke's law can be obtained immediately from (5.7)

$$\boldsymbol{\varepsilon}_\alpha = \frac{1}{\lambda_\alpha} \boldsymbol{\sigma}_\alpha, \qquad \alpha = 1, ..., \rho. \tag{5.16}$$

§6 Elastic energy

Substituting the material distribution (5.6) and (5.7) to the expression (3.1), we get the following expressions of elastic energy for any elastic body:

$$2\Phi = \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \tag{6.1}$$

$$= \lambda_1 \varepsilon_1^2 + \ \dots \ + \lambda_\rho \varepsilon_\rho^2 = \tag{6.2}$$

$$= \frac{1}{\lambda_1} \sigma_1^2 + \ \dots \ + \frac{1}{\lambda_\rho} \sigma_\rho^2, \tag{6.3}$$

where

$$\varepsilon_\alpha \equiv (\boldsymbol{\varepsilon}_\alpha \cdot \boldsymbol{\varepsilon}_\alpha)^{1/2}, \qquad \sigma_\alpha \equiv (\boldsymbol{\sigma}_\alpha \cdot \boldsymbol{\sigma}_\alpha)^{1/2}. \tag{6.4}$$

Hence, it follows the fundamental theorem:

Theorem 6. Elastic energy is positive for any strain tensor $\boldsymbol{\varepsilon} \neq 0$ if and only if

$$\boxed{\lambda_{\mathrm{I}} > 0, \dots, \lambda_\rho > 0}\,. \tag{6.5}$$

We pay attention to the extreme simplicity of the new **conditions for the positive determination of elastic energy** that have been found. It is useful to compare ρ inequalities (6.5) with the known Sylvester conditions superimposed on the components C_{ijkl} (see, for example, [10]).

If we accept some material tensor base $\boldsymbol{\omega}_{\mathrm{I}}, \dots, \boldsymbol{\omega}_{\mathrm{VI}}$, then

$$2\Phi = \lambda_{\mathrm{I}} \varepsilon_{\mathrm{I}}^2 + \dots + \lambda_{\mathrm{VI}} \varepsilon_{\mathrm{VI}}^2 = \tag{6.6}$$

$$= \frac{1}{\lambda_{\mathrm{I}}} \sigma_{\mathrm{I}}^2 + \dots + \frac{1}{\lambda_{\mathrm{VI}}} \sigma_{\mathrm{VI}}^2, \tag{6.7}$$

where

$$\varepsilon_K \equiv \boldsymbol{\varepsilon} \cdot \boldsymbol{\omega}_K, \qquad \sigma_K \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_K, \tag{6.8}$$

$$\lambda_K > 0, \qquad K = \mathrm{I}, \dots, \mathrm{VI}. \tag{6.9}$$

This representation of elastic energy allows the following geometric interpretation:

> surfaces of the energy constant value
>
> $$\Phi(\boldsymbol{\varepsilon}) = \mathrm{const} \tag{6.10}$$
>
> are 6-dimensional ellipsoids in the space of symmetric tensors $\mathcal{S}$, the axes of the ellipsoids have the directions of the elastic eigenstates of the body under consideration $\mathbf{C}$, and the lengths of the semi-axes are equal to the respective stiffness moduli.

When the values coincide $\lambda_K = \lambda_L$, the ellipsoids gain appropriate symmetry, and when $\lambda_\mathrm{I} = \dots = \lambda_\mathrm{VI} = \lambda$ (see (10.1)) they become spheres.

§7 Some material constants and elastic tensors

We introduce for an elastic body $\mathbf{C}$ two orthonormal material dyads

$$\mathbf{n}\otimes\mathbf{n}, \quad \mathbf{m}\otimes\mathbf{m}, \quad \mathbf{n}\cdot\mathbf{n}=\mathbf{m}\cdot\mathbf{m}=1, \quad \mathbf{n}\cdot\mathbf{m}=0, \tag{7.1}$$

i.e., two orthogonal directions of the material (two orthogonal fibers). We consider the well-known material constants of an elastic body (see, for example [5])

the modulus of volumetric compression K*

$$\frac{1}{K}\equiv\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{1}, \tag{7.2}$$

Young's modulus $E(\mathbf{n})$ in the direction $\mathbf{n}\otimes\mathbf{n}$:

$$\frac{1}{E(\mathbf{n})}\equiv(\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{n}), \tag{7.3}$$

Poisson's ratio $\nu(\mathbf{m},\mathbf{n})$ in the direction $\mathbf{m}\otimes\mathbf{m}$ at stretching in direction $\mathbf{n}\otimes\mathbf{n}$:

$$\frac{\nu(\mathbf{m},\,\mathbf{n})}{E(\mathbf{n})}\equiv-(\mathbf{m}\otimes\mathbf{m})\cdot\mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{n}) \tag{7.4}$$

and **modulus of shear stiffness** $G(\mathbf{m},\mathbf{n})$ under shear strain $(\mathbf{m}\otimes\mathbf{n})+(\mathbf{n}\otimes\mathbf{m})$:

$$\frac{1}{4G(\mathbf{m},\,\mathbf{n})}\equiv(\mathbf{m}\otimes\mathbf{n})\cdot\mathbf{S}\cdot(\mathbf{m}\otimes\mathbf{n}). \tag{7.5}$$

---

*Translator note: Caution should be exercised. The scalar determined by the formula (7.2) is today called the **compressibility factor (module)** and marked with the letter $\beta\equiv\mathbf{1}\cdot\mathbf{S}\cdot\mathbf{1}$. However, in the literature on the subject currently the most often the **coefficient (module) of volumetric elasticity** – **Bulk Modulus** is encountered, defined with the formula $K\equiv\frac{1}{9}\mathbf{1}\cdot\mathbf{C}\cdot\mathbf{1}$. In the general case, for the elastic anisotropic materials, the values of these modules **are not mutually reciprocal**. The values of these coefficients are mutually reciprocal for the so-called *volumetrically-isotropic* materials, i.e. those for which the tensor $\mathbf{1}$ is their elastic eigenstate. In particular, for *isotropic elastic* materials it is $K=\frac{E}{3(1-2\nu)}=1/\beta$.

Using the structural formula (3.39), we get the following elegant expressions,

$$\frac{1}{K} = \frac{\operatorname{tr}(\boldsymbol{\omega}_{\mathrm{I}})^2}{\lambda_{\mathrm{I}}} + \ldots + \frac{\operatorname{tr}(\boldsymbol{\omega}_{\mathrm{VI}})^2}{\lambda_{\mathrm{VI}}}, \tag{7.6}$$

$$\frac{1}{E(\mathbf{n})} = \frac{(\mathbf{n}\boldsymbol{\omega}_{\mathrm{I}}\mathbf{n})^2}{\lambda_{\mathrm{I}}} + \ldots + \frac{(\mathbf{n}\boldsymbol{\omega}_{\mathrm{VI}}\mathbf{n})^2}{\lambda_{\mathrm{VI}}}, \tag{7.7}$$

$$-\frac{\nu(\mathbf{m},\,\mathbf{n})}{E(\mathbf{n})} = \frac{(\mathbf{m}\boldsymbol{\omega}_{\mathrm{I}}\mathbf{m})(\mathbf{n}\boldsymbol{\omega}_{\mathrm{I}}\mathbf{n})}{\lambda_{\mathrm{I}}} + \ldots + \frac{(\mathbf{m}\boldsymbol{\omega}_{\mathrm{VI}}\mathbf{m})(\mathbf{n}\boldsymbol{\omega}_{\mathrm{VI}}\mathbf{n})}{\lambda_{\mathrm{VI}}}, \tag{7.8}$$

$$\frac{1}{4G(\mathbf{m},\,\mathbf{n})} = \frac{(\mathbf{m}\boldsymbol{\omega}_{\mathrm{I}}\mathbf{n})^2}{\lambda_{\mathrm{I}}} + \ldots + \frac{(\mathbf{m}\boldsymbol{\omega}_{\mathrm{VI}}\mathbf{n})^2}{\lambda_{\mathrm{VI}}}. \tag{7.9}$$

Here,

$$\operatorname{tr}(\boldsymbol{\omega}_K) \equiv \omega_{K\,ii}, \tag{7.10}$$

$$\mathbf{n}\boldsymbol{\omega}_K\mathbf{n} \equiv \omega_{K\,ij}n_i n_j, \tag{7.11}$$

$$2\mathbf{m}\boldsymbol{\omega}_K\mathbf{n} \equiv 2\omega_{K\,ij}m_i n_j. \tag{7.12}$$

This is nothing else, but the **relative change in volume**, the **relative elongation of the fiber $\mathbf{n}\otimes\mathbf{n}$**, and the shear angle of the fibers $\mathbf{m}\otimes\mathbf{m}$, $\mathbf{n}\otimes\mathbf{n}$, corresponding to the strain $\boldsymbol{\omega}_K$, $K = \mathrm{I}, \ldots, \mathrm{VI}$.

When the stiffness moduli have the same value $\lambda_K = \lambda_L$ the sums of the corresponding them expressions correspond to the terms $\lambda_\alpha \mathbf{P}_\alpha$ in the most general structural formula (3.27).

The obtained formulae agree well with the intuitive understanding of the elastic eigenstate $\boldsymbol{\omega}_K$:

1) it does not contribute to the body volumetric modulus K, when the corresponding volumetric strain $\operatorname{tr}(\boldsymbol{\omega}_K)$ is zero;

2) it does not contribute to Young's modulus $E(\mathbf{n})$, when the corresponding elongation $\mathbf{n}\boldsymbol{\omega}_K\mathbf{n}$ is zero;

3) it does not contribute to the corresponding Poisson's ratio $\nu(\mathbf{m},\mathbf{n})/E(\mathbf{n})$, when one of the elongations $\mathbf{m}\boldsymbol{\omega}_K\mathbf{m}$, $\mathbf{n}\boldsymbol{\omega}_K\mathbf{n}$ is zero;

4) it does not contribute to the shear modulus $G(\mathbf{m},\mathbf{n})$, when the corresponding shear angle $\mathbf{m}\boldsymbol{\omega}_K\mathbf{n}$ is zero.

The obtained formulae greatly facilitate the understanding and, apparently, help in determining the introduced moduli of stiffness and elastic eigenstates for various elastic bodies, especially composites. In my opinion, the whole game was "worth the candle", if only for the finding of these formulae.

In order to list the most basic, preliminary practical information on elastic bodies, it is worth having tools that allow simply, as far as possible, to compare the bodies according to their stiffness and anisotropy.

As useful measures of overall body stiffness there can be proposed, for example:

1) the norm of $\mathbf{C}$ treated as a linear operator (see, for example, [15])

$$l(\mathbf{C}) \equiv \sup_{\boldsymbol{\alpha}\in\mathcal{K}} |\mathbf{C}\cdot\boldsymbol{\alpha}|, \tag{7.13}$$

where $|\boldsymbol{\tau}| \equiv (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}$, and $\mathcal{K}$ is the unit sphere in $\mathcal{S}$;

2) an index proportional to the norm of $\mathbf{C}$ as a fourth-order tensor

$$m(\mathbf{C}) \equiv \frac{1}{\sqrt{6}}\|\mathbf{C}\|; \tag{7.14}$$

3) a linear invariant

$$n(\mathbf{C}) \equiv \frac{1}{6}\mathrm{tr}(\boldsymbol{\nu}) = \frac{1}{6}C_{ikik}, \tag{7.15}$$

where $\boldsymbol{\nu}$ is defined by the formula (7.21).

Applying the structural formula (3.39) we get immediately:

$$l(\mathbf{C}) = \max(\lambda_{\mathrm{I}}, ..., \lambda_{\mathrm{VI}}), \tag{7.16}$$

$$m(\mathbf{C}) = \left[\frac{1}{6}(\lambda_{\mathrm{I}}^2 + ... + \lambda_{\mathrm{VI}}^2)\right]^{1/2}, \tag{7.17}$$

$$n(\mathbf{C}) = \frac{1}{6}(\lambda_{\mathrm{I}} + ... + \lambda_{\mathrm{VI}}), \tag{7.18}$$

where $l(\mathbf{C})$ is the maximum, $m(\mathbf{C})$ is the mean square, and $n(\mathbf{C})$ is the arithmetic mean modulus of stiffness.

As a useful **measure of anisotropy** of the body $\mathbf{C}$ there can be proposed "relative, mean, square elastic anisotropy" introduced in [10], or the following more precise measure,

$$\delta(\mathbf{C}) \equiv \frac{d(\langle\mathbf{C}\rangle)}{m(\mathbf{C})}, \tag{7.19}$$

where

$$d(\langle \mathbf{C} \rangle) \equiv \sup_{\mathbf{X},\, \mathbf{Y} \in \langle \mathbf{C} \rangle} \|\mathbf{X} - \mathbf{Y}\| = \max_{\mathbf{Q} \in \mathcal{O}} \|\mathbf{C} - \mathbf{Q} * \mathbf{C}\|, \tag{7.20}$$

is the diameter of $\langle \mathbf{C} \rangle$. (This extreme problem was examined in [19]).

V.V. Novozhilov [9] drew attention to two unusual tensors $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ with components*

$$\mu_{ij} \equiv C_{ijkk}, \qquad \nu_{ij} \equiv C_{ikkj} = C_{ikjk}. \tag{7.21}$$

These are linear isotropic functions of $\mathbf{C}$, i.e. **material tensors** of the body. Each symmetric, second-order material tensor linearly dependent on $\mathbf{C}$, have the form $a\boldsymbol{\mu} + b\boldsymbol{\nu}$.

Tensor $\boldsymbol{\mu}$ describes the body's reaction to spherical deformation: if $\boldsymbol{\varepsilon} = \mathbf{1}$ then $\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{1} = \boldsymbol{\mu}$, cf. [9]. It can be expressed as follows

$$\boldsymbol{\mu} = \lambda_{\mathrm{I}} \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{I}})\, \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \lambda_{\mathrm{VI}} \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{VI}})\, \boldsymbol{\omega}_{\mathrm{VI}}. \tag{7.22}$$

Only non-deviatoric elastic eigenstates contribute to $\boldsymbol{\mu}$.

The tensor $\boldsymbol{\nu}$ plays a role in the dynamics of elastic waves. It can be expressed as follows

$$\boldsymbol{\nu} = \lambda_{\mathrm{I}}\, \boldsymbol{\omega}_{\mathrm{I}}^2 + \ldots + \lambda_{\mathrm{VI}}\, \boldsymbol{\omega}_{\mathrm{VI}}^2. \tag{7.23}$$

Note 4. The formulae (3.22) and (3.23) immediately show that proposed in [9], the interesting choice of the so-called main principal values of anisotropy, is not universal. It is easy to indicate many examples when the body is completely anisotropic, and at the same time tensors $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ either have a common axis of symmetry, or are spherical tensors at all.

The theory of elastic waves propagation is based on the so-called **Christoffel tensor $\boldsymbol{\chi}(\mathbf{n})$** having components $\chi_{il} = C_{ijkl} n_j n_k$**, where $\mathbf{n}$ denotes a wave normal (phase) vector. According to the structural formula (3.39),

$$\boldsymbol{\chi}(\mathbf{n}) = \lambda_{\mathrm{I}}\, \boldsymbol{\omega}_{\mathrm{I}}\mathbf{n} \otimes \boldsymbol{\omega}_{\mathrm{I}}\mathbf{n} + \ldots + \lambda_{\mathrm{VI}}\, \boldsymbol{\omega}_{\mathrm{VI}}\mathbf{n} \otimes \boldsymbol{\omega}_{\mathrm{VI}}\mathbf{n}. \tag{7.24}$$

This formula has a number of consequences that we do not dwell on here. We only note that the positive determination of $\boldsymbol{\chi}(\mathbf{n})$ is immediately apparent.

---

*Translator note: $\boldsymbol{\mu} \equiv \mathbf{C} \cdot \mathbf{1} \Rightarrow \mathrm{tr}(\boldsymbol{\mu}) = \mathbf{1} \cdot \mathbf{C} \cdot \mathbf{1}$, $\boldsymbol{\nu} \equiv (\langle 2, 3 \rangle \times \mathbf{C}) \cdot \mathbf{1} \Rightarrow \mathrm{tr}(\boldsymbol{\nu}) = \boldsymbol{\nu} \cdot \mathbf{1} = \mathrm{tr}(\mathbf{C})$.

**Translator note: $\boldsymbol{\chi} \equiv \mathbf{n}\mathbf{C}\mathbf{n}$.

§8 New information on elastic constants

Based on the structural formula, we analyze the question of determining the set of independent scalar parameters that continuously describe the diversity of elastic bodies.

According to (3.39) the problem comes down to the determination of a set of independent parameters that continuously describe the variety of tensorial material reference frames. Let us take the reference frame $\boldsymbol{\omega}_K$, $\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \delta_{KL}$, $K, L = \mathrm{I}, ..., \mathrm{VI}$. We describe each of the tensors $\boldsymbol{\omega}_K$ with three linearly independent invariants, for example, traces of tensors $\boldsymbol{\omega}_K$, $\boldsymbol{\omega}_K\boldsymbol{\omega}_K$, $\boldsymbol{\omega}_K\boldsymbol{\omega}_K\boldsymbol{\omega}_K$ and three parameters that determine the direction of the principal axes of $\boldsymbol{\omega}_K$ in the laboratory coordinate system, for example, Euler angles θ_K, φ_K, ψ_K. In this way, we replace each tensor $\boldsymbol{\omega}_K$ with six parameters, for example,

$$\begin{aligned} &\mathrm{tr}(\boldsymbol{\omega}_K), \quad \mathrm{tr}(\boldsymbol{\omega}_K^2), \quad \mathrm{tr}(\boldsymbol{\omega}_K^3), \quad \theta_K, \ \varphi_K, \ \psi_K, \\ &K = \mathrm{I}, ..., \mathrm{VI}. \end{aligned} \tag{8.1}$$

The 36 parameters obtained are bound together with the 21 orthonormal conditions; six of them are conditions for normalization $|\boldsymbol{\omega}_K|^2 \equiv \mathrm{tr}(\boldsymbol{\omega}_K^2) = 1$. The other 30 parameters, for example,

$$\mathrm{tr}(\boldsymbol{\omega}_K), \quad \mathrm{tr}(\boldsymbol{\omega}_K^3), \quad \theta_K, \ \varphi_K, \ \psi_K, \tag{8.2}$$

are bound with 15 orthogonality conditions

$$\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = 0 \qquad \text{for} \qquad K \neq L. \tag{8.3}$$

Let us organize the parameters as follows. As **linearly independent** parameters, let us choose, for example, 3 parameters determining the axes of the first tensor in the laboratory coordinate system,

$$\theta \equiv \theta_{\mathrm{I}}, \qquad \varphi \equiv \varphi_{\mathrm{I}}, \qquad \psi \equiv \psi_{\mathrm{I}}, \tag{8.4}$$

and, for example, 12 linearly independent tensor invariants*

$$\begin{aligned}\chi_1 \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{I}}), \ldots, \chi_6 \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{VI}}),\\ \chi_7 \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{I}}^3), \ldots, \chi_{12} \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{VI}}^3).\end{aligned} \tag{8.5}$$

The remaining 15 invariant parameters, for example,

$$\theta_K, \ \varphi_K, \ \psi_K, \qquad K = \mathrm{II}, \ldots, \mathrm{VI}, \tag{8.6}$$

we express from 15 conditions (8.3) through 15 linearly independent quantities θ, φ, ψ, $\chi_1, \ldots, \chi_{12}$.

Thus, the number of parameters in the structural formula (3.39), in the general case, is $6 + 12 + 3 = 21$. This corresponds to the number of non-zero components given in textbooks,

$$C_{ijkl}, \qquad i, j, k, l = 1, 2, 3,$$

of the stiffness tensor $\mathbf{C}$ in a laboratory coordinate system. This should also be the case, as both parameter sets describe a variety of stiffness tensors in a continuous manner.

Note 5. A reader with an analytical mindset can demand explanations from us, and he will be right. Here are the most important of them. The words "in general" should be understood as follows: the environment $\|\mathbf{C} - \mathbf{C}^0\| < \varepsilon$ of a certain tenor $\mathbf{C}^0$ is considered, in which the λ_K^0 are pairwise different and all ω_K^0 have pairwise different eigenvalues.

Now, using the implicit function theorem (see, for example [20]), one has to show that the conditions (8.3) do indeed define in the neigbourhood some mapping of 15 parameters (8.4) and (8.5) into 15 parameters (8.6). This tedious procedure can clearly be bypassed while remaining within the reasonable habit of working on the mechanics. Moreover, our considerations suggest that the set $\boldsymbol{\omega}_K$ is well-defined, i.e. the signs of $\boldsymbol{\omega}_K$ and the order of $\boldsymbol{\omega}_{\mathrm{I}}, \ldots, \boldsymbol{\omega}_{\mathrm{VI}}$ are known.

*Translator note: The typographical error $\chi_7 \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{I}}^2), \ldots, \chi_{12} \equiv \mathrm{tr}(\boldsymbol{\omega}_{\mathrm{VI}}^2)$ occurring in the original has been corrected. It is worth noting that proposed here by Jan Rychlewski set of invariants characterizing Hooke's tensor makes a simple illustrative example. Generally, one can construct infinitely many invariants of Hooke's tensor, but a maximum of 18 of them can be mutually linearly independent. The selection of a set of linearly independent invariants of a tensor, appropriate for a given modelling problem, is more an act of art than it is routine operation. For example, see the work of A. Ziółkowski [P14], where the problem of selection of the optimal set of (three) invariants for a symmetric second-order tensor interpreted as Cauchy stress is comprehensively discussed.

It is easy to demonstrate that it is viable in the environment of $\mathbf{C}^0$. Finally, in the neighbourhood of $\mathbf{C}^0$ it is possible to agree on the choice of signs of eigenvectors for each $\boldsymbol{\omega}_K$. Thus, the quantities are well defined θ_K, φ_K, ψ_K. ⧫

Note 6. The following points are often underestimated by a researcher in tasks of calculating the number of parameters that define a certain variety of bodies. Without imposing certain requirements on the parameters (continuity, algebraic character, etc.), the mere formulation of the problem about the number of parameters is pointless.

Indeed, Georg Kantor has already shown that n-dimensional space R^n is unambiguously mapped onto single R, and he constructed a specific technique for such mapping. It is pointless, for example, to ask how many linearly independent invariants define a symmetric tensor of the second-order with accuracy to rotation – **one** such invariant can always be constructed. Already at the stage of formulating the problem, we need a **continuity** of mapping $R^n \to \mathcal{T}$, understood in the sense of a norm $||\mathbf{C}||$ in a space $\mathcal{T}$. We have omitted it here and we continue to omit the related technical details. ⧫

Here is another calculation of the number of parameters in the structural formula (3.39), continuously describing the variety of elastic bodies:

$$6 + 5 + 4 + 3 + 2 + 1 + 0 = 21.$$

Indeed, in the general case, the number of stiffness moduli λ_K is 6, the number of free parameters of the tensor $\boldsymbol{\omega}_{\mathrm{I}}$, bound by the normalization condition, is 5, the number of free parameters of the tensor $\boldsymbol{\omega}_{\mathrm{II}}$ bound by the normalization condition and the orthogonality condition to the previous tensor is 4, ..., the number of free parameters of the tensor $\boldsymbol{\omega}_{\mathrm{VI}}$, bound by the normalization condition and conditions of orthogonality to the previous five tensors is zero.

Considering the fundamental nature of the problem of elastic constants, let us summarize.

> **Comment.** The variety of elastic bodies for which there exists an elastic potential, in the general case, can be continuously described by a set of 21 parameters consisting of the following three very different subsets.
>
> **Firstly**, these are 6 unambiguously defined, for a given elastic material, **dimensional material constants** (invariants of the stiffness tensor)
>
> $$\lambda_1, ..., \lambda_{\mathrm{VI}}, \qquad \text{(I)}$$

having the physical dimension of stress and positively defined. Looking at the formulae (5.11), (6.6), (7.16)–(7.18), we can see that these constants describe the degree of general stiffness of the material. These are the true **stiffness moduli** of the material. The **compliance moduli** correspond to them

$$\lambda_{\mathrm{I}}^{-1}, ..., \lambda_{\mathrm{VI}}^{-1}.$$

Secondly, these are 12 **dimensionless material constants** (invariants of the stiffness tensor),

$$\chi_1, ..., \chi_{12} \tag{II}$$

(defined, for example, by formulae (8.5)), constituting a functionally complete and irreducible system of invariants of a tensor material reference frame $\boldsymbol{\omega}_{\mathrm{I}}, ..., \boldsymbol{\omega}_{\mathrm{VI}}$. These invariants somehow distribute the stiffness over the fibers and planes of the material. Their most important property is that they are the same for the stiffness tensor $\mathbf{C}$ and the compliance tensor $\mathbf{S}$. We will call these dimensionless constants **stiffness distributors**.

Thirdly, these are 3 non-invariant parameters, for example, Euler angles

$$\theta, \varphi, \psi, \tag{III}$$

setting the orientation of a specific body made of the considered elastic material in relation to the laboratory reference frame.

Degenerate cases when some **stiffness moduli** start to coincide

$$\lambda_K = \lambda_L, \qquad \text{for some pairs} \quad K, L,$$

and/or **stiffness distributors** are subject to whatever constraints

$$f_i(\chi_1, ..., \chi_{12}) = 0, \qquad i = 1, ..., k \leq 12, \tag{8.7}$$

require separate examination. Sets (I)–(III) come down to a set

$$\begin{gathered} (\lambda_1, ..., \lambda_\rho; \qquad \chi_1, ..., \chi_t; \qquad \varphi_1, ..., \varphi_u) \\ \rho \leq 6, \qquad t \leq 12, \qquad u \leq 3, \end{gathered} \tag{8.8}$$

where λ_i are pairwise different, and $\chi_1, ..., \chi_t$ constitute a complete set of continuous invariants of projectors $\mathbf{P}_1, ..., \mathbf{P}_\rho$ from formula (3.27). The symbol,

$$[\rho + t + u] \tag{8.9}$$

we call the second structural index of the class of elastic bodies under consideration. We write the structural formula as follows

$$\begin{aligned} C_{ijkl} = \lambda_1 P_{(1)\,ijkl}(\chi_1, ..., \chi_t; \quad \varphi_1, ..., \varphi_u) + \\ ... + \lambda_\rho P_{(\rho)\,ijkl}(\chi_1, ..., \chi_t; \quad \varphi_1, ..., \varphi_u). \end{aligned} \tag{8.10}$$

As there is an extensive experimental material on the anisotropy of elastic materials (see, for example, [21–23]) developed as a function of component values C_{ijkl} in selected material bases, there arises a practical need to reverse these relationships

$$\lambda_\alpha = \lambda_\alpha(C_{ijkl}), \qquad \chi_b = \chi_b(C_{ijkl}), \qquad \varphi_c = \varphi_c(C_{ijkl}). \tag{8.11}$$

In the general case, there can be no question about specifying closed formulae – the sixth degree equation of characteristic equation (3.33) hinders this. Numerical calculations do not present any difficulties. However, for the most important cases, everything simplifies considerably. We show this in §10 using the example of transversely isotropic bodies.

§9 Elasticity of bodies with rigid constraints

The idea of the **incompressible** body enjoys well-deserved popularity in the mechanics of continuous media. In the case of an isotropic body, it is very natural. In the general case of anisotropy, the concept of incompressibility is clearly artificial. The problem of the elasticity of bodies with rigid internal constraints obtains a very clear physical interpretation within the framework of the approach developed here.

We take the elastic body

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \ldots + \lambda_\nu \mathbf{P}_\nu + \ldots + \lambda_\rho \mathbf{P}_\rho \tag{9.1}$$

and assume that with respect to elastic eigenstates from space $\mathbf{P}_\nu$ the body is much stiffer than for all others, i.e.

$$\lambda_\nu \gg \lambda_1, \ldots, \lambda_{\nu-1},\ \lambda_{\nu+1}, \ldots, \lambda_\rho. \tag{9.2}$$

In such a case, the following idealization may turn out to be warranted; **compliance in relation to all states from the subspace $\mathcal{P}_\nu$ is equal to zero, i.e.**

$$\lambda_\nu^{-1} = 0. \tag{9.3}$$

We call this condition the condition of l-**fold rigid constraints**, where $l \equiv q_\nu$. We give this idealization a correct form.

The compliance tensor corresponding to the body (9.1) with the condition (9.3) has the form

$$\mathbf{S} = \frac{1}{\lambda_1}\mathbf{P}_1 + \ldots + \frac{1}{\lambda_{\nu-1}}\mathbf{P}_{\nu-1} + \frac{1}{\lambda_{\nu+1}}\mathbf{P}_{\nu+1} + \ldots + \frac{1}{\lambda_\rho}\mathbf{P}_\rho. \tag{9.4}$$

For any deformation $\boldsymbol{\varepsilon}$ we get,

$$\boldsymbol{\varepsilon}_\nu \equiv \mathbf{P}_\nu \cdot \boldsymbol{\varepsilon} = \mathbf{P}_\nu \cdot (\mathbf{S} \cdot \boldsymbol{\sigma}) = \mathbf{0}, \tag{9.5}$$

as it should be. Substituting the formula for small deformations expressed by displacements $\mathbf{u}$, we obtain equations as a function of displacements,

$$\mathbf{P}_\nu \cdot (\nabla \mathbf{u}) = 0, \qquad P_{(\nu)\,ijkl}\, u_{k,\,l} = 0. \tag{9.6}$$

The number of independent scalar conditions is equal here $q_\nu = l$. If $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_a$ is some basis in $\mathcal{P}_\nu$, then these conditions according to (3.20) have the form,

$$\omega_{(1)\,pq}\, u_{p,q} = 0, \ldots, \omega_{(a)\,pq}\, u_{p,q} = 0. \tag{9.7}$$

We consider the problem of stresses. Hooke's law in the form (2.1) with the stiffness tensor (9.1) at (9.3) is incorrect in the term

$$\boldsymbol{\sigma} = \ldots + \lambda_\nu \mathbf{P}_\nu \cdot \boldsymbol{\varepsilon} + \ldots, \tag{9.8}$$

the written out element represents indeterminacy of the type $\infty \cdot 0$.

We introduce decomposition

$$\mathcal{S} = \mathcal{P}_\nu^\perp \oplus \mathcal{P}_\nu, \tag{9.9}$$

$$\mathcal{P}_\nu^\perp = \mathcal{P}_1 \oplus \ldots \oplus \mathcal{P}_{\nu-1} \oplus \mathcal{P}_{\nu+1} \oplus \ldots \oplus \mathcal{P}_\rho \tag{9.10}$$

and write down

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_a + \boldsymbol{\sigma}_\nu, \qquad \boldsymbol{\sigma}_a \in \mathcal{P}_\nu^\perp, \qquad \boldsymbol{\sigma}_\nu \in \mathcal{P}_\nu. \tag{9.11}$$

Hooke's law determines only the **active** part of stresses $\boldsymbol{\sigma}_a$. The **reactive** part of the stresses $\boldsymbol{\sigma}_\nu$ is not related to deformations and is determined only by the equations of motion (equilibrium) and the boundary conditions for stresses.

Thus, the stiffness tensor for an elastic body with rigid constraints (9.6) has the form

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + \ldots + \lambda_{\nu-1} \mathbf{P}_{\nu-1} + \lambda_{\nu+1} \mathbf{P}_{\nu+1} + \ldots + \lambda_\rho \mathbf{P}_\rho, \tag{9.12}$$

and Hooke's law takes the form

$$\boldsymbol{\sigma}_a = \mathbf{C} \cdot \boldsymbol{\varepsilon}_a. \tag{9.13}$$

We note that the operator $\mathbf{C}$ turns out to be the generalized reverse operator of the operator $\mathbf{S}$ [15], i.e.

$$\mathbf{C} \circ \mathbf{S} = \mathbf{I}^{(4s)\prime} \equiv \mathbf{I}^{(4s)} - \mathbf{P}_\nu. \tag{9.14}$$

Incompressibility is a special case of 1-dimensional rigid constraints, $l = 1$, when

$$\mathbf{P}_\nu = \frac{1}{3} \mathbf{1} \otimes \mathbf{1}. \tag{9.15}$$

Another special case of 1-dimensional rigid constraints, $l = 1$ is **inextensibility** along a certain direction $\mathbf{n} \otimes \mathbf{n}$. Then

$$\mathbf{P}_\nu = \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n}. \tag{9.16}$$

§10 First examples

Fine, a sophisticated reader can say, but the task of finding the appropriate elastic eigenstates for a given elastic body is not at all simple, at least computationally. My answer is: solving this task directly will be necessary only in exceptional cases. On the contrary, I think it is better to do the opposite:

> An elastic body should be **characterized** by its set of elastic eigenstates and the corresponding to them stiffness moduli.

We give the first simple examples.

10.1. Perfectly elastic bodies

> **Definition 1.** An elastic body is called **perfectly elastic** body if any symmetric tensor of the second-order is its elastic eigenstate.

We note that if the structural formula (3.27) contains more than one term then there are tensors that are not eigentensors. Therefore, for $\rho = 1$ the stiffness tensor and Hooke's law take the form*

$$\mathbf{C} = \lambda\,\mathbf{I}^{(4s)}, \qquad \boldsymbol{\sigma} = \lambda\,\boldsymbol{\varepsilon}, \qquad \lambda > 0. \tag{10.1}$$

Structural indexes are integers

$$\langle 6\rangle, \qquad [1+0+0]. \tag{10.2}$$

*Translator note: Note that the coefficient in formula (10.1) denoted by λ does not denote the Lamé coefficient λ. The case of a perfectly elastic body can be interpreted as the limiting case of an isotropic elastic body (cf. Section 10.2) below, for which the Lamé coefficient $\lambda = 0$. Then, it becomes clear that the coefficient λ present in the formula (10.1) should be interpreted as the equivalent to the Lamé coefficient 2μ.

The existence of the bodies in question is theoretically possible, but they are not a good idealization of real materials.

10.2. Isotropic elastic bodies

We adopt the following refreshing definition,

Definition 2. We call an **isotropic elastic body** an elastic body for which each pure shear is its elastic eigenstate[3].

In order to proceed from this definition to the known Hooke's law, we use the following lemma (valid for any symmetric operator in Euclidean space).

Lemma 2. Let $\mathcal{P}$, $\mathcal{D}$ be subspaces of the elastic eigenstates of the stiffness tensor $\mathbf{C}$. If $\mathcal{P}$ is not orthogonal to $\mathcal{D}$ then the entire simple sum $\mathcal{P} \oplus \mathcal{D}$ consists of elastic eigenstates.

Proof. Let $\lambda_{\mathcal{P}}$, $\lambda_{\mathcal{D}}$ be eigenvalues corresponding to the eigenstates from $\mathcal{P}$ and $\mathcal{D}$, respectively. Since $\mathcal{P}$ and $\mathcal{D}$ are not orthogonal, there exist $\boldsymbol{\pi} \in \mathcal{P}$, $\boldsymbol{\rho} \in \mathcal{D}$, such that $\boldsymbol{\pi} \cdot \boldsymbol{\rho} \neq 0$. Then

$$\lambda_{\mathcal{P}} \boldsymbol{\pi} \cdot \boldsymbol{\rho} = \boldsymbol{\rho} \cdot \mathbf{C} \cdot \boldsymbol{\pi} = \boldsymbol{\pi} \cdot \mathbf{C} \cdot \boldsymbol{\rho} = \lambda_{\mathcal{D}} \boldsymbol{\pi} \cdot \boldsymbol{\rho},$$

i.e. $\lambda_{\mathcal{P}} = \lambda_{\mathcal{D}}$. ♦

Theorem 7. The material decomposition (3.26) for an isotropic body has the form

$$\mathcal{S} = \mathcal{P} \oplus \mathcal{D}, \tag{10.3}$$

where $\mathcal{P}$ is 1-dimensional space of spherical tensors, $\mathcal{D}$ is 5-dimensional space of deviators.

Proof. Let us take some orthonormal basis $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and five pure shears

$$\begin{gathered}\boldsymbol{\tau}_1 \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\tau}_2 \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\tau}_3 \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \\ \boldsymbol{\tau}_4 \sim \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\tau}_5 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \end{gathered} \tag{10.4}$$

[3] I am not trying here to replace with this definition the well-known one. My task is to emphasize the unnoticed subtleties in this painting.

Of course that $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_5$ is a basis (not orthonormal!) for space of deviators $\mathcal{D}$. Since $\operatorname{Lin}\boldsymbol{\tau}_1$, $\operatorname{Lin}\boldsymbol{\tau}_2$ are spaces of eigenstates according to the very definition of an isotropic body, $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \neq 0$, then according to the lemma $\operatorname{Lin}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ consists of eigenstates. Continuing in the same vein, we find that $\operatorname{Lin}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_5) = \mathcal{D}$ is composed of eigenstates. But then the complement $\mathcal{P} = \mathcal{D}^{\perp}$ consists of eigenstates. ⧫

Since projectors for spherical tensors space and deviators space are well known*,

$$\mathbf{I}_{\mathcal{P}} = \frac{1}{3}\mathbf{1} \otimes \mathbf{1}, \qquad \mathbf{I}_{\mathcal{D}} = \mathbf{I}^{(4s)} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}, \tag{10.5}$$

so the stiffness tensor for an isotropic body can be expressed as follows

$$\begin{aligned} \mathbf{C} &= \lambda_{\mathcal{P}}\, \mathbf{I}_{\mathcal{P}} + \lambda_{\mathcal{D}}\, \mathbf{I}_{\mathcal{D}} = \\ &= \frac{1}{3}(\lambda_{\mathcal{P}} - \lambda_{\mathcal{D}})\, \mathbf{1} \otimes \mathbf{1} + \lambda_{\mathcal{D}}\, \mathbf{I}^{(4s)}. \end{aligned} \tag{10.6}$$

The decomposition (5.6) takes the form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathcal{P}} + \boldsymbol{\sigma}_{\mathcal{D}}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathcal{P}} + \boldsymbol{\varepsilon}_{\mathcal{D}}, \tag{10.7}$$

where, for example:

$$\boldsymbol{\sigma}_{\mathcal{P}} \equiv \mathbf{I}_{\mathcal{P}} \cdot \boldsymbol{\sigma} = \left(\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}\right)\mathbf{1}, \qquad \boldsymbol{\sigma}_{\mathcal{D}} \equiv \mathbf{I}_{\mathcal{D}} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathcal{P}}. \tag{10.8}$$

The orthogonal decomposition of Hooke's law (5.7) takes the form

$$\boldsymbol{\sigma}_{\mathcal{P}} = \lambda_{\mathcal{P}}\, \boldsymbol{\varepsilon}_{\mathcal{P}}, \qquad \boldsymbol{\sigma}_{\mathcal{D}} = \lambda_{\mathcal{D}}\, \boldsymbol{\varepsilon}_{\mathcal{D}}. \tag{10.9}$$

We see, as we announced in §5, that all these formulae, well known for the isotropic body, are in fact special cases of our general relations.

Lamé constants can be expressed in terms of the stiffness moduli $\lambda_{\mathcal{P}}$, $\lambda_{\mathcal{D}}$ as follows:

$$\lambda = \frac{1}{3}(\lambda_{\mathcal{P}} - \lambda_{\mathcal{D}}), \qquad \mu = \frac{1}{2}\lambda_{\mathcal{D}}. \tag{10.10}$$

For modulus of volumetric compression and the Poisson ratio, we have:

$$K = \frac{1}{3}\lambda_{\mathcal{P}}, \qquad \nu = \frac{\lambda_{\mathcal{P}} - \lambda_{\mathcal{D}}}{2\lambda_{\mathcal{P}} + \lambda_{\mathcal{D}}}. \tag{10.11}$$

*Translator note: $\mathbf{I}_{\mathcal{P}} \perp \mathbf{I}_{\mathcal{D}}$.

Structural indexes for the class of isotropic bodies are equal

$$\langle 1+5\rangle, \qquad [2+0+0]. \tag{10.12}$$

The parameters of the stiffness distributors $\chi_1, \ldots, \chi_{12}$ and the orientational angles with respect to the laboratory coordinate system θ, φ, ψ are absent because **there is nothing** to distribute and **nothing** to orientate: there is not a single fiber of material in the body to be distinguished.

Comparing (10.6) with (10.1), we see that

> **perfectly elastic body is an isotropic elastic body with the Poisson's ratio equal to zero***.

Note 7. If we know in advance that the stiffness tensor of an isotropic body has the form (10.6), then it is immediately clear that the spherical tensors and deviators are elastic eigenstates. This is demonstrated (without using the concept of elastic eigenstate!) in almost every good textbook of the theory of elasticity. ⧫

Assuming $\lambda_{\mathcal{P}}^{-1} = 0$ we get an **incompressible** isotropic body

$$\mathbf{C} = \lambda_{\mathcal{D}}\, \mathbf{I}_{\mathcal{D}}. \tag{10.13}$$

The case $\lambda_{\mathcal{D}}^{-1} = 0$ corresponds to an elastic body, completely rigid in shape. Then

$$\mathbf{C} = \lambda_{\mathcal{P}}\, \mathbf{I}_{\mathcal{P}}, \tag{10.14}$$

$$\boldsymbol{\varepsilon} = \left(\frac{1}{3}\mathrm{tr}\,\boldsymbol{\varepsilon}\right)\mathbf{1} \tag{10.15}$$

(not to be confused with the ideal liquid!). This case, apparently, is more difficult to carry out in practice.

10.3. Volumetrically-isotropic elastic bodies

It is worthwhile to distinguish the following family of elastic bodies.

> **Definition 3.** A **volumetrically-isotropic** body is called any elastic body for which the spherical tensor is its elastic eigenstate.

*Translator note: $\lambda = \lambda_{\mathcal{P}} - \lambda_{\mathcal{D}} = 0 \Rightarrow \nu = 0$.

Theorem 2 for volumetrically isotropic bodies has the following form:

$$\mathcal{S} = \mathcal{P} \oplus (\mathcal{D}_1 \oplus ... \oplus \mathcal{D}_\gamma), \tag{10.16}$$

$$\mathbf{C} = K(\mathbf{1} \otimes \mathbf{1}) + (\mu_1 \mathbf{P}_1 + ... + \mu_\gamma \mathbf{P}_\gamma), \tag{10.17}$$

where $\mathcal{P}$ is the 1-dimensional space of spherical tensors, K is the modulus of volumetric stiffness, $\mathcal{D}_1, ..., \mathcal{D}_\gamma$ is the orthogonal decomposition of the space of deviators $\mathcal{D}$, $\mathbf{P}_1, ..., \mathbf{P}_\gamma$ is the corresponding to it decomposition of the unit tensor $\mathbf{I}_\mathcal{D}$ of the space $\mathcal{D}$,

$$\mathbf{P}_1 + ... + \mathbf{P}_\gamma = \mathbf{I}_\mathcal{D}, \qquad \gamma \leq 5. \tag{10.18}$$

We do not go into the details of describing volumetrically-isotropic bodies. We note only the following interesting theorem:

> **Theorem 8.** If not a single pure shear is an elastic eigenstate of a volumetrically-isotropic body, then $\gamma = 5$, i.e. the first structural index has the form $\langle 1 + 1 + 1 + 1 + 1 + 1 \rangle$.

Proof. We execute it through the opposite. We show that in any 2-dimensional subspace of the deviators space there is pure shear. Indeed, let us take non-proportional but otherwise arbitrary deviators $\boldsymbol{\pi}$, $\boldsymbol{\rho}$ of course different from pure shears, i.e.[4] $\mathrm{tr}(\boldsymbol{\pi}^3) \neq 0$, $\mathrm{tr}(\boldsymbol{\rho}^3) \neq 0$. We consider a non-zero linear combination $\boldsymbol{\tau} = a\boldsymbol{\pi} + b\boldsymbol{\rho}$, $a^2 + b^2 \neq 0$ and demand that $\boldsymbol{\tau}$ be pure shear,

$$\begin{aligned} \mathrm{tr}(\boldsymbol{\tau}^3) = a^3\,\mathrm{tr}(\boldsymbol{\pi}^3) + b^3\,\mathrm{tr}(\boldsymbol{\rho}^3) + a^2 b\,\mathrm{tr}(\boldsymbol{\pi}^2\boldsymbol{\rho}) +& \\ + ab^2\,\mathrm{tr}(\boldsymbol{\pi}\,\boldsymbol{\rho}^2) = 0.& \end{aligned} \tag{10.19}$$

Because $b \neq 0$ then by introducing $t \equiv a/b$ we get the equation

$$t^3\,\mathrm{tr}(\boldsymbol{\pi}^3) + t^2\,\mathrm{tr}(\boldsymbol{\pi}^2\boldsymbol{\rho}) + t\,\mathrm{tr}(\boldsymbol{\pi}\,\boldsymbol{\rho}^2) + \mathrm{tr}(\boldsymbol{\rho}^3) = 0, \tag{10.20}$$

which has at least one real, non-zero solution. ♦

Definition 2 and Theorem 8 draw our attention to the special role of pure shears. Expressing it somewhat imprecisely, but visually, one can say that the pure shear eigenstate is a kind of a trace of isotropy in an anisotropic body.

[4]The necessary and sufficient condition for the tensor to be pure shear has the form $\mathrm{tr}(\boldsymbol{\tau}) = \mathrm{tr}(\boldsymbol{\tau}^3) = 0$.

10.4. Transversely isotropic elastic bodies

We start with a special example. We describe the elastic properties of the composite shown in Figure 4. The isotropic matrix is reinforced with a family of thin parallel flat layers of another isotropic material and a bundle of elastic fibers perpendicular to the layers. The direction of the fibers is determined by the dyad $\mathbf{k} \otimes \mathbf{k}$. We use an orthonormal basis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 = \mathbf{k}$.

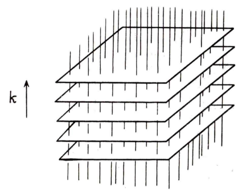

Fig. 4. An example of a transversely-isotropic composite with an isotropic matrix reinforced with a family of thin parallel flat layers of another isotropic material and a bundle of elastic fibers perpendicular to the layers.

Firstly, it is clear that because the layers are thin and the joints between the fibers and the layers are not rigid, in the event of whichever (small!) shear deformation

$$\begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ p & q & 0 \end{pmatrix} \tag{10.21}$$

the layers and fibers behave like rigid bodies, i.e. only the matrix deforms[5]. Since the matrix is isotropic, any shear is its eigenstate, with one and the same stiffness modulus. Therefore, any **shear (10.21) is an elastic eigenstate of the considered composite**. We have obtained a $\mathcal{P}_3$, 2-dimensional space of elastic eigenstates of the form (10.21).

An orthonormal basis in $\mathcal{P}_3$ would be, for example, a pair

$$\boldsymbol{\omega}_{\mathrm{III}} = \frac{\sqrt{2}}{2}(\mathbf{n}_1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_1) \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{10.22}$$

$$\boldsymbol{\omega}_{\mathrm{IV}} = \frac{\sqrt{2}}{2}(\mathbf{n}_2 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_2) \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{10.23}$$

[5]Therefore, it makes no sense to strengthen a rod that is twisted in the direction of the axis $\mathbf{k} \otimes \mathbf{k}$!

The projektor on $\mathcal{P}_3$ has the form*

$$\begin{aligned}\mathbf{P}_3 &= \boldsymbol{\omega}_{\mathrm{III}} \otimes \boldsymbol{\omega}_{\mathrm{III}} + \boldsymbol{\omega}_{\mathrm{IV}} \otimes \boldsymbol{\omega}_{\mathrm{IV}} = \\ &= \frac{1}{2}(\sigma_1 + \sigma_2 - \varepsilon) \times [\mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{k} \otimes \mathbf{k} - \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{1}] + \frac{1}{2}\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k},\end{aligned} \tag{10.24}$$

where $\varepsilon \equiv \langle 1\,2\,3\,4 \rangle$, $\sigma_1 \equiv \langle 1\,3\,2\,4 \rangle$, $\sigma_2 \equiv \langle 1\,4\,3\,2 \rangle$**.

Secondly, in the case of whichever shear deformation in the plane of the layers, i.e. deformation of the type

$$\begin{pmatrix} u & v & 0 \\ v & -u & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10.25}$$

the fibers move like rigid bodies, and only the matrix and layers deform. Since both the matrix and the layers are isotropic, the shears (10.25) are their eigenstates. Therefore, **any shear (10.25) is an elastic eigenstate of the considered composite**. We obtained $\mathcal{P}_4$, a 2-dimensional space of eigenshears of the form (10.25), $\mathcal{P}_4 \perp \mathcal{P}_3$.

An orthonormal basis in $\mathcal{P}_4$ is, for example

$$\boldsymbol{\omega}_{\mathrm{V}} = \frac{\sqrt{2}}{2}(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2) \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10.26}$$

$$\boldsymbol{\omega}_{\mathrm{VI}} = \frac{\sqrt{2}}{2}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1) \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10.27}$$

The projector $\mathcal{P}_4$ has the form***:

$$\begin{aligned}\mathbf{P}_4 &= \boldsymbol{\omega}_{\mathrm{V}} \otimes \boldsymbol{\omega}_{\mathrm{V}} + \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}} = \\ &= \frac{1}{2}(\sigma_1 + \sigma_2) \times [\mathbf{1} \otimes \mathbf{k} \otimes \mathbf{k} - \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{1}] - 2\mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{k},\end{aligned} \tag{10.28}$$

where $\sigma_1 \equiv \langle 1\,3\,2\,4 \rangle$, $\sigma_2 \equiv \langle 1\,4\,3\,2 \rangle$.

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*Translator note: $\mathbf{P}_3$ can be expressed equivalently – more transparently, in the form, $\mathbf{P}_3 = \frac{1}{2}(\mathbf{n}_1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_1) \otimes (\mathbf{n}_1 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_1) + \frac{1}{2}(\mathbf{n}_2 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_2) \otimes (\mathbf{n}_2 \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{n}_2)$, $\mathbf{n}_1 \cdot \mathbf{n}_2 = \mathbf{n}_1 \cdot \mathbf{k} = \mathbf{n}_2 \cdot \mathbf{k} = 0$ (where $\mathbf{n}_1$ and $\mathbf{n}_2$ are free).

**Translator note: ε, σ_1, σ_2 denote the permutations of the indices.

***Translator note: $\mathbf{P}_4$ can be expressed equivalently – more transparently in the form, $\mathbf{P}_4 = \frac{1}{2}(\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2) \otimes (\mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2) + \frac{1}{2}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1) \otimes (\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1)$.

Thirdly, we consider $\mathcal{P} \equiv (\mathcal{P}_3 \oplus \mathcal{P}_4)^{\perp}$. Recalling the definitions (10.21) and (10.25) of spaces $\mathcal{P}_3$ and $\mathcal{P}_4$ we see that the orthogonal complement $\mathcal{P}$ is 2-dimensional and consists of tensors of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}. \tag{10.29}$$

In the case of our composite, as for any other elastic body, the decomposition of space $\mathcal{S}$ into the sum of the space of eigenstates should be complete, (3.26). Therefore, either the whole $\mathcal{P}$ consists of elastic eigenstates, or $\mathcal{P}$ **divides into two 1-dimensional subspaces of eigenstates**, $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$. In general case, i.e. for any set of matrix, layers and fibers stiffnesses and specific volume fractions of the fiber plies, there is no justification for the first possibility. There is also no justification for distinguishing any pair $\mathcal{P}_1$, $\mathcal{P}_2$. Therefore, it is necessary to consider **all** orthogonal decompositions $\mathcal{P}$, i.e. **all** orthonormal pairs $\boldsymbol{\omega}_{\mathrm{I}}$, $\boldsymbol{\omega}_{\mathrm{II}}$ in $\mathcal{P}$. It is easy to notice that they constitute a one-parameter set that can be conveniently represented as[6]:

$$\begin{aligned} \boldsymbol{\omega}_{\mathrm{I}} &= \frac{\sqrt{2}}{2}[\sin(\chi)\mathbf{1} + \sqrt{3}\sin(\chi_0 - \chi)\mathbf{k}\otimes\mathbf{k}] \\ &\sim \frac{\sqrt{2}}{2}\begin{pmatrix} \sin(\chi) & 0 & 0 \\ 0 & \sin(\chi) & 0 \\ 0 & 0 & \sqrt{2}\cos(\chi) \end{pmatrix}, \end{aligned} \tag{10.30}$$

$$\begin{aligned} \boldsymbol{\omega}_{\mathrm{II}} &= \frac{\sqrt{2}}{2}[\cos(\chi)\mathbf{1} + \sqrt{3}\cos(\chi_0 - \chi)\mathbf{k}\otimes\mathbf{k}] \\ &\sim \frac{\sqrt{2}}{2}\begin{pmatrix} \cos(\chi) & 0 & 0 \\ 0 & \cos(\chi) & 0 \\ 0 & 0 & -\sqrt{2}\sin(\chi) \end{pmatrix}. \end{aligned} \tag{10.31}$$

Here χ is a parameter of the family, and without losing generality it can be assumed that

$$0 \le \chi < \pi/2, \tag{10.32}$$

[6]The form $\boldsymbol{\omega}_{\mathrm{I}} = g(s)\,[\mathbf{1} + s\,\mathbf{k}\otimes\mathbf{k}]$, $\boldsymbol{\omega}_{\mathrm{II}} = g(t)\,[\mathbf{1} + t\,\mathbf{k}\otimes\mathbf{k}]$ is also good, where $g(x) = (x^2 + 3x + 3)^{1/2}$, $3 + (t + s) + st = 0$.

and χ_0 is determined by the condition $\boldsymbol{\omega}_{\mathrm{I}} = \frac{\sqrt{3}}{3}\mathbf{1}$, i.e. $\mathrm{tg}(\chi_0) = \sqrt{2}^*$. Thus, we obtained for the considered composite a 1-dimensional space of eigenstates $\mathcal{P}_1$ proportional to $\boldsymbol{\omega}_{\mathrm{I}}$ and a 1-dimensional space of eigenstates $\mathcal{P}_2$ proportional to $\boldsymbol{\omega}_{\mathrm{II}}$. For projectors on $\mathcal{P}_1$, $\mathcal{P}_2$ we have

$$\begin{aligned}\mathbf{P}_1 = \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} = \frac{1}{2}\sin^2(\chi)\,\mathbf{1}\otimes\mathbf{1} + \\ + \frac{\sqrt{3}}{2}\sin(\chi)\sin(\chi_0-\chi)\,[\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{k} + \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{1}] + \\ + \frac{3}{2}\sin^2(\chi_0-\chi)\,\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k},\end{aligned} \tag{10.33}$$

$$\begin{aligned}\mathbf{P}_2 = \boldsymbol{\omega}_{\mathrm{II}} \otimes \boldsymbol{\omega}_{\mathrm{II}} = \frac{1}{2}\cos^2(\chi)\,\mathbf{1}\otimes\mathbf{1} + \\ - \frac{\sqrt{3}}{2}\cos(\chi)\cos(\chi_0-\chi)\,[\mathbf{1}\otimes\mathbf{k}\otimes\mathbf{k} + \mathbf{k}\otimes\mathbf{k}\otimes\mathbf{1}] + \\ + \frac{3}{2}\cos^2(\chi_0-\chi)\,\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{k}.\end{aligned} \tag{10.34}$$

Thus, all elastic eigenstates of the considered composite were determined.

When looking for elastic states, we referred to the mechanical concepts related specifically to the composite shown in Figure 4. However, if you look closely, it can be noticed that singling out of the spaces of relevant shears (10.21) and (10.25) is correct for any structure with axial symmetry. This can be rigorously demonstrated [24], however, while maintaining a uniform methodology, we assume here the following,

> **Definition 4.** A **transversely-isotropic** elastic body we call an elastic body for which a direction can be indicated $\mathbf{k}\otimes\mathbf{k}$, such that any shear
>
> $$\boldsymbol{\tau} = \mathbf{a}\otimes\mathbf{k} + \mathbf{k}\otimes\mathbf{a}, \qquad \mathbf{a}\mathbf{k} = 0, \tag{10.35}$$
>
> and any shear
>
> $$\boldsymbol{\tau} = \mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}, \quad \mathbf{a}\mathbf{k} = \mathbf{b}\mathbf{k} = \mathbf{a}\mathbf{b} = 0, \tag{10.36}$$
>
> are elastic eigenstates.

From this definition, we conclude the form of the stiffness tensor. First of all, we note that the shears (10.35) create a linear subspace in $\mathcal{S}$. Similarly, the shears (10.36) form a linear subspace in $\mathcal{S}$. These subspaces are nothing but $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively. Now, using the lemma, we immediately prove that

*Translator note: $\sin(\chi_0) = \frac{\sqrt{2}}{\sqrt{3}}$, $\cos(\chi_0) = \frac{1}{\sqrt{3}}$.

the shears (10.35) have a common stiffness modulus, the same as the shears (10.36). In other words $\mathcal{P}_3$ and $\mathcal{P}_4$ are spaces of eigenstates. But also $\mathcal{P}_1$, $\mathcal{P}_2$ for a certain χ, already dependent on a given body are spaces of eigenstates. Thus, we proved the following theorem:

Theorem 9. For any transversely-isotropic body, the structural decomposition has the form

$$\mathcal{S} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \mathcal{P}_4, \tag{10.37}$$

where $\mathcal{P}_1$, $\mathcal{P}_2$ depend on χ and $\mathbf{k} \otimes \mathbf{k}$, while $\mathcal{P}_3$, $\mathcal{P}_4$ on $\mathbf{k} \otimes \mathbf{k}$. This means that the stiffness tensor can be expressed by the formula

$$\begin{aligned} \mathbf{C} = \lambda_1 \mathbf{P}_1(\chi,\ \mathbf{k}\otimes\mathbf{k}) + \lambda_2 \mathbf{P}_2(\chi,\ \mathbf{k}\otimes\mathbf{k}) + \\ + \lambda_3 \mathbf{P}_3(\mathbf{k}\otimes\mathbf{k}) + \lambda_4 \mathbf{P}_4(\mathbf{k}\otimes\mathbf{k}), \end{aligned} \tag{10.38}$$

where $\mathbf{P}_1, ..., \mathbf{P}_4$ are defined by the formulae (10.33), (10.34), (10.24) and (10.28).

Instead of the argument $\mathbf{k} \otimes \mathbf{k}$ one can enter two angles

$$\mathbf{k} \otimes \mathbf{k} \leftrightarrow \varphi_1,\ \varphi_2, \tag{10.39}$$

fixing $\mathbf{k} \otimes \mathbf{k}$ relative to the laboratory coordinate system. The structural indexes of the class of transversely-isotropic bodies, in the general case, have the form

$$\langle 1 + 1 + 2 + 2 \rangle, \qquad [4 + 1 + 2]. \tag{10.40}$$

The eigenstates of a transversely-isotropic body are shown in Figure 5.

We have found 5 material constants of a transversely-isotropic body

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \chi. \tag{10.41}$$

The meaning of the first four is perfectly clear; in particular λ_3, λ_4 are simply shear moduli (Kirchhoff moduli) for (10.35) and (10.36). A bit more complicated is the mechanical interpretation of **the stiffness distribution parameter** χ. The transversely-isotropic material is volumetrically-isotropic if and only if

$$\chi = \chi_0, \quad \text{where} \quad \sin(\chi_0) = \frac{\sqrt{6}}{3}, \qquad \cos(\chi_0) = \frac{\sqrt{3}}{3}, \tag{10.42}$$

and as a consequence

$$\boldsymbol{\omega}_{\mathrm{I}} = \frac{\sqrt{3}}{3}\mathbf{1}, \qquad \boldsymbol{\omega}_{\mathrm{II}} = \frac{\sqrt{6}}{6}(\mathbf{1} - 3\mathbf{k}\otimes\mathbf{k}). \tag{10.43}$$

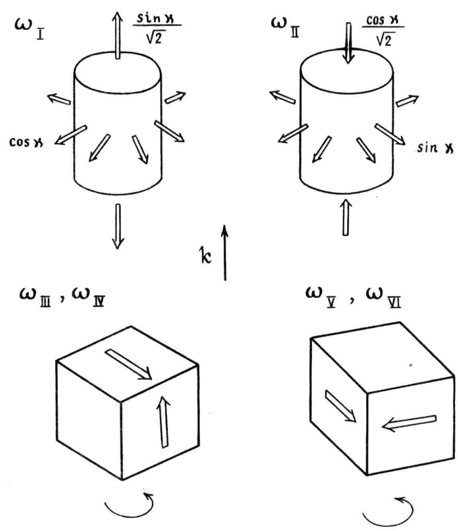

Fig. 5. Graphical illustration of the elastic eigenstates of a transversely-isotropic material.

From formulae (7.6)–(7.8) it follows that

$$\frac{1}{3K} = \frac{\cos^2(\chi_0 - \chi)}{\lambda_1} + \frac{\sin^2(\chi_0 - \chi)}{\lambda_2}, \tag{10.44}$$

$$\frac{1}{E(n)} = \frac{\cos^2(\chi)}{\lambda_1} + \frac{\sin^2(\chi)}{\lambda_2}, \tag{10.45}$$

$$-\frac{\nu(k)}{E(k)} = \frac{\sqrt{2}}{4}\sin(2\chi)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right). \tag{10.46}$$

We highlight the following characteristic of the parameter χ: as soon as $\lambda_1 = \lambda_2$, the value χ is no longer relevant*.

We express the new constants of the transversely-isotropic body (10.41) by the conventional ones, which are non-zero components of C_{ijkl} in the basis $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3 = \mathbf{k}$:

$$\begin{aligned} &C_{11} \equiv C_{1111} = C_{2222} \equiv C_{22}, && C_{12} \equiv C_{1122}, \\ &C_{13} \equiv C_{1133} = C_{2233} \equiv C_{23}, && C_{33} \equiv C_{3333}, \\ &C_{44} \equiv C_{2323} = C_{1313} \equiv C_{55}, && \frac{1}{2}(C_{11} - C_{12}) = C_{1212} \equiv C_{66}, \end{aligned} \tag{10.47}$$

*Translator note: $\lambda_1 = \lambda_2 \Leftrightarrow C_{13} \equiv 0$

(see, for example, [5, 10])*. The easiest way to do it is like this. We write Hooke's law in the form (5.11). We have for stresses:

$$
\begin{aligned}
\sigma_{\mathrm{I}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{I}} = \frac{1}{\sqrt{2}}(\sigma_{11}+\sigma_{22})\sin(\chi)+\sigma_{33}\cos(\chi),\\
\sigma_{\mathrm{II}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{II}} = \frac{1}{\sqrt{2}}(\sigma_{11}+\sigma_{22})\cos(\chi)-\sigma_{33}\sin(\chi),\\
\sigma_{\mathrm{III}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{III}} = \frac{1}{\sqrt{2}}\sigma_{13},\\
\sigma_{\mathrm{IV}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{IV}} = \frac{1}{\sqrt{2}}\sigma_{23},\\
\sigma_{\mathrm{V}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{V}} = \frac{1}{\sqrt{2}}(\sigma_{11}-\sigma_{22}),\\
\sigma_{\mathrm{VI}} &\equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\mathrm{VI}} = \frac{1}{\sqrt{2}}\sigma_{12}.
\end{aligned}
\tag{10.48}
$$

We obtain similar formulae for $\varepsilon_{\mathrm{I}},...,\varepsilon_{\mathrm{VI}}$. Hooke's law in the form (5.11) for a transversely-isotropic body takes the form

$$
\begin{gathered}
\left[\frac{1}{\sqrt{2}}(\sigma_{11}+\sigma_{22})\sin(\chi)+\sigma_{33}\cos(\chi)\right]=\lambda_1\left[\frac{1}{\sqrt{2}}(\varepsilon_{11}+\varepsilon_{22})\sin(\chi)+\varepsilon_{33}\cos(\chi)\right],\\
\left[\frac{1}{\sqrt{2}}(\sigma_{11}+\sigma_{22})\cos(\chi)-\sigma_{33}\sin(\chi)\right]=\lambda_2\left[\frac{1}{\sqrt{2}}(\varepsilon_{11}+\varepsilon_{22})\cos(\chi)-\varepsilon_{33}\sin(\chi)\right],\\
\sigma_{13}=\lambda_3\varepsilon_{13},\qquad \sigma_{23}=\lambda_3\varepsilon_{23},\\
\sigma_{11}-\sigma_{22}=\lambda_4(\varepsilon_{11}-\varepsilon_{22}),\qquad \sigma_{12}=\lambda_4\varepsilon_{12}.
\end{gathered}
\tag{10.49}
$$

We pay attention to the first of these relations. Clearly, one can consider that σ_{I} and ε_{I} in the transversely-isotropic body play the role, which in the isotropic body is played by the average pressure $\mathrm{tr}(\boldsymbol{\sigma})$ and corresponding to it volumetric deformation $\mathrm{tr}(\boldsymbol{\varepsilon})$.

*Translator note: Transversely-isotropic materials are uniquely characterized by 5 linearly independent elastic constants.

By comparing (10.49) with the standard notation

$$
\begin{aligned}
&\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33},\\
&\sigma_{22} = C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{13}\varepsilon_{33},\\
&\sigma_{33} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33},\\
&\sigma_{23} = 2C_{44}\varepsilon_{23}, \quad \sigma_{13} = 2C_{44}\varepsilon_{13}, \quad \sigma_{12} = (C_{11} - C_{12})\varepsilon_{12},
\end{aligned}
\tag{10.50}
$$

we get*:

$$
\begin{aligned}
&\lambda_1 = \sqrt{2}\,C_{13}\,\mathrm{tg}(\chi) + C_{33},\\
&\lambda_2 = -\sqrt{2}\,C_{13}\,\mathrm{tg}(\chi) + C_{11} + C_{12},\\
&\lambda_3 = 2C_{44},\\
&\lambda_4 = C_{11} - C_{12},
\end{aligned}
\tag{10.51}
$$

where $\mathrm{tg}(\chi)$ is the positive root of the equation

$$
\sqrt{2}\,C_{13}(\mathrm{tg}(\chi))^2 + (C_{33} - C_{11} - C_{12})\,\mathrm{tg}(\chi) - \sqrt{2}\,C_{13} = 0^{**}. \tag{10.52}
$$

When $C_{11} + C_{12} = C_{33} + C_{13}$ then $\mathrm{tg}(\chi) = \sqrt{2}$, i.e. the body is volumetrically-isotropic.

The set of material constants (10.41) is determined for composites on the basis of the material parameters of their components. For example, for the previously considered composite (10.41):

1) λ_3 is shear modulus of matrix,
2) λ_4 depends on the shear modulus of matrix and volume fraction of the plies,
3) λ_1, λ_2 and χ depends on all the parameters of the composite, but for a small volume fraction of reinforcement, the influence of Poisson's ratio of layers and fibers can be neglected.

---

*Translator note: The following mistaken original formulae have been corrected $\lambda_2 = -\sqrt{2}\,\mathrm{tg}(\chi) + C_{11} + C_{12}$, $\lambda_4 = (C_{11} - C_{22})$. It can be shown using (10.52), that the following formulae are also valid $\lambda_1 = \sqrt{2}\,C_{13}\frac{1}{\mathrm{tg}(\chi)} + (C_{11} + C_{12})$, $\lambda_2 = -\sqrt{2}\,C_{13}\frac{1}{\mathrm{tg}(\chi)} + C_{33}$. For clarity: $2C_{44} = 2C_{55}$ and $C_{11} - C_{12} = 2C_{66}$.

**Translator note: An explicit solution to this equation with respect to $\mathrm{tg}(\chi)$ takes the form $2\sqrt{2}C_{13}\cdot\mathrm{tg}(\chi) = (C_{11} + C_{12} - C_{33}) + \sqrt{(C_{11} + C_{12} - C_{33})^2 + 8C_{13}{}^2}$; $0 \le \chi < \pi/2$.

The structural formula (10.38) allows distinguishing a number of interesting special cases of transversely-isotropic bodies. We pay attention to some of them:

1) $\lambda_3 = \lambda_4$ – structural indexes of this class have the form

$$\langle 1 + 1 + 4\rangle, \qquad [3 + 1 + 2]; \tag{10.53}$$

this is the case, for example, for composite shown in Figure 6;

2) $\lambda_1 = \lambda_2$ – structural indexes of this class have the form

$$\langle 2 + 2 + 2\rangle, \qquad [3 + 0 + 2]; \tag{10.54}$$

this is a special case of a volumetrically-isotropic body*;

3) $\chi = 0$, $\lambda_1^{-1} = 0$; this is the case, for example, for a composite with inextensible fibers;

4) $\chi = 0$, $\lambda_2^{-1} = \lambda_4^{-1} = 0$; this is the case for the composite shown in Figure 4 with inextensible layers;

5) $\chi = 0$, $\lambda_1^{-1} = \lambda_2^{-1} = \lambda_4^{-1} = 0$; this is the case for a composite with inextensible fibers and layers.

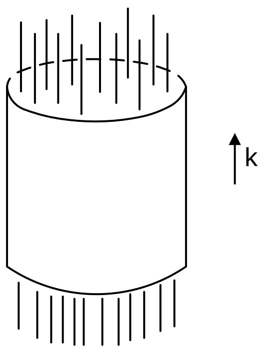

Fig. 6. An example of a transversely-isotropic composite with an isotropic matrix reinforced with a bundle of parallel elastic fibers.

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*Translator note: The above statement is incorrect. The condition $\lambda_1 = \lambda_2$ leads to the constraint equation $\mathrm{tg}(\chi)^2 = -1$, cf. formulas (10.51) and the note under formula (10.52) – the fulfillment of which is impossible. Hence, for the case $\lambda_1 = \lambda_2$ formulas (10.52) do not apply. A transversely-isotropic material is volumetrically-isotropic if and only if $\chi = \chi_0$, cf. (10.42). For a transversely-isotropic material and at the same time volumetrically-isotropic one it is $\lambda_1 = 2\,C_{13} + C_{33}$, $\lambda_2 = -C_{13} + C_{33}$.

Note 8. A **shear space** is any subspace $\mathcal{P} \subset \mathcal{S}$ sconsisting of pure shear, i.e. tensors $\boldsymbol{\tau}$ that satisfy the conditions $\mathrm{tr}(\boldsymbol{\tau}) = \det(\boldsymbol{\tau}) = 0$.

The following interesting theorem can be proved:

Theorem 10.

1. Any 2-dimensional shear space is either a plane deviators space (10.36), or an axial shears space (10.35).
2. Any pair of mutually orthogonal 2-dimensional shear spaces has the form (10.35) and (10.36) when selected appropriately.
3. There are no 3-dimensional shear spaces in $\mathcal{S}$.

This statement points to a special place occupied by transversely-isotropic bodies among anisotropic bodies.

10.5. Orthotropic elastic bodies

We start with a definition.

Definition 5. An **orthotropic elastic body** we call any elastic body for which there are such three mutually orthogonal directions $\mathbf{k} \otimes \mathbf{k}$, $\mathbf{l} \otimes \mathbf{l}$, $\mathbf{m} \otimes \mathbf{m}$, $\mathbf{mk} = \mathbf{ml} = \mathbf{kl} = 0$ that shears,

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{k} \otimes \mathbf{l} + \mathbf{l} \otimes \mathbf{k}, \\ \boldsymbol{\nu} &= \mathbf{l} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{l}, \\ \boldsymbol{\mu} &= \mathbf{k} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{k}, \end{aligned} \tag{10.55}$$

are elastic eigenstates.

An engineering intuition alone suffices to say that this will be the case, for example, in the case of the composite shown in Figure 7. The isotropic matrix is here reinforced with two plane-parallel mutually orthogonal families of thin layers.

We derive a structural formula. We take an orthonormal basis $\mathbf{n}_i$ oriented in the directions indicated in the definition. We introduce an orthonormal system of pure shears:

$$\boldsymbol{\omega}_{\mathrm{III}} \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega}_{\mathrm{IV}} \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\boldsymbol{\omega}_{\mathrm{V}} \sim \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{10.56}$$

According to Definition 5, we get 1-dimensional spaces of eigenshears $\mathcal{P}_K$ proportional to $\boldsymbol{\omega}_K$, $K = \mathrm{III}, \mathrm{IV}, \mathrm{V}$.

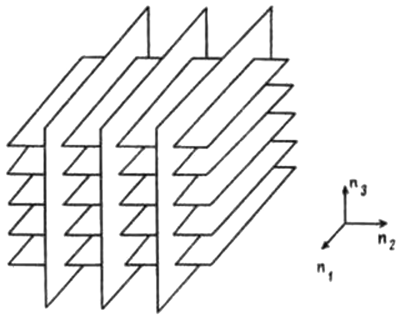

Fig. 7. An example of an orthotropic composite with an isotropic matrix reinforced with two plane-parallel and mutually orthogonal families of thin layers.

We consider $\mathcal{P} \equiv (\mathcal{P}_{\mathrm{III}} \oplus \mathcal{P}_{\mathrm{IV}} \oplus \mathcal{P}_{\mathrm{V}})^{\perp}$. This space is 3-dimensional and consists of tensors of the form

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}, \tag{10.57}$$

that is, from all tensors for which the directions specified in the definition are the principal directions. We obtain **all** orthogonal decompositions $\mathcal{P}$ by considering the set of **all** orthonormal bases in $\mathcal{P}$. The latter, it is convenient to write down as follows,

$$\boldsymbol{\omega}_K \sim \begin{pmatrix} \omega_{K1} & 0 & 0 \\ 0 & \omega_{K2} & 0 \\ 0 & 0 & \omega_{K3} \end{pmatrix}, \qquad K = \mathrm{I}, \mathrm{II}, \mathrm{III}, \tag{10.58}$$

where

$$\boldsymbol{\omega}_K \cdot \boldsymbol{\omega}_L = \omega_{Ki}\omega_{Li} = \delta_{KL}. \tag{10.59}$$

By identifying $\boldsymbol{\omega}_K$ with the numerical triple $(\omega_{K1}, \omega_{K2}, \omega_{K3})$, we see that these triples should constitute an orthonormal basis in the space of numerical triples R^3 with the usual dot product. This basis is defined by three parameters

$$\chi_1, \chi_2, \chi_3, \tag{10.60}$$

their roles, for example, can be taken by,

$$\chi_K \equiv \mathrm{tr}(\boldsymbol{\omega}_K) = \omega_{K1} + \omega_{K2} + \omega_{K3}, \tag{10.61}$$

or let us say Euler's angles relative to the standard basis (1, 0, 0), (0, 1, 0), (0, 0, 1).

Each choice $\boldsymbol{\omega}_{\mathrm{I}}$, $\boldsymbol{\omega}_{\mathrm{II}}$, $\boldsymbol{\omega}_{\mathrm{III}}$ from the indicated 3-parameter set corresponds to a decomposition $\mathcal{P}$ into three spaces $\mathcal{P}_L$, each of which is a 1-dimensional space of eigenstates proportional to $\boldsymbol{\omega}_L$, $L = \mathrm{I}, \mathrm{II}, \mathrm{III}$.

We showed that for an **orthotropic elastic body**

$$\mathcal{S} = \mathcal{P}_{\mathrm{I}} \oplus \ldots \oplus \mathcal{P}_{\mathrm{VI}}, \tag{10.62}$$

and the stiffness tensor has the form

$$\mathbf{C} = \lambda_{\mathrm{I}}\, \boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \lambda_{\mathrm{VI}}\, \boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}, \tag{10.63}$$

where $\boldsymbol{\omega}_K$ are defined by formulae (10.58) and (10.56).

The set of elastic parameters consists of the following elements

$$\lambda_{\mathrm{I}}, \ldots, \lambda_{\mathrm{VI}}; \qquad \chi_1,\ \chi_2,\ \chi_3; \qquad \varphi_1,\ \varphi_2,\ \varphi_3, \tag{10.64}$$

where λ_{IV}, λ_{V}, λ_{VI} denote shear moduli. The structural indices of the class of orthotropic bodies are equal to

$$\langle 1+1+1+1+1+1 \rangle, \qquad [6+3+3]. \tag{10.65}$$

The elastic eigenstates of an orthotropic body are shown in Figure 8.

10.6. One class of totally asymmetric elastic bodies

We consider the following generalization of orthotropic bodies. We take $\mathcal{P}_{\mathrm{I}}$, $\mathcal{P}_{\mathrm{II}}$, $\mathcal{P}_{\mathrm{III}}$ as for an orthotropic body, i.e. as an orthogonal decomposition of the

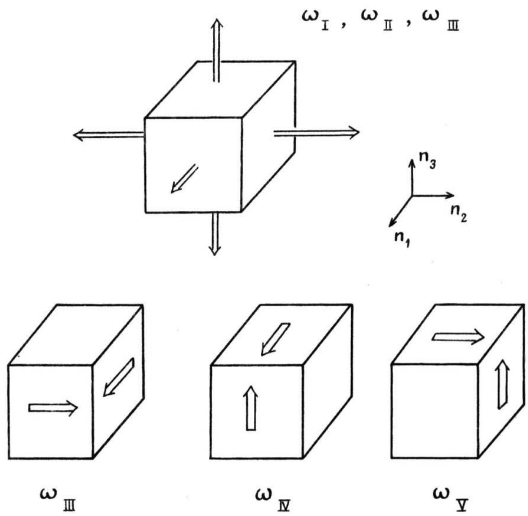

Fig. 8. Graphical illustration of the elastic eigenstates of an orthotropic material.

space of tensors with fixed principal axes $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. The orthogonal complement of this space is 3-dimensional and consists of the deviators of the form*

$$\begin{pmatrix} 0 & u & v \\ u & 0 & w \\ v & w & 0 \end{pmatrix}. \tag{10.66}$$

All orthonormal bases in this space can be expressed by formulae:

$$\boldsymbol{\omega}_L \sim \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & \gamma_{L1} & \gamma_{L2} \\ \gamma_{L1} & 0 & \gamma_{L3} \\ \gamma_{L2} & \gamma_{L3} & 0 \end{pmatrix}, \tag{10.67}$$

$$\boldsymbol{\omega}_L \cdot \boldsymbol{\omega}_K = \frac{1}{2}\gamma_{Li}\gamma_{Ki} = \delta_{LK}, \tag{10.68}$$

where $L = \mathrm{IV}, \mathrm{V}, \mathrm{VI}$. The triple (10.67) can be represented, similarly like the triple (10.58), by three parameters

$$\chi_4,\ \chi_5,\ \chi_6. \tag{10.69}$$

*Translator note: In the original formula (10.66) it was incorrectly shown an asymmetric matrix. The error has been corrected in the English translation.

Now, substituting in formula (3.39) $\boldsymbol{\omega}_{\mathrm{I}}$, $\boldsymbol{\omega}_{\mathrm{II}}$, $\boldsymbol{\omega}_{\mathrm{III}}$ according to the formulae (10.58) and $\boldsymbol{\omega}_{\mathrm{IV}}$, $\boldsymbol{\omega}_{\mathrm{V}}$, $\boldsymbol{\omega}_{\mathrm{VI}}$, applying the formulae (10.67), we obtain an extensive class of elastic materials, the structural indices of which, in the general case, are equal to

$$\langle 1+1+1+1+1+1\rangle, \qquad [6+6+3]. \tag{10.70}$$

An interesting subclass here is the collection of materials for which

$$\lambda_{\mathrm{I}} = \lambda_{\mathrm{II}} = \lambda_{\mathrm{III}}, \tag{10.71}$$

i.e. any tensor whose principal axes coincide with the fixed basis $\mathbf{n}_i$ is an eigentensor. The stiffness tensor has the form

$$\mathbf{C} = \lambda_{\mathrm{I}}\mathbf{P}_{\mathrm{I}} + (\lambda_{\mathrm{IV}}\mathbf{P}_{\mathrm{IV}} + \lambda_{\mathrm{V}}\mathbf{P}_{\mathrm{V}} + \lambda_{\mathrm{VI}}\mathbf{P}_{\mathrm{VI}}), \tag{10.72}$$

where

$$\mathbf{P}_{\mathrm{I}} \equiv \mathbf{n}_1\otimes\mathbf{n}_1\otimes\mathbf{n}_1\otimes\mathbf{n}_1 + \mathbf{n}_2\otimes\mathbf{n}_2\otimes\mathbf{n}_2\otimes\mathbf{n}_2 + \mathbf{n}_3\otimes\mathbf{n}_3\otimes\mathbf{n}_3\otimes\mathbf{n}_3, \tag{10.73}$$

$\mathbf{P}_{\mathrm{IV}}$, $\mathbf{P}_{\mathrm{V}}$, $\mathbf{P}_{\mathrm{VI}}$ are determined by the parameters of the stiffness distributors χ_4, χ_5, χ_6. The structural indices of this class are equal to

$$\langle 1+1+1+3\rangle, \qquad [4+3+3]. \tag{10.74}$$

In the aforementioned examples, we presented a certain uniform methodology. A mutually equivocal relation (3.27):

$$(\lambda_1,\dots,\lambda_\rho,\;\; \mathbf{P}_1,\dots,\mathbf{P}_\rho) \leftrightarrow \mathbf{C}, \tag{10.75}$$

where λ_i are pairwise different, and $\mathbf{P}_i$ make an orthogonal decomposition of unity, which has been obtained here "from left to right". For the class of elastic bodies defined by a specific set of elastic eigenstates (in the examples they were sets of pure shears), we determined all eigenstates and on these grounds we obtained the general form of the stiffness tensor, i.e. the general form of Hooke's law for this class. Let us add that according to (3.39) **any** set of six non-negative parameters $\lambda_{\mathrm{I}},\dots,\lambda_{\mathrm{VI}}$, which are not necessarily different, with **any** orthonormal basis $\boldsymbol{\omega}_{\mathrm{I}},\dots,\boldsymbol{\omega}_{\mathrm{VI}}$, defines some theoretically possible elastic body

$$(\lambda_{\mathrm{I}},\dots,\lambda_{\mathrm{VI}},\;\; \boldsymbol{\omega}_{\mathrm{I}},\dots,\boldsymbol{\omega}_{\mathrm{VI}}) \leftrightarrow \mathbf{C}, \tag{10.76}$$

for which $\boldsymbol{\omega}_K$ are eigenstates, and λ_K are the stiffness moduli.

§11 On classification of elastic materials

A judicious classification of elastic bodies is essential for a wide variety of applications. It is used, in particular, to limit and improve the work of the experimenter who determines the elastic properties of real materials and the work of the engineer selecting the material for the designed structure.

The only comprehensive classification of elastic bodies, we have today, is the classification by symmetry. Despite all its advantages, it has nowhere to catch on when symmetry disappears. It also does not distinguish between the properties of bodies with the same symmetry.

The structural formula naturally reveals completely new possibilities for comparing and distinguishing bodies according to elastic properties.

Classifications of elastic bodies based on the structural formula have not yet been built. Here we limit ourselves to presenting the problem and a few comments.

We denote by $\mathcal{F} \subset \mathcal{T}$ the entire set of tensors having symmetry

$$\sigma \times \mathbf{C} = \mathbf{C} \qquad \text{for every} \qquad \sigma = \langle 2\ 1\ 3\ 4 \rangle, \langle 1\ 2\ 4\ 3 \rangle, \langle 3\ 4\ 1\ 2 \rangle. \tag{11.1}$$

It is a 21-dimensional subspace of 36-dimensional space $\mathcal{T}$. The subject of our research is the set of all elastic bodies, i.e. the set $\mathcal{G} \subset \mathcal{T}$ of all possible stiffness tensors. According to requirement of the non-negativeness of elastic energy $\mathcal{G}$ **consists of all non-negative tensors from** $\mathcal{F}$ [*],

$$\boldsymbol{\omega} \cdot \mathbf{C} \cdot \boldsymbol{\omega} \geq 0 \qquad \text{for every} \qquad \boldsymbol{\omega} \in \mathcal{S}. \tag{11.2}$$

For convenience, we assume that also $\mathcal{O} \in \mathcal{G}$[**].

[*] Translator note: $\mathcal{G} \subset \mathcal{F} \subset \mathcal{T}$.

[**] Translator note: It is about $\mathcal{O}$-orbits of tensors $\mathbf{C}$, cf. text above formula (2.4).

It is obvious that $\mathcal{G}$ has the following properties,

1) for every $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{G}$ also $\mathbf{C}_1 + \mathbf{C}_2 \in \mathcal{G}$,

2) for every $\mathbf{C} \in \mathcal{G}$, $\alpha \geq 0$ also $\alpha\mathbf{C} \in \mathcal{G}$,

3) for any non-zero $\mathbf{C} \in \mathcal{G}$ we have $-\mathbf{C} \notin \mathcal{G}$.

In other words, the set of elastic bodies creates a **cone** in a subspace $\mathcal{F}$ [15].

Note 9. It should not be considered that for any $\mathbf{C} \in \mathcal{F}$ an alternative is true, $\mathbf{C}$ or $-\mathbf{C}$ belongs to $\mathcal{G}$. ♦

So, we have to describe the cone somehow $\mathcal{G} \subset \mathcal{F}$. In fact, this task can and should be narrowed down.

First of all, it is perfectly clear that we need to classify **elastic materials** $\langle\mathbf{C}\rangle$ (orbits of group $\mathcal{O}$ in $\mathcal{T}$), and not the elastic bodies $\mathbf{C}$ themselves. The cone $\mathcal{G}$ is composed of materials. **The set of elastic materials**, i.e. the set of orbits on a cone $\mathcal{G}$ is denoted by $\mathcal{H}$. Moreover, it is useful to collect materials in the following separate classes.

We say that bodies $\mathbf{C}$ and $\mathbf{C}'$ belong to one elastic type, if their systems of eigenspaces ($\mathcal{P}_1, ..., \mathcal{P}_\rho$ and $\mathcal{P}'_1, ..., \mathcal{P}'_\rho$) coincide with an accuracy to a certain rotation.

In other words, an **elastic type** to which belongs a certain body $\mathbf{C}$ with a structural formula

$$\mathbf{C} = \lambda_1 \mathbf{P}_1 + ... + \lambda_\rho \mathbf{P}_\rho, \tag{11.3}$$

is the set $[\mathbf{C}]$, consisting of all elastic bodies of the form

$$\mathbf{C}' = x_1 \mathbf{Q} * \mathbf{P}_1 + ... + x_\rho \mathbf{Q} * \mathbf{P}_\rho, \tag{11.4}$$

where $\mathbf{Q}$ runs through the whole group $\mathcal{O}$, and $(x_1, ..., x_\rho)$ runs through all possible combinations of ρ parameters – positive and pairwise different.

The set of materials $\mathcal{H}$ is divided into intersecting elastic types.

As it can be seen, the elastic type $[\mathbf{C}]$ is a $\rho(\mathbf{C})$ parametric set of materials, $\rho(\mathbf{C}) \leq 6$.

The simplest elastic type consists of ideal elastic bodies (10.1). This is the only 1-parameter type. In space $\mathcal{T}$ it has the form of a ray $\lambda\mathbf{I}^{(4s)}$, $\lambda > 0$.

The next simplest elastic type is the set of all isotropic elastic bodies with a non-zero Poisson's ratio. It is a 2-parameter type (10.6). In space, the right angle corresponds to it

$$\lambda_\mathcal{P} \mathbf{I}_\mathcal{P} + \lambda_\mathcal{D} \mathbf{I}_\mathcal{D}, \qquad \lambda_\mathcal{P}, \lambda_\mathcal{D} > 0, \tag{11.5}$$

located on the 2-dimensional plane of isotropic tensors from $\mathcal{F}$. A bundle of ideal materials separates at this angle an acute angle $\lambda_{\mathcal{P}} \geq \lambda_{\mathcal{D}} > 0$, in which visibly all the real isotropic bodies are located (Figure 9).

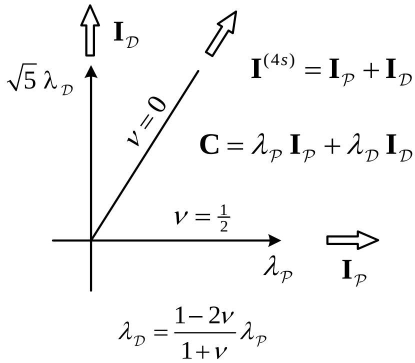

Fig. 9. Graphical illustration of a 2-dimensional parametric space of isotropic materials. The skew line shows a **perfectly elastic** material for which Poisson's ratio is zero $\nu = 0$.

We note that all the other elastic types are represented in $\mathcal{F}$ by curved sets.

The set of all transversely isotropic elastic bodies with a given $\chi = \text{const}$ consists of 15 elastic types: one 4-parameter, six 3-parameter, seven 2-parameter and one 1-parameter. The set of all, in general, transversely-isotropic elastic bodies consists, conventionally speaking, of $15 \cdot \infty$ elastic types, because $\chi \in [0, \pi/2)$.

It is obvious that it is necessary to further refine the division of the set of elastic bodies. This can be done in different ways.

For example, all elastic types with the same first structural index can be grouped together. We call such sets **elastic modes**. The elastic mode can simply be equated with the first structural index of entering it elastic bodies.

We shall say that an elastic mode $\langle k_1+...+k_t\rangle$ is subordinate to an elastic mode $\langle m_1 + ... + m_u\rangle$ if $t < u$, with k_i either being equal to some of m_i, or being their sums. In total there are 11 elastic modes. The scheme of their subordination is shown in Figure 10. The mod $\mathcal{B}$ is subordinate to the mode $\mathcal{A}$, if from $\mathcal{A}$ to $\mathcal{B}$ one can go according to the arrows shown. The k-th level of the scheme consists of all materials with pairwise different values of stiffness moduli $\lambda_1, ..., \lambda_k$, the transition from k-th to the $(k-1)$-th level is done by equating the values of two moduli from among $\lambda_1, ..., \lambda_k$; the number of possible combinations of equations is shown in Figure 10, next to the arrows.

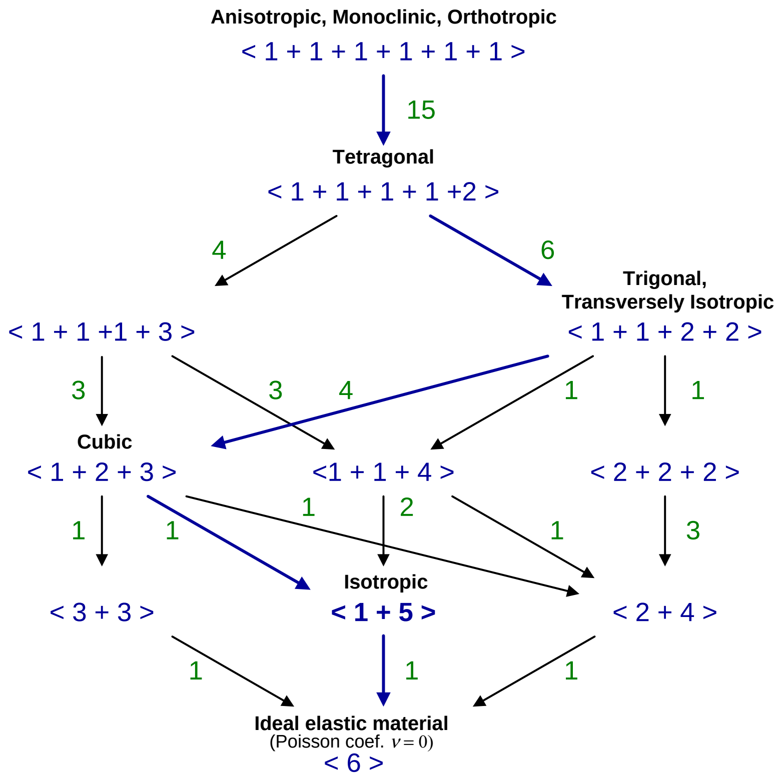

Fig. 10. Graphical illustration of the classification of types of elastic materials according to the first structural index, i.e. the number and multiplicity of linearly independent true moduli of elasticity. There are 11 such types.
(Descriptions of symmetry classes, colours and bolding of arrows in Fig. 10 have been added by translator.)

Elastic mode $\langle 6\rangle$ is, of course, the one elastic type $[\mathbf{I}^{(4s)}]$. Elastic modes above the first level consist of an infinite number of elastic types. For example, an elastic mode $\langle 1+5\rangle$ consists of all elastic bodies

$$\mathbf{C} = \lambda_\omega\, \boldsymbol{\omega}\otimes\boldsymbol{\omega} + \lambda\Big(\mathbf{I}^{(4s)} - \boldsymbol{\omega}\otimes\boldsymbol{\omega}\Big), \tag{11.6}$$

where $\lambda_\omega,\ \lambda > 0$, and $\boldsymbol{\omega}$ is an arbitrarily determined symmetric tensor. Here $\boldsymbol{\omega}$ and any tensor orthogonal to $\boldsymbol{\omega}$ are elastic eigenstates. Isotropic bodies are a special case of bodies of the type $\langle 1+5\rangle$, when

$$\boldsymbol{\omega} = \frac{\sqrt{3}}{3}\mathbf{1} \quad\rightarrow\quad \boldsymbol{\omega}\otimes\boldsymbol{\omega} = \mathbf{I}_{\mathcal{P}}. \tag{11.7}$$

Materials of the same elastic type may differ significantly in the nature of eigenstates and in symmetry, because the first structural index takes into account only the dimension of the eigenstates space.

Other classification methods based on a much more precise consideration of eigenstates require much more subtle considerations, which we do not dwell on here.

Note 10. Throughout this section, we have assumed that the body is not rigidly constrained. The classification of bodies with rigid constraints should be carried out through compliance tensors $\mathbf{S}$, allowing for the possibility of zeroing of some of the compliance moduli λ_i^{-1}. ⧫

§12 Symmetry and elastic eigenstates

At the beginning, we recall two fundamental definitions.

1. **A symmetry group of tensor** of p-order $\mathbf{A} \in T_p$ is a subgroup $\mathcal{O}_\mathbf{A} \subset \mathcal{O}$ consisting of those $\mathbf{Q} \in \mathcal{O}$, under the influence of which $\mathbf{A}$ does not change,

$$\mathbf{Q} * \mathbf{A} = \mathbf{A}.$$

2. Let $G \subset \mathcal{O}$ be a subgroup of the group $\mathcal{O}$. A subspace $\mathcal{P}$ of the space of p-order tensors T_p is called G-stable, if the action of any $\mathbf{Q} \in G$ does not takes out from $\mathcal{P}$, i.e. from $\mathbf{A} \in \mathcal{P}$ it follows that $\mathbf{Q} * \mathbf{A} \in \mathcal{P}$. Subspaces G-stable and not containing G-stable eigenspaces are called **irreducible**. The decomoposition $\mathcal{P}_1, ..., \mathcal{P}_\rho$ of space $T_p = \mathcal{P}_1 \oplus ... \oplus \mathcal{P}_\rho$ is called G-stable if all $\mathcal{P}_i$ are G-stable; in this case $\mathcal{P}_i$ are pairwise orthogonal in terms of the dot product:

$$(\mathbf{A},\ \mathbf{B}) \to \mathbf{A} \cdot \mathbf{B} \equiv A_{i\,...\,j} B_{i\,...\,j}.$$

The G-stable decomposition is called **irreducible** if all $\mathcal{P}_i$ are irreducible.

The **group of elastic symmetry of a body** is usually called the symmetry group $\mathcal{O}_\mathbf{C}$ of its stiffness tensor $\mathbf{C}$. Applying the structural formula, we get the following **theorem on elastic symmetry**:

Theorem 11. The group of elastic symmetry of a body is equal to the common part of symmetry groups of its eigenprojectors

$$\boxed{\mathcal{O}_\mathbf{C} = \mathcal{O}_{\mathbf{P}_1} \cap ... \cap \mathcal{O}_{\mathbf{P}_\rho}}\,. \tag{12.1}$$

Proof. Let the structural formula of the body $\mathbf{C}$ be $\mathbf{C} = \lambda_1 \mathbf{P}_1 + ... + \lambda_\rho \mathbf{P}_\rho$ and let

$$\mathbf{Q} \in \mathcal{O}_{\mathbf{P}_1} \cap ... \cap \mathcal{O}_{\mathbf{P}_\rho}, \quad \text{i.e.} \quad \mathbf{Q} * \mathbf{P}_i = \mathbf{P}_i, \tag{12.2}$$

then

$$\mathbf{Q} * \mathbf{C} = \lambda_1 \mathbf{Q} * \mathbf{P}_1 + \ldots = \lambda_1 \mathbf{P}_1 + \ldots = \mathbf{C}, \tag{12.3}$$

i.e. $\mathbf{Q} \in \mathcal{O}_\mathbf{C}$. The other way around, let $\mathbf{Q} \in \mathcal{O}_\mathbf{C}$, then

$$\lambda_1 \mathbf{Q} * \mathbf{P}_1 + \ldots = \lambda_1 \mathbf{P}_1 + \ldots \tag{12.4}$$

Since $\mathbf{P}_1, \ldots$ is an orthogonal decompostion of unity, then also $\mathbf{Q} * \mathbf{P}_1, \ldots$ is an orthogonal decompostion of unity, i.e. on both sides of the equality the material decompostion of a certain stiffness tensor is written down. Since the material decomposition is unambiguous, then $\mathbf{Q} * \mathbf{P}_i = \mathbf{P}_i$ for $i = 1, \ldots, \rho$. ♦

In the symmetry formula (12.1) you can, of course, skip one (exactly one!) projector

$$\mathcal{O}_\mathbf{C} = \mathcal{O}_{\mathbf{P}_1} \cap \ldots \cap \mathcal{O}_{\mathbf{P}_{\nu-1}} \cap \mathcal{O}_{\mathbf{P}_{\nu+1}} \cap \ldots \cap \mathcal{O}_{\mathbf{P}_\rho}, \tag{12.5}$$

because $\mathbf{P}_\nu = \mathbf{I}^{(4s)} - (\mathbf{P}_1 + \ldots + \mathbf{P}_{\nu-1} + \mathbf{P}_{\nu+1} + \ldots + \mathbf{P}_\rho)$ and $\mathbf{Q} * \mathbf{I}^{(4s)} = \mathbf{I}^{(4s)}$ for any $\mathbf{Q} \in \mathcal{O}$.

We illustrate this with a simple but important example.

Example. We consider a body of the type $\langle 1 + 5 \rangle$, i.e.

$$\mathbf{C} = \lambda\, \boldsymbol{\omega} \otimes \boldsymbol{\omega} + \lambda' \,(\mathbf{I}^{(4s)} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}). \tag{12.6}$$

According to (12.5) the elastic symmetry of this body is described by a very simple formula:

$$\mathcal{O}_\mathbf{C} = \mathcal{O}_{\boldsymbol{\omega}\otimes\boldsymbol{\omega}}. \tag{12.7}$$

This group consists of all those $\mathbf{Q} \in \mathcal{O}$, satisfying the equation

$$\mathbf{Q} * (\boldsymbol{\omega} \otimes \boldsymbol{\omega}) \equiv \mathbf{Q} * \boldsymbol{\omega} \otimes \mathbf{Q} * \boldsymbol{\omega} = \boldsymbol{\omega} \otimes \boldsymbol{\omega}, \tag{12.8}$$

i.e.

$$\mathbf{Q} * \boldsymbol{\omega} \equiv \mathbf{Q} \boldsymbol{\omega}\, \mathbf{Q}^T = \pm \boldsymbol{\omega}. \tag{12.9}$$

The "minus" sign in this formula is possible only when $\boldsymbol{\omega}$ is pure shear. Indeed, if $\mathbf{Q} * \boldsymbol{\omega} = \boldsymbol{\omega}$, then

$$\begin{aligned} \det(\boldsymbol{\omega}) &= \det(\mathbf{Q} * \boldsymbol{\omega}) = -\det(\boldsymbol{\omega}), \\ \operatorname{tr}(\boldsymbol{\omega}) &= \operatorname{tr}(\mathbf{Q} * \boldsymbol{\omega}) = -\operatorname{tr}(\boldsymbol{\omega}), \end{aligned} \tag{12.10}$$

i.e.

$$\det(\boldsymbol{\omega}) = \operatorname{tr}(\boldsymbol{\omega}) = 0. \tag{12.11}$$

We express $\boldsymbol{\omega}$ by eigenvectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ and eigenvalues a, b, c,

$$\begin{gathered}\boldsymbol{\omega} = a\,\mathbf{l}\otimes\mathbf{l} + b\,\mathbf{m}\otimes\mathbf{m} + c\,\mathbf{n}\otimes\mathbf{n},\\ a^2 + b^2 + c^2 = 1.\end{gathered} \tag{12.12}$$

In the case of (12.11) one of the eigenvalues, let us say c, is equal to zero, and we have,

$$a = \frac{1}{\sqrt{2}}, \qquad b = -\frac{1}{\sqrt{2}}, \qquad c = 0. \tag{12.13}$$

Now, if $\boldsymbol{\omega}$ is not pure shear, then the elastic symmetry group is just equal to the elastic symmetry group of the eigenstate $\boldsymbol{\omega}$

$$\mathcal{O}_{\mathbf{C}} = \mathcal{O}_{\boldsymbol{\omega}}. \tag{12.14}$$

In the general case, when $a \neq b \neq c \neq a$ then $\mathcal{O}_{\boldsymbol{\omega}}$ is a symmetry group of a triaxial ellipsoid, i.e. $\mathcal{O}_{\boldsymbol{\omega}} = D_{2h}$, and then the body is orthotropic. The axes of orthotropy are the axes of the tensor $\boldsymbol{\omega}$. When $a = b \neq c$, then the body is transversely isotropic with the axis $\mathbf{n}$, and when $a = b = c$, then the body is isotropic.

If the eigenstate $\boldsymbol{\omega}$ is pure shear

$$\boldsymbol{\omega} = \frac{1}{\sqrt{2}}(\mathbf{l}\otimes\mathbf{l} - \mathbf{m}\otimes\mathbf{m}), \tag{12.15}$$

then all solutions $\mathbf{Q}$ to equation (12.9) constitute a symmetry group of the pyramid with a square base with the axis $\mathbf{n}$, $\mathcal{O}_{\mathbf{C}} = D_{4h}$ (tetragonal symmetry).

We consider a larger class of bodies

$$\mathbf{C} = \lambda_\omega\,\boldsymbol{\omega}\otimes\boldsymbol{\omega} + \lambda_\tau\,\boldsymbol{\tau}\otimes\boldsymbol{\tau} + \lambda'\left(\mathbf{I}^{(4s)} - \boldsymbol{\omega}\otimes\boldsymbol{\omega} - \boldsymbol{\tau}\otimes\boldsymbol{\tau}\right). \tag{12.16}$$

The structural indexes are equal here

$$\langle 1+1+4\rangle, \qquad [3+6+3]. \tag{12.17}$$

The symmetry formula (12.5) has the form

$$\mathcal{O}_{\mathbf{C}} = \mathcal{O}_{\boldsymbol{\omega}\otimes\boldsymbol{\omega}} \cap \mathcal{O}_{\boldsymbol{\tau}\otimes\boldsymbol{\tau}}. \tag{12.18}$$

As long as the eigenstates $\boldsymbol{\omega}$, $\boldsymbol{\tau}$ do not have any common eigenvectors, this intersection is trivial, and the body is completely asymmetric (triclinic symmetry). This example illustrates the rapid loss of elastic symmetry as the structure becomes more complex.

Let us return to the general Theorem 11. We pose the following problem:

> determine all elastic eigenstates of all elastic bodies that are symmetric with respect to a given subgroup $G \subset \mathcal{O}$, i.e. satisfying the relation
>
> $$G \subset \mathcal{O}_{\mathbf{C}}. \tag{12.19}$$

In other words, we talk about determination of all elastic eigenstates of the crystals with a given symmetry G.

Theorem 11 on elastic symmetry (12.1) makes that this problem is not only noticeable but also completely solvable for all crystals. First of all, we note that, according to this theorem, the aforementioned problem can be reformulated as follows:

> find all orthogonal decompositions of unity
>
> $$\mathbf{P}_1 + \ldots + \mathbf{P}_\rho = \mathbf{I}^{(4s)}, \qquad \rho \leq 6, \tag{12.20}$$
>
> invariant to a given subgroup $G \subset \mathcal{O}$, i.e. satisfying relations
>
> $$G \subset \mathcal{O}_{\mathbf{P}_i} \qquad \text{for all} \qquad i = 1, \ldots, \rho. \tag{12.21}$$

After solving this problem, we can easily obtain a general form of stiffness tensors $\mathbf{C}$ for the bodies under consideration.

Moreover, it is not difficult to prove the following lemma.

> **Lemma 3.** An orthogonal projector $\mathbf{P}$ on a subspace $\mathcal{P} \subset \mathcal{S}$ is symmetric with respect to G, i.e. $G \subset \mathcal{O}_{\mathbf{P}}$, if and only if $\mathcal{P}$ is G-stable.

Proof. Let $G \subset \mathcal{O}_{\mathbf{P}}$. Let us take any $\mathbf{Q} \in G$ and $\boldsymbol{\omega} \in \mathcal{P}$. Because $\mathbf{P} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}$ then

$$\mathbf{P} \cdot (\mathbf{Q} * \boldsymbol{\omega}) = (\mathbf{Q} * \mathbf{P}) \cdot (\mathbf{Q} * \boldsymbol{\omega}) = \mathbf{Q} * (\mathbf{P} \cdot \boldsymbol{\omega}) = \mathbf{Q} * \boldsymbol{\omega}, \tag{12.22}$$

i.e. also $\mathbf{Q} * \boldsymbol{\omega} \in \mathcal{P}$. That means that $\mathcal{P}$ is G-stable. Conversely, if $\mathcal{P}$ is G-stable, i.e. for any $\mathbf{Q} \in G$, $\boldsymbol{\omega} \in \mathcal{P}$ and $\mathbf{Q} * \boldsymbol{\omega} \in \mathcal{P}$, then

$$\mathbf{Q} * \boldsymbol{\omega} = \mathbf{Q} * (\mathbf{P} \cdot \boldsymbol{\omega}) = (\mathbf{Q} * \mathbf{P}) \cdot (\mathbf{Q} * \boldsymbol{\omega}), \tag{12.23}$$

i.e.

$$(\mathbf{Q} * \mathbf{P}) \cdot \boldsymbol{\tau} = \boldsymbol{\tau} \quad \text{for any} \quad \boldsymbol{\tau} \in \mathcal{P}. \tag{12.24}$$

Since the orthogonal complement $\mathcal{P}^{\perp}$ is also G-stable, then for any $\boldsymbol{\mu} \in \mathcal{P}^{\perp}$ we have $\mathbf{Q}^T * \boldsymbol{\mu} \in \mathcal{P}^{\perp}$, and therefore

$$(\mathbf{Q} * \mathbf{P}) \cdot \boldsymbol{\mu} = (\mathbf{Q} * \mathbf{P}) \cdot [\mathbf{Q} * (\mathbf{Q}^T * \boldsymbol{\mu})] = \mathbf{Q} * [\mathbf{P} \cdot (\mathbf{Q}^T * \boldsymbol{\mu})] = 0. \qquad (12.25)$$

From (12.24) and (12.25) it follows that $\mathbf{Q} * \mathbf{P}$ is a projector on $\mathcal{P}$, $\mathbf{Q} * \mathbf{P} = \mathbf{P}$, i.e. $\mathbf{Q} \in \mathcal{O}_{\mathbf{P}}$. ⧫

The proven lemma reduces the problem of finding all the decompositions of unity invariant upon operation of G to the problem of finding all G-stable decompositions of the space $\mathcal{S}$. But all such decompositions will be known if we find all irreducible decompositions. Thus, **the task of finding all elastic eigenstates of crystals with a given symmetry G is equivalent to the following simple task**:

obtain all irreducible G-stable decompositions

$$\mathcal{S} = \mathcal{P}_1 \oplus \ldots \oplus \mathcal{P}_\rho. \qquad (12.26)$$

As soon as they are found, Hooke's law is given by the formula (3.27).

When $G = \mathcal{O}$, this task is trivial. Indeed, the only irreducible $\mathcal{O}$-stable decomposition of $\mathcal{S}$ is the decomposition into the sum of the space of spherical tensors $\mathcal{P}$ and the space of deviators $\mathcal{D}$, i.e. decomposition (10.3), from which it directly follows (10.6). This sentence is perhaps a record-short derivation of Hooke's law for an isotropic body (see for example [25]).

We also pay attention to the following fact, which makes it much easier to find the space of elastic eigenstates of crystals.

Theorem 12. Let $\mathcal{P}$ be a space of all elastic eigenstates corresponding to the stiffness modulus λ of a body $\mathbf{C}$, symmetric in terms of elastic properties with respect to the group G. We consider a linear shell of G-orbits of tensor $\boldsymbol{\omega} \in \mathcal{P}$

$$\mathcal{R}(\boldsymbol{\omega}) \equiv \mathrm{Lin}(G * \boldsymbol{\omega}). \qquad (12.27)$$

So, for any $\boldsymbol{\omega} \in \mathcal{P}$

$$\mathcal{R}(\boldsymbol{\omega}) \subset \mathcal{P}. \qquad (12.28)$$

If $\mathcal{P}$ is irreducible, then

$$\mathcal{P} = \mathcal{R}(\boldsymbol{\omega}). \qquad (12.29)$$

Proof. Any tensor $\boldsymbol{\alpha} \in \mathcal{R}$ has the form

$$\boldsymbol{\alpha} = a_1\mathbf{Q}_1 * \boldsymbol{\omega} + \ldots + a_k\mathbf{Q}_k * \boldsymbol{\omega}, \tag{12.30}$$

where $\mathbf{Q}_i \in \mathcal{O}$. Because

$$\mathbf{C} \cdot \boldsymbol{\omega} = \lambda\,\boldsymbol{\omega}, \qquad \mathbf{Q}_i * \mathbf{C} = \mathbf{C}, \tag{12.31}$$

it is

$$\begin{aligned} \mathbf{C} \cdot \boldsymbol{\alpha} &= a_1\mathbf{C} \cdot (\mathbf{Q}_1 * \boldsymbol{\omega}) + \ldots = \\ &= a_1(\mathbf{Q}_1 * \mathbf{C}) \cdot (\mathbf{Q}_1 * \boldsymbol{\omega}) + \ldots = \\ &= a_1\mathbf{Q}_1 * (\mathbf{C} \cdot \boldsymbol{\omega}) + \ldots = \\ &= \lambda\,[a_1\mathbf{Q}_1 * \boldsymbol{\omega} + \ldots] = \lambda\,\boldsymbol{\alpha}, \end{aligned} \tag{12.32}$$

i.e. $\boldsymbol{\alpha} \in \mathcal{P}$, what proves (12.28). When $\mathcal{P}$ is irreducible, then $\mathcal{P} = \mathcal{R}(\boldsymbol{\omega})$ because $\mathcal{R}(\boldsymbol{\omega})$, of course, it is G-stable. ⧫

For example, the space of deviators $\mathcal{D}$ can be written as

$$\mathcal{D} = \mathrm{Lin}(\mathcal{O} * \boldsymbol{\tau}), \tag{12.33}$$

where $\boldsymbol{\tau}$ is any fixed shear!

In work [24] the posed problem (12.26) was solved by us for all the crystals.

§13 Plasticity and other generalizations

A reader who has had the patience to trace the presented lecture to this point of course understands that it can be played with another main character.

We consider the **flow of rigid-plastic anisotropic bodies with a quadratic plastic potential** (see, for example, [26])

$$2\Psi(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}. \tag{13.1}$$

The **flow law** has the form:

$$\boldsymbol{\delta} = \lambda\, \partial_{\boldsymbol{\sigma}} \Psi = \lambda\, \mathbf{H} \cdot \boldsymbol{\sigma}, \tag{13.2}$$

where $\boldsymbol{\delta} \equiv \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathrm{T}})$ is the tensor of deformation velocity, $\mathbf{v}$ the particles velocity field*

$$\lambda = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\delta}}{\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma}} \geq 0. \tag{13.3}$$

Just note the following:

> The plastic anisotropy tensor $\mathbf{H}$ has exactly the same internal symmetry as the stiffness tensor $\mathbf{C}$**,
>
> $$H_{ijkl} = H_{jikl} = H_{klij}.$$

Therefore, **all the basic formulae of this work are also valid for the tensor H**. We pay attention only to the most important ones. The structural formulae for a tensor $\mathbf{H}$ have the form:

$$\begin{aligned} \mathbf{H} &= \frac{1}{k_1^2}\mathbf{P}_1 + \ldots + \frac{1}{k_\rho^2}\mathbf{P}_\rho = \\ &= \frac{1}{k_I^2}\boldsymbol{\omega}_{\mathrm{I}} \otimes \boldsymbol{\omega}_{\mathrm{I}} + \ldots + \frac{1}{k_{\mathrm{VI}}^2}\boldsymbol{\omega}_{\mathrm{VI}} \otimes \boldsymbol{\omega}_{\mathrm{VI}}, \end{aligned} \tag{13.4}$$

*Translator note: Typographical error in the original formula (13.3) has been corrected by replacement $\boldsymbol{\varepsilon} \to \boldsymbol{\delta}$.

**Translator note: The $\mathbf{H}$ rather corresponds to the compliance tensor $\mathbf{S}$.

where $\boldsymbol{\omega}_{\mathrm{I}}, ..., \boldsymbol{\omega}_{\mathrm{VI}}$ are the **plastic eigenstates** of the body under consideration, and $k_{\mathrm{I}}, ..., k_{\mathrm{VI}}$ are the **limits of plastic flow**.

Quadratic **flow condition for a body with arbitrary anisotropy**

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = 1 \tag{13.5}$$

gets the following form:

$$\boxed{\left(\frac{|\boldsymbol{\sigma}_1|}{k_1}\right)^2 + ... + \left(\frac{|\boldsymbol{\sigma}_\rho|}{k_\rho}\right)^2 = 1}\,, \tag{13.6}$$

where

$$\boldsymbol{\sigma}_\alpha \equiv \mathbf{P}_\alpha \cdot \boldsymbol{\sigma} \tag{13.7}$$

expresses the essence of the projection of the stress tensor onto the spaces of plastic eigenstates $\mathcal{P}_\alpha$, $\alpha = 1, ..., \rho$. If we use quantities $\sigma_K \equiv \boldsymbol{\sigma} \cdot \boldsymbol{\omega}_K$, $K = \mathrm{I}, ..., \mathrm{VI}$, then, the plastic flow condition can be written as

$$\left(\frac{|\boldsymbol{\sigma}_{\mathrm{I}}|}{k_{\mathrm{I}}}\right)^2 + ... + \left(\frac{|\boldsymbol{\sigma}_{\mathrm{VI}}|}{k_{\mathrm{VI}}}\right)^2 = 1. \tag{13.8}$$

> The set of constants describing in the most general case the plastic flow of a body with a square flow condition and the associated flow law will consist of 6 **yield limits**, 12 **flow distributors** and 3 orientation **angles** with respect to the laboratory coordinate system.

In special cases, the number of constants is reduced.

For example, for a **transversely-isotropic body**, the number of constants is 5,

$$k_1,\ k_2,\ k_3,\ k_4,\ \varphi \tag{13.9}$$

(cf. (10.41)), where k_i are the flow boundaries, and φ, $0 \le \varphi < \pi/2$, is the flow distributor. The quadratic flow condition for a transversely-isotropic body has, due to (10.48) and $k_{\mathrm{III}} = k_{\mathrm{IV}}$, $k_{\mathrm{V}} = k_{\mathrm{VI}}$, the following form:

$$\begin{aligned} &\frac{1}{k_1^2}\left[(\sigma_{11} + \sigma_{22})\sin(\varphi) + \sqrt{2}\,\sigma_{33}\cos(\varphi)\right]^2 + \\ &\quad + \frac{1}{k_2^2}\left[(\sigma_{11} + \sigma_{22})\cos(\varphi) - \sqrt{2}\,\sigma_{33}\sin(\varphi)\right]^2 + \\ &\quad + \frac{1}{k_3^2}(\sigma_{13}^2 + \sigma_{23}^2) + \frac{1}{k_4^2}\left[(\sigma_{11} - \sigma_{22})^2 + \sigma_{12}^2\right] = 2. \end{aligned} \tag{13.10}$$

For an **isotropic body** according to (10.6) we get

$$\mathbf{H} = \frac{1}{k_{\mathcal{P}}^2}\mathbf{I}_{\mathcal{P}} + \frac{1}{k_{\mathcal{D}}^2}\mathbf{I}_{\mathcal{D}}. \tag{13.11}$$

The flow condition (13.6) takes the form

$$\left(\frac{\sigma_{\mathcal{P}}}{k_{\mathcal{P}}}\right)^2 + \left(\frac{\sigma_{\mathcal{D}}}{k_{\mathcal{D}}}\right)^2 = 1. \tag{13.12}$$

If we assume that no hydrostatic pressure can lead to plastic flow, then it should be taken

$$\frac{1}{k_{\mathcal{P}}} = 0. \tag{13.13}$$

Then we get:

$$|\boldsymbol{\sigma}_{\mathcal{D}}|^2 = k_{\mathcal{D}}, \tag{13.14}$$

$$\mathbf{H} = \frac{1}{k_{\mathcal{D}}^2}\mathbf{I}_{\mathcal{D}}, \tag{13.15}$$

$$\boldsymbol{\delta} = \frac{\boldsymbol{\delta}_{\mathcal{D}} \cdot \boldsymbol{\sigma}_{\mathcal{D}}}{k_{\mathcal{D}}^2}\boldsymbol{\sigma}_{\mathcal{D}}. \tag{13.16}$$

This is the usual variant of the theory of a rigid-plastic body with the Huber--Mises flow condition, where $k_{\mathcal{D}}/\sqrt{2}$ denotes the yield limit for shear.

The following generalization of the theory of flow with a quadratic potential is imposing (13.1), (13.2), (13.4), (13.5). We take determined in advance material decomposition of unity

$$\mathbf{P}_1 + \ldots + \mathbf{P}_\rho = \mathbf{I}^{(4s)}, \tag{13.17}$$

we introduce projections

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \ldots + \boldsymbol{\sigma}_\rho, \qquad \boldsymbol{\sigma}_\alpha \equiv \mathbf{P}_\alpha \cdot \boldsymbol{\sigma}, \tag{13.18}$$

and we assume the flow condition in the form

$$|\boldsymbol{\sigma}_\alpha|^2 \leq k_\alpha^2, \qquad \alpha = 1, \ldots, \rho. \tag{13.19}$$

The associated flow law here has the form

$$\boldsymbol{\delta} = \sum_{\alpha=1}^{\rho} \lambda_\alpha \boldsymbol{\sigma}_\alpha, \tag{13.20}$$

where

$$\lambda_\alpha = 0 \quad \text{for any} \quad |\boldsymbol{\sigma}_\alpha| < k_\alpha. \tag{13.21}$$

Combining (2.1) with (13.2), respectively, we obtain the **theory of elastic-perfectly-plastic bodies**. Here, the question arises what is the relationship between $\mathbf{C}$ and $\mathbf{H}$, i.e. the relationship between elastic and plastic properties. The simplest special hypothesis is that $\mathbf{C}$ and $\mathbf{H}$ have common eigenstates.

We do not bother the reader with further obvious applications of the ideas developed by us, for example in terms of **viscoelasticity**.

Instead, we highlight the following non-trivial generalization of our approach to nonlinear elasticity. One of the simple, but extremely common, classes of nonlinear elastic bodies is the class

$$\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon}, \tag{13.22}$$

where

$$\mathbf{C}(\boldsymbol{\varepsilon}) = \lambda_1(\boldsymbol{\varepsilon})\mathbf{P}_1 + \ldots + \lambda_\rho(\boldsymbol{\varepsilon})\mathbf{P}_\rho, \tag{13.23}$$

where projectors $\mathcal{P}_\nu$ constitute a priori material decomposition of unity

$$\mathbf{P}_1 + \ldots + \mathbf{P}_\rho = \mathbf{I}^{(4s)}, \tag{13.24}$$

independent of $\boldsymbol{\varepsilon}$. It seems to me that this case will be carried out for **small deformations of physically nonlinear elastic bodies**. The eigenstates in (13.23) are fixed, and only the stiffness moduli depend on the deformation.

On the basis of (13.22) and (13.23), it is clearly possible to construct a nice variant of the theory of small, **elastic-plastic deformations**.

In the special case for isotropy $\rho = 2$, $\mathbf{P}_1 = \mathbf{I}_{\mathcal{P}}$, $\mathbf{P}_2 = \mathbf{I}_{\mathcal{D}}$ (see (10.6)) it is the Hencky–Ilyushin theory.

The relationship (13.22) and (13.23) between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ is the natural to call **quasi-linear**.

§14 Concluding remarks

Let us consider some of the open problems.

1. The proposed approach focuses primarily on the description of "how an elastic body is built". It is obviously useful in studying general qualitative problems of the theory of elasticity of anisotropic bodies. We can also expect its effectiveness on some boundary problems. If we use the full decomposition of $\boldsymbol{\sigma}$ over the elastic eigenstates (5.9), the equation of motion can be written in the following form

$$\operatorname{div}(\sigma_{\mathrm{I}}\boldsymbol{\omega}_{\mathrm{I}} + \ldots + \sigma_{\mathrm{VI}}\boldsymbol{\omega}_{\mathrm{VI}}) + \rho\,\mathbf{b} = \rho\,\mathbf{a},$$

   where $\mathbf{b}$ denotes intensity of the mass forces, $\mathbf{a}$ is acceleration. When $\boldsymbol{\omega}_K$ are homogeneous, then the equation of motion expressed in stresses has the form

$$\boldsymbol{\omega}_{\mathrm{I}}\nabla\sigma_{\mathrm{I}} + \ldots + \boldsymbol{\omega}_{\mathrm{VI}}\nabla\sigma_{\mathrm{VI}} + \rho\mathbf{b} = \rho\mathbf{a},$$

   where $\nabla\sigma_K$ is the gradient of the scalar field $\sigma_K \equiv \boldsymbol{\sigma}\cdot\boldsymbol{\omega}_K$. Substituting here Hooke's law in the form (5.11) and using the relationship ε_K with the displacement vector $\mathbf{u}$

$$\varepsilon_K = \boldsymbol{\varepsilon}\cdot\boldsymbol{\omega}_K = \boldsymbol{\omega}_K\cdot\nabla\mathbf{u},$$

   we get the equations of motion expressed in displacements.

2. Returning to the main topic of our findings, it should be noted that the success in their application seems to be determined by the progress of works in two directions:

   a) in developing theoretical approaches linking the disclosed **mathematical structure** of an elastic body, described by its elastic eigenstates and stiffness moduli with the **physical structure** described, let us say, in the case of composites, by material constants of components, geometric characteristics of their mutual location, contact and joining parameters, etc.;

b) in the development of effective and economical experimental procedures for determining the eigenstates of elasticity and stiffness moduli directly from macroscopic tests, using solutions to standard boundary problems.

3. The physical implementation of the spectral theorem of the theory of linear operators in the form of the structural formula of an elastic body can be used by mathematicians as an excellent illustration, such as, say, planar incompressible fluid flows serve as illustrations of the conformal mapping theory.

I am pleased to express my deep appreciation to the management of the USSR Academy of Sciences and the Institute of Problems in Mechanics for providing the conditions under which this work could be completed. The work was discussed at seminars at the Institute of Problems in Mechanics of the USSR Academy of Sciences, at the departments of elasticity theory and plasticity theory at Moscow University, at the Institute of Crystallography of the USSR Academy of Sciences, and earlier at the Institute of Fundamental Technological Research, Polish Academy of Sciences. I am grateful to the leaders of these seminars A.Yu. Ishlinsky, A.A. Ilyushin, V.D. Klyushnikov, V.L. Indenbom, W. Szczepiński and all participants of the discussions for their kind and friendly comments. I am grateful for the friendly interest in the obtained results by my colleagues, A. Blinowski and J. Ostrowska.

Warsaw, 1969 – Moscow, 1983

Appendices

Appendix A. Euclidean tensors

In continuum mechanics, except for some specific areas (relativistic continuum mechanics, continual dislocation theory, etc.), only Euclidean tensors are used. We highlight some of the most important points of their algebraic theory, not always clearly perceived by users of the so-called “tensor calculus”, often reduced to simply juggling with indices.

Euclidean tensors of p-order are elements of the p-fold tensor product [I–III]

$$T_p = \underbrace{E \otimes \ldots \otimes E}_{p \text{ times}}, \tag{A.1}$$

where E is the original Euclidean vector space, which we take as 3-dimensional; for $p = 0$ these elements are numbers, for $p = 1$ vectors $\mathbf{a}$, from the space E itself. In other words, T_p is a linear space, which basis make 3^p simple tensors

$$\mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j, \qquad i, \ldots, j = 1, 2, 3, \tag{A.2}$$

where $\mathbf{e}_i$ is the basis in E, and the symbol of tensor multiplication $\otimes$ has the property of multilinearity. Each tensor $\mathbf{A} \in T_p$ can therefore be written down as

$$\mathbf{A} = A^{i \cdots j} \mathbf{e}_i \otimes \ldots \otimes \mathbf{e}_j. \tag{A.3}$$

All the usual rules for converting a tensor representation $A^{i \cdots j}$ of tensor $\mathbf{A}$ from one basis to another follow from this formula.

In the space T_p a group of automorphisms of space E opearates, i.e. an **orthogonal group** $\mathcal{O}$ of a 3-dimensional space. For any such mapping $\mathbf{Q} \in \mathcal{O}$ that converts a vector $\mathbf{a}$ into a vector $\mathbf{Qa}$ and for any tensor $\mathbf{A} \in T_p$, we have,

$$\mathbf{A} \to \mathbf{Q} * \mathbf{A}, \tag{A.4}$$

where $\mathbf{Q}*$ is a linear operation, defined for simple tensors according to the formula

$$\mathbf{Q} * (\mathbf{a}_1 \otimes ... \otimes \mathbf{a}_p) \equiv \mathbf{Q}\,\mathbf{a}_1 \otimes ... \otimes \mathbf{Q}\,\mathbf{a}_p. \tag{A.5}$$

In T_p **the group of permutations** Σ of natural numbers $1, ..., p$ also operates. For any permutation $\sigma \in \Sigma$ and any tensor $\mathbf{A} \in T_p$ we have,

$$\mathbf{A} \to \sigma \times \mathbf{A}, \tag{A.6}$$

where $\sigma\times$ is a linear operation defined on simple tensors according to the formula

$$\sigma \times (\mathbf{a}_1 \otimes ... \otimes \mathbf{a}_p) \equiv \mathbf{a}_{\sigma(1)} \otimes ... \otimes \mathbf{a}_{\sigma(p)}. \tag{A.7}$$

In the set of all Euclidean tensors of all orders, the so-called **tensorial operations** are introduced, i.e. mappings invariant with respect to the operation (A.4).

For any tensorial operation

$$(\mathbf{A}_1, ..., \mathbf{A}_s) \to f(\mathbf{A}_1, ..., \mathbf{A}_s),$$

where

$$\mathbf{A}_i \in T_p, \qquad i = 1, ..., s, \qquad f(\mathbf{A}_1, ..., \mathbf{A}_s) \in T_p,$$

by definition, identity is satisfied

$$f(\mathbf{Q} * \mathbf{A}_1, ..., \mathbf{Q} * \mathbf{A}_s) = \mathbf{Q} * f(\mathbf{A}_1, ..., \mathbf{A}_s),$$

for any $\mathbf{Q} \in \mathcal{O}$. In other words, any tensor operation is by definition an isotropic function. For example, for

$$f(\mathbf{C}, \boldsymbol{\omega}) \equiv \mathbf{C} \cdot \boldsymbol{\omega},$$

$$s = 2, \qquad p_1 = 4, \qquad p_2 = 2, \qquad p = 2,$$

we have

$$(\mathbf{Q} * \mathbf{C}) \cdot (\mathbf{Q} * \boldsymbol{\omega}) = \mathbf{Q} * (\mathbf{C} \cdot \boldsymbol{\omega}).$$

In particular in T_p various operations are introduced that define structures invariant with respect to (A.4). For example in T_2 an operation is introduced $(\boldsymbol{\omega}, \boldsymbol{\tau}) \to \boldsymbol{\omega}\boldsymbol{\tau}$, with respect to which T_2 is **a ring**. It is convenient in any space T_p to introduce an operation

$$(\mathbf{A}, \mathbf{B}) \to \mathbf{A} \cdot \mathbf{B} \tag{A.8}$$

by the bilinear definition, and defined for simple tensors by the formula

$$(\mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_p) \cdot (\mathbf{b}_1 \otimes \ldots \otimes \mathbf{b}_p) \equiv (\mathbf{a}_1 \cdot \mathbf{b}_1) \ldots (\mathbf{a}_p \cdot \mathbf{b}_p), \tag{A.9}$$

where $\mathbf{a} \cdot \mathbf{b}$ denotes the dot product in E. You can immediately see that it is a bilinear operation,

1) symmetric, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$,

2) positively defined, $\mathbf{A} \cdot \mathbf{A} > 0$ for $\mathbf{A} \neq 0$,

3) invariant to the group $\mathcal{O}$

$$(\mathbf{Q} * \mathbf{A}) \cdot (\mathbf{Q} * \mathbf{B}) = \mathbf{A} \cdot \mathbf{B}.$$

In other words, (A.8) turns out to be a correctly defined **dot product**, invariant with respect to $\mathcal{O}$. A space T_p with an inner product (A.8) is a **3^p-dimensional Euclidean space**. This structure in T_p is fully consistent (due to the third of the above mentioned properties of the dot product) with the structure of the p-fold tensor product[7].

By narrowing the dot product to a certain tensorial (i.e. constant with respect to $\mathcal{O}$, [I]) subspace $\mathcal{S} \subset T_p$, we obtain the Euclidean space of a lower dimension, consistent with the tensor structure.

Note. The scalar product $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ in T_2 **does not need** to be changed when passing to the subspace of symmetric tensors $\mathcal{S} \subset T_2$ just as it is not necessary, for example, to change the dot product $x_1x_2+y_1y_2+z_1z_2$ in R^3 when passing, say, to a 2-dimensional plane $x = y$. Then $x_1x_2+y_1y_2+z_1z_2 = 2x_1x_2+z_1z_2 = \xi_1\xi_2+z_1z_2$, where $\xi \equiv \sqrt{2}x = \sqrt{2}y$. This example explains well "the problem with 2's" in the formula $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \alpha_{11}\beta_{11} + \alpha_{22}\beta_{22} + \alpha_{33}\beta_{33} + 2(\alpha_{12}\beta_{12} + \alpha_{13}\beta_{13} + \alpha_{23}\beta_{23})$. ♦

Let us also draw attention to the following circumstance, completely ignored in the classical presentation of the theory of tensors. In many cases, it is convenient to carry out the following identifications,

$$T_p = T_{q_1} \otimes \ldots \otimes T_{q_s}, \qquad p = q_1 + \ldots + q_s. \tag{A.10}$$

We illustrate this on the example of the fourth-order tensors. Here

$$T_4 = T_1 \otimes T_1 \otimes T_1 \otimes T_1 = T_1 \otimes T_3 = T_3 \otimes T_1 = T_2 \otimes T_2. \tag{A.11}$$

[7]Of course, (A.8) is not the only dot product that is invariant in T_p with respect to $\mathcal{O}$, cf. §4.

Let us take the basis $\mathbf{e}_i \in E$. Of course, the set $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ is the basis in T_3. However, we can use in T_3 as the basis any linearly independent system of $3^3 = 27$ third-order tensors $\mathbf{E}_K$, $K = 1, ..., 27$. Then a set of 81 tensors

$$\mathbf{e}_i \otimes \mathbf{E}_K, \qquad i = 1, 2, 3, \qquad K = 1, ..., 27,$$

is (from the very definition of a tensor product $U \otimes V$ of finite-dimensional linear spaces U, V) the basis in $T_4 = T_1 \otimes T_3$, i.e. any fourth-order tensor can be represented in the form)

$$\mathbf{A} = \sum_{K=1}^{27} a^i_K \, \mathbf{e}_i \otimes \mathbf{E}_K \tag{A.12}$$

(summation over i is implied).

If, in turn, in T_2 the basis $\boldsymbol{\omega}_K$, $K = 1, ..., 9$ is adopted, then the set of 81 tensors

$$\boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_L, \qquad K, L = 1, ..., 9, \tag{A.13}$$

is also a good basis in T_4, i.e.

$$\mathbf{A} = \sum_{K,L=1}^{9} a_{KL} \, \boldsymbol{\omega}_K \otimes \boldsymbol{\omega}_L. \tag{A.14}$$

These formulae are valid with the relevant contractions in the tensor multiplication of tensor subspaces, for example for $\mathcal{T} = \mathcal{S} \otimes \mathcal{S}$, in the main part of the Report.

Note. In Cartesian index notation the formulae (A.12), (A.14) have the form:

$$A_{ijkl} = \sum_{K=1}^{27} a_{iK} E_{Kjkl}, \tag{A.12^1}$$

$$A_{ijkl} = \sum_{K,L=1}^{9} a_{KL} \, \omega_{Kij} \otimes \omega_{Lkl}. \quad \blacklozenge \tag{A.14^1}$$

Tensors from

$$T_{p+q} = T_p \otimes T_q \tag{A.15}$$

can be identified with linear transformations of space T_q into space T_p.

We introduce an operation

$$(\mathbf{L}, \mathbf{A}) \to \mathbf{L} \cdot \mathbf{A}, \tag{A.16}$$

where $\mathbf{L} \in T_{p+q}$, $\mathbf{A} \in T_q$, $\mathbf{L} \cdot \mathbf{A} \in T_p$, as bilinear operation, defined for simple tensors by the formula

$$\begin{gathered}(\mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_p \otimes \mathbf{a}_{p+1} \otimes \ldots \otimes \mathbf{a}_{p+q}) \cdot (\mathbf{b}_1 \otimes \ldots \otimes \mathbf{b}_q) \equiv \\ \equiv (\mathbf{a}_{p+1} \cdot \mathbf{b}_{p+1}) \ldots (\mathbf{a}_{p+q} \cdot \mathbf{b}_{p+q})(\mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_p).\end{gathered} \tag{A.17}$$

Thus, for each linear mapping $l : T_q \to T_p$ there is exactly one tensor $\mathbf{L} \in T_{p+q}$ such that for any $\mathbf{A} \in T_q$,

$$l(\mathbf{A}) = \mathbf{L} \cdot \mathbf{A}. \tag{A.18}$$

When $\mathbf{E}_K$, $\mathbf{E}^L$ are two bases in T_q, related by relationships

$$\mathbf{E}_K \cdot \mathbf{E}^L = \delta_K^L, \tag{A.19}$$

then a specific tensor $\mathbf{L}$, realizing a linear mapping l, can be expressed by a formula

$$\mathbf{L} = \sum_{K=1}^{3^q} l(\mathbf{E}_K) \otimes \mathbf{E}^K. \tag{A.20}$$

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Appendix B. Block-symmetric tensors as linear mappings

During the lectures on mechanics of materials, which I gave in October 1969 at the Faculty of Mechanics and Management of the Leningrad University of Technology, I said ([11], s. 54):

"We will pay attention to the following possibility of describing the properties of any even-valence tensor space T_{2p}. We can consider this space as $T_p \otimes T_p$ and accept T_p as the Euclidean vector space with the dimension n^p. Then $\mathbf{L} \in T_{2p}$ can be treated as linear mapping of T_p into itself

$$\mathbf{A} \in T_p \to \mathbf{L} \cdot \mathbf{A} \in T_p$$

and applying most of the previous theorems with a change of $T_1 = E_n$ to T_p and T_2 to T_{2p}.

For example, for tensors from T_{2p} with symmetry

$$\sigma \times \mathbf{L} = \mathbf{L},$$

where

$$\sigma = (p+1, ..., 2p, 1, ..., p) \in \Sigma_{2p}$$

(in Cartesian index notation $L_{i\,...\,jp\,...\,q} = L_{p\,...\,qi\,...\,j}$, where each ellipse replaces the $p-2$ indices) we get the spectral decomposition

$$\boxed{\mathbf{L} = \sum_{i=1}^{n^p} L_i \, \mathbf{l}_{\langle i \rangle} \otimes \mathbf{l}_{\langle i \rangle}}\,, \tag{B.1}$$

where

$$\mathbf{l}_{\langle i \rangle} \in T_p, \qquad \mathbf{l}_{\langle i \rangle} \cdot \mathbf{l}_{\langle j \rangle} = \delta_{\langle i \rangle \langle j \rangle}\text{''}.$$

Formula (B.1) is the foundation of all our work. The structural formula (3.39) is a special case of (B.1) for $n = 3$ and $p = 2$. I used this particular case to describe Hooke's law, which was of great interest to Anatoly Isakovich Lurie. Unfortunately, I did not keep my promise given to him to print it quickly.

Appendix C. Matrix approach

In crystal physics, the matrix method of writing Hooke's law enjoys well-deserved popularity (see e.g. [6, 10]). Typically, the following scheme is used to replace the stress and strain tensors with a numerical vector representation:

$$\begin{aligned} &\sigma_1 \equiv \sigma_{11}, \quad \sigma_2 \equiv \sigma_{22}, \quad \sigma_3 \equiv \sigma_{33}, \\ &\sigma_4 \equiv \sigma_{23}, \quad \sigma_5 \equiv \sigma_{13}, \quad \sigma_6 \equiv \sigma_{12}, \end{aligned} \tag{C.1}$$

and

$$\begin{aligned} &\varepsilon_1 \equiv \varepsilon_{11}, \qquad \varepsilon_2 \equiv \varepsilon_{22}, \qquad \varepsilon_3 \equiv \varepsilon_{33}, \\ &\varepsilon_4 \equiv 2\,\varepsilon_{23}, \quad \varepsilon_5 \equiv 2\,\varepsilon_{13}, \quad \varepsilon_6 \equiv 2\,\varepsilon_{12}, \end{aligned} \tag{C.2}$$

Hooke's law then takes the form

$$\sigma_\alpha = C_{\alpha\beta}\,\varepsilon_\beta, \qquad \alpha, \beta = 1, \dots, 6, \tag{C.3}$$

where

$$C_{\alpha\beta} \qquad \text{equals} \quad C_{ijkl} \tag{C.4}$$

with an appropriate change of indexes. Be warned that the eigentensors problem (2.8) **is not** a problem for the six roots of the matrix $C_{\alpha\beta}$, because the transformation laws (C.1) and (C.2) are different. In order to transform (2.8) into a matrix problem the equivalents for any tensor $\boldsymbol{\alpha} \in \mathcal{S}$ have to be taken

$$\begin{aligned} &\alpha_1 \equiv \alpha_{11}, \qquad \alpha_2 \equiv \alpha_{22}, \qquad \alpha_3 \equiv \alpha_{33}, \\ &\alpha_4 \equiv \sqrt{2}\,\alpha_{23}, \quad \alpha_5 \equiv \sqrt{2}\,\alpha_{13}, \quad \alpha_6 \equiv \sqrt{2}\,\alpha_{12}. \end{aligned}$$

The matrix obtained from the matrix C_{ijkl} we denote by C_{KL}, $K, L = 1, \dots, 6$. Then (2.8) is equivalent to

$$\sum_{L=1}^{6} C_{KL}\,\omega_L = \lambda\,\omega_K, \tag{C.5}$$

$K = 1, \dots, 6$, i.e. the matrix problem on eigenvectors and eigenvalues*.

Of course, in this case, it is necessary to check here that everything has an adequate invariance with respect to rotations in the space under consideration, but it is so. We do not stop at obtaining very opaque formulae in this manner.

Note. As I was informed at a seminar at the Institute of Crystallography of the USSR Academy of Sciences, the eigenvalues of the matrix C_{KL} were considered in the unpublished dissertation of K.S. Aleksandrov.

*Translator note: A typographical error in the original formula (C.5) of incorrect summation over two subscripts K and L has been corrected.

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Extended Commentary to English Translation

Andrzej Ziółkowski

Addendum 1

Re. §1. Introductory Remarks, section Notation, page 3 – Isotropic tensors of fourth-order.

The Report predominantly uses the absolute notation (index-free), nevertheless independent reproduction of the proofs of formulated in the Report theorems, as well as possible analytical and/or numerical calculations, requires executing operations on components of tensors, and therefore proficient use of the index notation. It is very helpful in this craft to write down, depict graphically and learn about the properties of *"unit" (isotropic) fourth-order tensors* often employed as generators of various subspaces, operators, projectors – unit tensors also in this sense that they are sums of permuted tensor products of second-order unit tensors $(\mathbf{1}, \sim\delta_{ij})$.

Definition P1.* *Cartesian tensors* are such tensors for which rectangular coordinate systems are adopted, i.e. orthonormal bases. Only such tensors are discussed here. For example, any second-order Cartesian tensor generated by vectors of a 3-dimensional Euclidean space can always be presented in the form,

$$\begin{gathered}\boldsymbol{\omega} = \sum_{i,j=1,2,3} \omega_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{\omega} \in T_2, \quad \mathbf{e}_i \in E_3, \\ \{\mathbf{e}_i\},\ \ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij};\ \ \{\mathbf{e}_i \otimes \mathbf{e}_j\},\ \ (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (\mathbf{e}_k \otimes \mathbf{e}_l) = \delta_{ik}\delta_{jl};\ \ i,j,k,l = 1,2,3,\end{gathered} \tag{P.1}$$

where $\boldsymbol{\omega}$ is the second-order tensor, the set of three unit vectors $\{\mathbf{e}_i\}$ is the orthonormal (Cartesian) basis of the 3-dimensional Euclidean vector space E_3 generating the second-order tensors space T_2 (the Euclidean space is a linear space with a defined scalar product operation), the set of 9 dyads $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ is the orthonormal basis of the space T_2. The coefficients ω_{ij} (3×3 matrix) are components of the tensor $\boldsymbol{\omega}$ representation in the tensorial basis $\mathbf{e}_i \otimes \mathbf{e}_j$.

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*Editorial note: In the Translator's appendix, all definitions, theorems and references are additionally marked with a prefix P for distinction.

It is worth noting that a tensor, which formally, mathematically is defined as a certain *algebraic structure*, when used to describe real physical phenomena, can be, and it is convenient to interpret it as a certain *geometric object* (like a point, line or plane), *invariant* in this sense that regardless of the manner of its description – including the choice of the basis (coordinate system) of the vector space generating the tensor space to which a given tensor belongs (here E_3), see, for example, chapter 2 in Stanisław Gołąb's book [P5]. In the light of this interpretation, it becomes clear that the tensor is an integrated (inseparable) entity consisting of the *tensor representation* and the *tensor basis*, in which this representation has been written out, see definition $(\text{P.1})_1$. Considering the tensor representation only – for example the matrix representation in isolation from the basis – may lead to erroneous conclusions, which is discussed in more detail in Addendum 3.

Remark. All (matrix) representations of tensors discussed in Addendum 1, unless explicitly stated otherwise, have been written out in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$, see $(\text{P.1})_3$.

In Appendix A of the Report, a *group of permutations* operating in a tensor space T_p (on p-order tensors) is discussed, and the permutation operation itself is defined with formula (A.7) to be *linear operation.* Functionality of this operation seems to be *significantly underestimated* in the literature on the subject. Due to the importance of the *permutation operation* for the considerations presented below, its definition and properties are discussed in more detail.

Definition P2. A *Permutation* $\sigma\times$ of a *tensor* $\mathbf{A}$ we call a linear mapping defined by the following formula:

$$\sigma \times \mathbf{A}: \ \mathbf{A} = A_{1\,2\,\dots\,p}\, \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \dots \otimes \mathbf{e}_p \to \sigma \times \mathbf{A} = A_{1\,2\,\dots\,p}\, \mathbf{e}_{\sigma(1)} \otimes \mathbf{e}_{\sigma(2)} \otimes \dots \otimes \mathbf{e}_{\sigma(p)},$$
$$\sigma \equiv \langle \sigma(1) \dots \sigma(p) \rangle, \qquad \mathbf{A},\ \sigma \times \mathbf{A} \in T_p, \tag{P.2}$$

where $\sigma(1), \sigma(2), \dots, \sigma(p)$ represents the given permutation of the first p natural numbers $1, 2, \dots, p$, while $A_{1\,2\,\dots\,p}$ denote the components of the tensor $\mathbf{A}$ of order p in tensorial basis $\mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \dots \otimes \mathbf{e}_p$. Permutation of a tensor means changing the ordering of the components of its tensorial basis.

The permutation operation can be interpreted completely equivalently, as a permutation of the components of the tensor representation written out in a fixed basis,

$$\sigma \times \mathbf{A} \equiv \langle \sigma(1)\ \sigma(2)\ \dots\ \sigma(p) \rangle \times \mathbf{A} = A_{\sigma(1)\sigma(2)\dots\sigma(p)}\, \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \dots \otimes \mathbf{e}_p \in T_p. \tag{P.3}$$

For the permutation operation σ of a tensor, it is convenient to introduce the following more compact notation $\sigma \times \mathbf{A} \equiv \langle \sigma(1) \dots \sigma(p) \rangle \times \mathbf{A} \equiv \mathbf{A}^{\langle (\sigma(1) \dots \sigma(p) \rangle}$. In the case where it is known that the ordering of only two indices is changed it is convenient to specify only those indices that are swapped, for example, in the case of tensors of the fourth-order $\mathbf{A}^{\langle 4\,2 \rangle}$ instead of $\mathbf{A}^{\langle 1\,4\,3\,2 \rangle}$.

The operation of permutation is an *automorphism*, i.e. a reversible linear transformation of a tensor space T_p onto itself ($\sigma : T_p \xrightarrow{on} T_p$).

The concept of a group is one of the most important concepts widely used in building theories (models) of real physical phenomena.

Definition P3. An algebraic structure $G \equiv (\{G\}, \lozenge)$ composed of a non-empty set of *elements* $\{G\}$, and an *operation* (*mapping*) „$\lozenge$" assigning an element from $\{G\}$ to any pair of elements from $\{G\}$ ($\lozenge : (g, h) \in \{G\} \times \{G\} \Rightarrow g \lozenge h \in \{G\}$) is called a group when the $\lozenge$ satisfies the following axioms:

$$
\begin{aligned}
&\text{(i)} && \bigwedge_{g_1, g_2, g_3 \in G} && g_1 \lozenge (g_2 \lozenge g_3) = (g_1 \lozenge g_2) \lozenge g_3, \\
&\text{(ii)} && \bigvee_{e \in G} \bigwedge_{g \in G} && e \lozenge g = g \lozenge e = g, \\
&\text{(iii)} && \bigwedge_{g \in G} \bigvee_{h \in G} && g \lozenge h = h \lozenge g = e,
\end{aligned}
\tag{P.4}
$$

i.e. the operation $\lozenge$ it is *associative* (i), there exist a *neutral element* of the group (ii), for each element of the group exists an *inverse element* (iii).

A group is called *commutative* (*Abelian group*), when operation $\lozenge$ is commutative

$$
\text{(iv)} \qquad \bigwedge_{g, h \in G} \quad g \lozenge h = h \lozenge g.
$$

The set of all permutation transformations operating in a fixed-order tensor space constitutes a group ($\mathcal{P}^\sigma$), which allows to introduce the concept of *internal symmetry of tensors*.

Definition P4. *A group of internal symmetry of a tensor* $\mathbf{A} \in T_p$ it is called a subset of the group $\mathcal{P}^\sigma$, the elements of which satisfy the condition

$$
\mathcal{P}^\sigma_{\mathbf{A}} \equiv \{\sigma \in \mathcal{P}^\sigma; \quad \sigma \times \mathbf{A} = \mathbf{A}\} \quad \mathcal{P}^\sigma_{\mathbf{A}} \subset \mathcal{P}^\sigma. \tag{P.5}
$$

Examples

Tensor $\mathbf{A}$ is *(internally) symmetric* over a pair of indexes (α, β), if the following condition is satisfied, $\mathbf{A} = \mathbf{A}^{\langle \beta\, \alpha \rangle}$, $\sim A_{\dots\alpha\dots\beta\dots} = A_{\dots\beta\dots\alpha\dots}$, i.e. when the components of the tensor $\mathbf{A}$ representation, in any fixed basis, upon swapping the indexes (α, β) are the same.

In the case of fourth-order tensors, the symmetry with respect to the permutation operation $\langle 1\ 3\ 2\ 4\rangle\times$ means that $\mathbf{A} = \mathbf{A}^{\langle 1\ 2\ 3\ 4\rangle} = \mathbf{A}^{\langle 1\ 3\ 2\ 4\rangle}$, i.e. $A_{ijkl} = A_{ikjl}$ in any fixed basis.

The permutation operations very conveniently allow the introduction of many useful tensors, such as *operators of symmetrization* and/or *projectors*. For example, for fourth-order tensors, the following permutation operators are very useful:

$$\mathfrak{l} = id = \langle 1\ 2\ 3\ 4\rangle, \quad \mathfrak{c} = \frac{1}{2}[\langle 1\ 3\ 2\ 4\rangle + \langle 1\ 4\ 3\ 2\rangle], \quad \mathfrak{s} = \frac{1}{3}\langle \mathfrak{l} + 2\mathfrak{c}\rangle. \tag{P.6}$$

The permutation operator $\mathfrak{c}$ applied to the tensor $\mathbf{1}\otimes\mathbf{1}$ makes it possible to obtain a fourth-order tensor $\mathbf{I}^{(4s)} \equiv \mathfrak{c}\times(\mathbf{1}\otimes\mathbf{1})$ symmetrizing any second-order tensor, cf. (P.18) below. The operator $\mathfrak{s}$ transforms any fourth-order tensor into an absolutely symmetric tensor, i.e. symmetric when any pair of indexes are swapped, cf. also (P.52) below.

Definition P5. A tensor is *absolutely (internally) symmetric*, when the group of its internal symmetries is the entire set of permutations $\mathcal{P}^{\sigma}_{\mathbf{A}} = \mathcal{P}^{\sigma}$, i.e., it is symmetric over each pair of indices.

More very interesting information about the utility and applications of permutation operations can be found in Rychlewski's work on the linear decomposition of fourth-order tensors [P12].

Definition P6. A set of second-order tensors $\mathcal{O}$ with properties

$$\mathcal{O} = \{\mathbf{Q}\in T_2;\ \ \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{1}, \quad \det(\mathbf{Q}) = \pm 1\} \tag{P.7}$$

is a group and is called the *group of orthogonal tensors*.

A subset of orthogonal tensors for which $\det(\mathbf{Q}) = 1$

$$\mathcal{R} = \{\mathbf{Q}\in T_2;\ \ \mathbf{Q}\mathbf{Q}^T = \mathbf{1}, \quad \det(\mathbf{Q}) = +1\} \subset \mathcal{O} \tag{P.8}$$

is called the *proper orthogonal group* or the *rotation group $SO(3)$*.

It is interesting to note that the proper orthogonal group $\mathcal{R}$ in 3D is not an Abelian group, but a group of rotations around the fixed axis is an Abelian group (2D).

Orthogonal tensors can be interpreted as *automorphisms* – linear transformations of space $E_3 \xrightarrow{on} E_3$. In the case of tensors $\mathbf{Q}$ from the group $\mathcal{O}$ these are rotations and mirror images of vectors from a space E_3, while in the case of tensors from the group $\mathcal{R}$ these are only rotations.

Transformations determined by orthogonal tensors allow us to introduce the concept of *external symmetry of tensors*.

Definition P7. A *group of external symmetry of tensor* $\mathbf{A} \in T_p$ we call a subset of all orthogonal tensors $\mathbf{Q}$, which satisfy the following condition

$$\mathcal{O}_T = \{\mathbf{Q} \in \mathcal{O}; \quad \mathbf{Q} * \mathbf{A} = \mathbf{A}\}. \tag{P.9}$$

Tensors $\mathbf{A}$ that satisfy condition (P.9) are called *symmetric* (*invariant*, *stable*) tensors with respect to orthogonal transformations $\mathbf{Q} \in \mathcal{O}_T \subset \mathcal{O}$.

It has been proved that eight groups of external symmetry of Hooke's tensors exist, cf. Forte and Vianello [P4]. These groups are described in Figure 10 of the Report. It is the concept of external symmetry of tensors that forms the basis for the division of Hooke's tensors into *equivalence classes* adopted in the Report and for specification of an appropriate spectral distribution of a tensor depending on its belonging to a specific class/group of external symmetry.

Definition P8. A relation operating in a certain set X is called an *equivalence* relation $\widehat{R} \subseteq X \times X$, if and only if it is: i) *reflexive*, i.e. for any given $x \in X$ it holds $x\,\widehat{R}\,x$, ii) *symmetric*, for any given $x, y \in X$ it holds $x\,\widehat{R}\,y \Rightarrow y\,\widehat{R}\,x$, iii) *transitive*, i.e. for any given $x, y, z \in X$ it holds $(x\,\widehat{R}\,y) \wedge (y\,\widehat{R}\,z) \Rightarrow y\,\widehat{R}\,z$.

Two elements $x, y \in X$ such that $x, y \in \widehat{R}$ are called *equivalent* and are often symbolically denoted $x \sim y$.

Frequently, in the existing literature on material behavior modeling the basic definition of different types of symmetry is omitted, so that the less experienced reader can easily get impression that the symmetry property is a property of the tensor representation. In the light of the recalled above definitions, it is easy to conclude that this is not true. The symmetry property of different types is property of a tensor – integrated object that consists of a basis and its representation in this basis. Possessing certain symmetry by a given tensor enforces the existence of specific constraints between the components of its representation in a specific tensor basis. Such constraints between the representation components written out in different tensor bases can have and have different explicit form. Naturally, when a given tensor simultaneously has two different types of symmetry, for example a specific external symmetry and a specific internal symmetry, then these different types of symmetry also impose constraints between its representation components, generally different even in the same tensor basis. This type of situation is discussed in more detail in Addendum 3 below the formula (P.52) on the example of an elastic material having external symmetry of the monoclinic type and absolute internal symmetry.

In this Addendum much attention is focused on unit tensors, which are *isotropic tensors*.

Definition P9. *Isotropic tensors* are such tensors, for which the group of external symmetry is a whole set of orthogonal tensors $\mathcal{O}_T = \mathcal{O}$, cf. (P.7).

An isotropic tensor *does not have identical components in any orthonormal basis* of a tensor space to which it belongs. Examples of different representations of a unit tensor of the second-order $\mathbf{1} \in T_2$ – see (P.11), in different orthonormal bases of symmetric, second-order tensors space $\mathcal{S}$, and different representations of unit fourth-order tensors in different orthonormal bases of the space of symmetric, fourth-order tensors $\mathcal{S} \otimes \mathcal{S}$ re given in Addendum 3 below.

Remark. The *isotropic* tensors have *identical* representation components in *all* bases *isometric* with respect to a *proper orthogonal group*, but isotropic tensors *do not have* the same representation in all *orthonormal bases.*

Definition P10. Two orthonormal bases are *isometric*, with respect to a *proper orthogonal group*, when a rotation tensor $\mathbf{Q} \in \mathcal{R}$ exists, cf. (P.8), such that

$$\mathbf{p}_\alpha = \delta^i_\alpha \mathbf{Q}\,\mathbf{e}_i, \quad (\mathbf{p}_\alpha \otimes \mathbf{p}_\beta = \delta^i_\alpha \mathbf{Q}\,\mathbf{e}_i \otimes \delta^j_\beta \mathbf{Q}\,\mathbf{e}_j, ..., \text{etc.}), \quad \mathbf{e}_i, \mathbf{p}_\alpha \in E_3, \tag{P.10}$$

cf. e.g. chapter 4 in Ostrowska-Maciejewska text book [P10].

Remark. *Not all* orthonormal tensor bases are isometric with respect to the *proper orthogonal group*, see also the text below the formula (P.46). It is worth noting that *all* (single-handed) orthonormal bases in 3-dimensional Euclidean space E_3 are isometric.

A unit tensor in the space of second-order tensors can be presented in the following form:

$$\mathbf{1} \equiv \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 \in T_2, \quad 1_{ij} = \delta_{ij} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{P.11}$$

The tensor $\mathbf{1}$ is the neutral tensor of a tensor space T_2 $\boldsymbol{\omega}\mathbf{1} = \boldsymbol{\omega}$ $(\omega_{is}\delta_{sj} = \omega_{ij})$ for any $\boldsymbol{\omega} \in T_2$. Tensor $\mathbf{1}$ is isotropic tensor and it generates a 1-dimensional subspace of second-order isotropic tensors $\boldsymbol{\omega}^{iso} = a\mathbf{1} \in T_2$, where a stands for any real number. *There are no non-trivial isotropic vectors.* The only isotropic vector is the zero vector ($\mathbf{v} = \mathbf{0} \in T_1$).

Let us move on to the issue of unit (isotropic) fourth-order tensors. A maximum of *three fourth-order isotropic tensors* are linearly independent, see e.g. the book by Jeffreys [P7].

In order to construct three linearly independent, fourth-order "unit" tensors there are used the permutation operations and tensor $\mathbf{1}\otimes\mathbf{1}$. The most commonly encountered in the literature set of three linearly independent unit tensors of the fourth-order has the following form:

$$
\begin{aligned}
\mathbf{1}\otimes\mathbf{1} &\equiv \langle 1\,2\,3\,4\rangle \times \mathbf{1}\otimes\mathbf{1}, \quad \sim (1\otimes 1)_{ijkl} = \delta_{ij}\delta_{kl},\\
(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} &\equiv \langle 1\,3\,2\,4\rangle \times \mathbf{1}\otimes\mathbf{1}, \quad \sim (1\otimes 1)^{\langle 32\rangle}{}_{ijkl} = \delta_{ik}\delta_{jl},\\
(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle} &\equiv \langle 1\,4\,3\,2\rangle \times \mathbf{1}\otimes\mathbf{1}, \quad \sim (1\otimes 1)^{\langle 42\rangle}{}_{ijkl} = \delta_{il}\delta_{kj},
\end{aligned}
\tag{P.12}
$$

$$
\mathbf{Q}*(\mathbf{1}\otimes\mathbf{1})=\mathbf{1}\otimes\mathbf{1}, \quad \mathbf{Q}*(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}=(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}, \quad \mathbf{Q}*(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}=(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}.
$$

The set of tensors (P.12) is a generator of the 3-dimensional subspace of isotropic fourth-order tensors.

The isotropy of the tensor $\mathbf{1}\otimes\mathbf{1}$ is proved, for the remaining "units" the proof is analogous. Applying the rotation transformation $\mathbf{Q} \sim q_{ij}$, $\mathbf{Q}\mathbf{Q}^T = \mathbf{1} \sim q_{im}q_{jm} = \delta_{ij}$ to the coordinate system (orthonormal basis) of the 3-dimensional vector Euclidean space E_3 – generating a given tensor space T_4, the components of the tensor $\mathbf{1}\otimes\mathbf{1}$ transform according to the following formula $(1\otimes 1)^Q{}_{ijkl} = q_{im}q_{jn}q_{kp}q_{lq}(1\otimes 1)_{mnpq}$. Substituting the components of the tensor $\mathbf{1}\otimes\mathbf{1}$ into this formula, we get $(1\otimes 1)^Q{}_{ijkl} = q_{im}q_{jn}q_{kp}q_{lq}\delta_{mn}\delta_{pq} = q_{in}q_{jn}q_{kp}q_{lp} = \delta_{ij}\delta_{kl} = (1\otimes 1)_{ijkl}$. ♦

In order to simplify the notation of the components of the second- and fourth-order symmetric tensors in the vector and/or matrix form, respectively, frequently the mapping of index pairs in the representations of these tensors is used. The following index pairs mapping is most commonly used:

(i,j)	11	22	33	23	13	12	32	31	21
K	1	2	3	4	5	6	7	8	9

$$
\omega_{ij} \to \omega_K, \qquad A_{ijkl} \to A_{KL}, \qquad i,j,k,l = 1,2,3, \qquad K,L = 1,\ldots,9. \tag{P.13}
$$

In the case of symmetric tensors, when using a compact notation of their representations, the range of indices is reduced $(K,L = 1,\ldots,6)$. See also, for example Moakher's paper [P8].

Tensors (P.12) as fourth-order tensors generally have 81 components ($\mathbf{A} \to A_{ijkl}$, $i,j,k,l = 1,2,3$) but only 9 components for each of these tensors are non-zero. The components of tensors (P.12) can be presented visually in the form of 9×9 matrixes, as shown graphically in (P.14); for clarity only non-zero components of respective tensors representations were entered:

$$\mathbf{1} \otimes \mathbf{1} \quad \sim (1 \otimes 1)_{ijkl} = \delta_{ij}\delta_{kl}, \quad ijkl = 1, 2, 3$$

$ij \downarrow kl \rightarrow$	11	22	33	23	32	13	31	12	21
11	1	1	1						
22	1	1	1						
33	1	1	1						
23									
32									
13									
31									
12									
21									

, (P.14)$_1$

$$(\mathbf{1} \otimes \mathbf{1})^{\langle 32 \rangle} \quad \sim I^{(4)}_{ijkl} \equiv (1 \otimes 1)^{\langle 32 \rangle}_{ijkl} = \delta_{ik}\delta_{jl}$$

$ij \downarrow kl \rightarrow$	11	22	33	23	32	13	31	12	21
11	1								
22		1							
33			1						
23				1					
32					1				
13						1			
31							1		
12								1	
21									1

, (P.14)$_2$

$$(\mathbf{1} \otimes \mathbf{1})^{\langle 42 \rangle} \quad \sim (1 \otimes 1)^{\langle 42 \rangle}_{ijkl} = \delta_{il}\delta_{kj}$$

$ij \downarrow kl \rightarrow$	11	22	33	23	32	13	31	12	21
11	1								
22		1							
33			1						
23					1				
32				1					
13							1		
31						1			
12									1
21								1	

. (P.14)$_3$

In the matrix representations (P.14) respective columns and rows were ordered not according to the natural order $(P.13)_2$ but according to the order $(1,2,3,4,7,5,8,6,9)$.

Indices of non-zero components of tensors: $\mathbf{1}\otimes\mathbf{1}$, $(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}$, $(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}$ are,

$$\begin{aligned}
\mathbf{1}\otimes\mathbf{1} \quad &\to \{(1111),(1122),(1133),(2211),(2222),(2233),(3311),(3322),(3333)\},\\
(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} &\to \{(1111),(1212),(1313),(2121),(2222),(2323),(3131),(3232),(3333)\},\\
(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle} &\to \{(1111),(1221),(1331),(2112),(2222),(2332),(3113),(3223),(3333)\}.
\end{aligned} \tag{P.15}$$

As it is easy to see thanks to the graphic representation $(P.14)_2$ a tensor $(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} \in T_4$ treated as a linear mapping of the space of second-order tensors onto itself $T_2 \xrightarrow{on} T_2$ (automorphism) is a unit operator because it transforms any second-order tensor into the same tensor – it is a neutral element of the space T_4.

The following relations are valid:

$$\begin{gathered}
\mathbf{I}^{(4)}\cdot\boldsymbol{\omega} = \boldsymbol{\omega} \quad \left(I^{(4)}_{ijkl}\omega_{kl} = \omega_{ij}\right), \quad \boldsymbol{\omega}\in T_2,\\
(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}\circ\mathbf{A} = \mathbf{A} \quad \left(I^{(4)}_{ijst}A_{stkl} = A_{ijkl}\right), \quad \mathbf{A}\in T_4,\\
\mathbf{I}^{(4)} \equiv (\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}\in T_4, \quad \sim I^{(4)}_{ijkl} = \delta_{ik}\delta_{jl}.
\end{gathered} \tag{P.16}$$

Due to the property (P.16) the tensor $(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle}$ has been additionally marked here with the symbol $\mathbf{I}^{(4)}\in T_4$.

The most general form of the fourth-order isotropic tensor can be represented by tensors (P.12) as follows:

$$\begin{gathered}
\mathbf{A}^{iso} = a\,(\mathbf{1}\otimes\mathbf{1}) + b\,(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} + c\,(\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}\in T_4,\\
A^{iso}_{ijkl} = a\,\delta_{ij}\delta_{kl} + b\,\delta_{ik}\delta_{jl} + c\,\delta_{il}\delta_{kj},
\end{gathered} \tag{P.17}$$

where a, b, c are scalars.

Tensors (P.12) are commonly used to construct other fourth-order tensors having the desired properties and useful in various applications. For example, in order to separate the symmetric part and the antisymmetric part of the second-order tensors $\boldsymbol{\omega}\in T_2$, the tensors–projectors, defined as follows are useful:

$$
\begin{gathered}
\mathbf{I}^{(4s)} \equiv \frac{1}{2}\Big[(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} + (\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}\Big], \quad \mathbf{I}^{(4a)} \equiv \frac{1}{2}\Big[(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} - (\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}\Big], \\
\mathbf{I}^{(4s)} \sim I^{(4s)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \quad \mathbf{I}^{(4a)} \sim I^{(4a)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj}), \\
\mathbf{I}^{(4)} = \mathbf{I}^{(4s)} + \mathbf{I}^{(4a)}, \quad \mathbf{I}^{(4s)} \circ \mathbf{I}^{(4a)} = \mathbf{0}, \quad \mathbf{I}^{(4s)} \circ \mathbf{I}^{(4s)} = \mathbf{I}^{(4s)}, \quad \mathbf{I}^{(4a)} \circ \mathbf{I}^{(4a)} = \mathbf{I}^{(4a)}.
\end{gathered}
\tag{P.18}
$$

The components of the projectors $\mathbf{I}^{(4s)}$, $\mathbf{I}^{(4a)}$ can be presented in a convenient matrix representation (9×9 matrix) as follows:

$$
\mathbf{I}^{(4s)} \sim I^{(4s)}_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})
$$

$$
\begin{array}{c|ccc:cc:cc:cc}
ij\downarrow kl\rightarrow & 11 & 22 & 33 & 23 & 32 & 13 & 31 & 12 & 21 \\
\hline
11 & 1 & & & & & & & & \\
22 & & 1 & & & & & & & \\
33 & & & 1 & & & & & & \\
\hdashline
23 & & & & \frac{1}{2} & \frac{1}{2} & & & & \\
32 & & & & \frac{1}{2} & \frac{1}{2} & & & & \\
\hdashline
13 & & & & & & \frac{1}{2} & \frac{1}{2} & & \\
31 & & & & & & \frac{1}{2} & \frac{1}{2} & & \\
\hdashline
12 & & & & & & & & \frac{1}{2} & \frac{1}{2} \\
21 & & & & & & & & \frac{1}{2} & \frac{1}{2}
\end{array},
\tag{P.19$_1$}
$$

$$
\mathbf{I}^{(4a)} \sim I^{(4a)}_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{kj})
$$

$$
\begin{array}{c|ccc:cc:cc:cc}
ij\downarrow kl\rightarrow & 11 & 22 & 33 & 23 & 32 & 13 & 31 & 12 & 21 \\
\hline
11 & 0 & & & & & & & & \\
22 & & 0 & & & & & & & \\
33 & & & 0 & & & & & & \\
\hdashline
23 & & & & \frac{1}{2} & -\frac{1}{2} & & & & \\
32 & & & & -\frac{1}{2} & \frac{1}{2} & & & & \\
\hdashline
13 & & & & & & \frac{1}{2} & -\frac{1}{2} & & \\
31 & & & & & & -\frac{1}{2} & \frac{1}{2} & & \\
\hdashline
12 & & & & & & & & \frac{1}{2} & -\frac{1}{2} \\
21 & & & & & & & & -\frac{1}{2} & \frac{1}{2}
\end{array}.
\tag{P.19$_2$}
$$

The properties (P.18)$_{7,8}$ indicating that tensors $\mathbf{I}^{(4s)}$, $\mathbf{I}^{(4a)}$ are projectors are easy to prove by simply multiplying their matrix representations (P.19)$_1$ and (P.19)$_2$, respectively, by themselves.

Tensors $\mathbf{1} \otimes \mathbf{1}$, $\mathbf{I}^{(4)}$, $\mathbf{I}^{(4s)}$, $\mathbf{I}^{(4a)}$ have the following properties:

$$\operatorname{tr}(\boldsymbol{\omega})\,\mathbf{1} = (\mathbf{1} \otimes \mathbf{1}) \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \mathbf{I}^{(4)} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega}^s = \mathbf{I}^{(4s)} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega}^a = \mathbf{I}^{(4a)} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} \in T_2, \tag{P.20}$$

where $\boldsymbol{\omega}$ is any second-order tensor, $\operatorname{tr}(\boldsymbol{\omega}) = \omega_{11} + \omega_{22} + \omega_{33}$ is the trace of tensor $\boldsymbol{\omega}$, $\boldsymbol{\omega}^s$ is the symmetric part of the tensor $\boldsymbol{\omega}$, and $\boldsymbol{\omega}^a$ its antisymmetric part.

Properties (P.20)$_1$ and (P.20)$_3$ can be presented visually by adopting the matrix representation of the fourth-order tensor (9×9 matrix) and the vector representation of the second-order tensor (vector 9×1) as follows:

$$\begin{bmatrix} \operatorname{tr}(\boldsymbol{\omega}) \\ \operatorname{tr}(\boldsymbol{\omega}) \\ \operatorname{tr}(\boldsymbol{\omega}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{33} \\ \omega_{23} \\ \omega_{32} \\ \omega_{13} \\ \omega_{31} \\ \omega_{12} \\ \omega_{21} \end{bmatrix}, \tag{P.21$_1$}$$

$$\begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{33} \\ \frac{1}{2}(\omega_{23} + \omega_{32}) \\ \frac{1}{2}(\omega_{23} + \omega_{32}) \\ \frac{1}{2}(\omega_{13} + \omega_{31}) \\ \frac{1}{2}(\omega_{13} + \omega_{31}) \\ \frac{1}{2}(\omega_{12} + \omega_{21}) \\ \frac{1}{2}(\omega_{12} + \omega_{21}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{33} \\ \omega_{23} \\ \omega_{32} \\ \omega_{13} \\ \omega_{31} \\ \omega_{12} \\ \omega_{21} \end{bmatrix}. \tag{P.21$_2$}$$

The components of the antisymmetric part $\boldsymbol{\omega}^a = \mathbf{I}^{(4a)} \cdot \boldsymbol{\omega}$ of tensor $\boldsymbol{\omega}$ are:

$$\begin{aligned}
&\omega^a_{11} = \omega^a_{22} = \omega^a_{33} = 0, && \omega^a_{23} = \frac{1}{2}(\omega_{23} - \omega_{32}) = -\omega^a_{32},\\
&\omega^a_{13} = \frac{1}{2}(\omega_{13} - \omega_{31}) = -\omega^a_{31}, && \omega^a_{12} = \frac{1}{2}(\omega_{12} - \omega_{21}) = -\omega^a_{21}.
\end{aligned} \tag{P.22}$$

It can be noticed that the tensors $\mathbf{1} \otimes \mathbf{1}$, $\mathbf{I}^{(4)}$, $(\mathbf{1} \otimes \mathbf{1})^{\langle 42 \rangle}$ are not mutually orthogonal in the sense of dot product. There are also not orthogonal tensors $\mathbf{1} \otimes \mathbf{1}$ and $\mathbf{I}^{(4s)}$ $((\mathbf{1} \otimes \mathbf{1}) \cdot \mathbf{I}^{(4s)} \neq 0)$. This is one of the reasons why it is most commonly found in the literature (for example in material behavior modeling), the following decomposition of the space of *fourth-order isotropic tensors* composed of *mutually orthogonal projectors*:

$$\begin{gathered}
\mathbf{A}^{iso} = a_1 \mathbf{I}_{\mathcal{P}} + b_1 \mathbf{I}_{\mathcal{D}} + c_1 \mathbf{I}^{(4a)} \in T_4,\\
\mathbf{I}_{\mathcal{P}} \equiv \frac{1}{3} \mathbf{1} \otimes \mathbf{1}, \qquad \mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1},\\
\mathbf{I}_{\mathcal{P}} \boldsymbol{\omega} = \frac{1}{3} \mathrm{tr}(\boldsymbol{\omega}) \mathbf{1} \equiv \boldsymbol{\omega}^{sph}, \qquad \mathbf{I}_{\mathcal{D}} \boldsymbol{\omega} = \boldsymbol{\omega}^s - \frac{1}{3} \mathrm{tr}(\boldsymbol{\omega}) \mathbf{1} \equiv \boldsymbol{\omega}^d.
\end{gathered} \tag{P.23}$$

Projectors $\mathbf{I}_{\mathcal{P}}$, $\mathbf{I}_{\mathcal{D}}$, $\mathbf{I}^{(4a)}$ allow decomposing the space of second-order tensors into mutually orthogonal subspaces of spherical tensors $\boldsymbol{\omega}^{sph} (\mathrm{tr}(\boldsymbol{\omega}) \neq 0)$ and deviatoric tensors $(\mathrm{tr}(\boldsymbol{\omega}) = 0)$ – symmetric $(\boldsymbol{\omega}^d)$ and antisymmetric $(\boldsymbol{\omega}^a)$.

All fourth-order tensors occurring in the decomposition are mutually orthogonal and are projectors, because

$$\begin{gathered}
\mathbf{I}_{\mathcal{P}} \cdot \mathbf{I}_{\mathcal{D}} = 0, \qquad \mathbf{I}_{\mathcal{P}} \cdot \mathbf{I}^{(4a)} = 0, \qquad \mathbf{I}_{\mathcal{D}} \cdot \mathbf{I}^{(4a)} = 0,\\
\mathbf{I}_{\mathcal{P}} \circ \mathbf{I}_{\mathcal{P}} = \mathbf{I}_{\mathcal{P}}, \qquad \mathbf{I}_{\mathcal{D}} \circ \mathbf{I}_{\mathcal{D}} = \mathbf{I}_{\mathcal{D}}, \qquad \mathbf{I}_{\mathcal{P}} \circ \mathbf{I}_{\mathcal{D}} = \mathbf{0}, \qquad \mathbf{I}^{(4a)} \circ \mathbf{I}^{(4a)} = \mathbf{I}^{(4a)}.
\end{gathered} \tag{P.24}$$

Formula (10.6) in the Report is an example of the decomposition (P.23), where, due to the premises of physical nature, the deviatoric antisymmetric part was assumed to be identically equal to zero $(a_1 = \lambda_{\mathcal{P}} = 3\lambda + 2\mu = 3K$, $b_1 = \lambda_{\mathcal{D}} = 2\mu$, $c_1 = 0)$ – λ, μ denote Lamé constants.

More information on Euclidean (Cartesian) isotropic tensors of any order, as well as proofs of their properties, can be found for example in chapter 7 of H. Jeffreys book [P7] and/or in chapter 1, section 1.2.5 of R. Ogden book [P9].

Addendum 2

Re. §3. Structural formula (3.33), page 16 – Coefficients of characteristic equation of fourth-order symmetric tensor.

As stated in the Report, the Kelvin stiffness moduli λ_i $(i = 1, ..., 6)$, the real moduli of elastic stiffness, are the roots of the characteristic equation of Hooke's elastic stiffness tensor ($\mathbf{C}$), cf. formula (3.33). However, the Report does not provide explicit formulae for the coefficients of the characteristic equation a_i expressed by the basic invariants (traces) of the powers of the tensor $\mathbf{C}$.

The coefficients of the characteristic equation of Hooke's tensor – generally any symmetric, fourth-order tensor from a tensor space generated by a linear vector space of dimension 3, can be expressed in terms of traces of powers of this tensor, as follows:

$$
\begin{aligned}
&p(\lambda) = \det(\mathbf{C} - \lambda\mathbf{I}^{(4s)}) = a_0\lambda^6 + a_1(\mathbf{C})\lambda^5 + ... + a_5(\mathbf{C})\lambda + a_6(\mathbf{C}) = 0,\\
&\quad a_0 = 1, \qquad a_1 = \operatorname{tr}\mathbf{C}, \qquad a_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2],\\
&\quad a_3 = \frac{1}{6}[(\operatorname{tr}\mathbf{C})^3 - 3\operatorname{tr}\mathbf{C}^2\operatorname{tr}\mathbf{C} + 2\operatorname{tr}\mathbf{C}^3],\\
&\quad a_4 = \frac{1}{24}[(\operatorname{tr}\mathbf{C})^4 + 8\operatorname{tr}\mathbf{C}^3\operatorname{tr}\mathbf{C} - 6\operatorname{tr}\mathbf{C}^2(\operatorname{tr}\mathbf{C})^2 + 3(\operatorname{tr}\mathbf{C}^2)^2 - 6\operatorname{tr}\mathbf{C}^4],\\
&\quad a_5 = \frac{1}{120}[(\operatorname{tr}\mathbf{C})^5 - 30\operatorname{tr}\mathbf{C}^4\operatorname{tr}\mathbf{C} + 15(\operatorname{tr}\mathbf{C}^2)^2\operatorname{tr}\mathbf{C} - 20\operatorname{tr}\mathbf{C}^3\operatorname{tr}\mathbf{C}^2\\
&\qquad\quad - 10\operatorname{tr}\mathbf{C}^2(\operatorname{tr}\mathbf{C})^3 + 20\operatorname{tr}\mathbf{C}^3(\operatorname{tr}\mathbf{C})^2 + 24\operatorname{tr}\mathbf{C}^5],\\
&\quad a_6 = \frac{1}{720}[(\operatorname{tr}\mathbf{C})^6 + 144\operatorname{tr}\mathbf{C}^5\operatorname{tr}\mathbf{C} - 120\operatorname{tr}\mathbf{C}^3\operatorname{tr}\mathbf{C}^2\operatorname{tr}\mathbf{C} - 15\operatorname{tr}\mathbf{C}^2(\operatorname{tr}\mathbf{C})^4\\
&\qquad\quad + 90\operatorname{tr}\mathbf{C}^4\operatorname{tr}\mathbf{C}^2 + 40\operatorname{tr}\mathbf{C}^3(\operatorname{tr}\mathbf{C})^3 - 15(\operatorname{tr}\mathbf{C}^2)^3 - 90\operatorname{tr}\mathbf{C}^4(\operatorname{tr}\mathbf{C})^2\\
&\qquad\quad + 40(\operatorname{tr}\mathbf{C}^3)^2 + 45(\operatorname{tr}\mathbf{C}^2)^2(\operatorname{tr}\mathbf{C})^2 - 120\operatorname{tr}\mathbf{C}^6] = \det(\mathbf{C}).
\end{aligned}
\tag{P.25}
$$

In another notation, explicit formulae for the coefficients $a_i(\mathbf{C})$ of the characteristic equation of the fourth-order symmetric tensor are given in J. Betten work [P1].

In the case when the trace of the elastic stiffness tensor is equal to zero ($\operatorname{tr}(\mathbf{C}) = 0$), the expressions for the coefficients of the characteristic equation are significantly simplified and can be presented in the following form:

$$
\begin{aligned}
&\operatorname{tr}\mathbf{C} = 0 \Rightarrow p(\lambda) = \lambda^6 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0,\\
&a_2 = -\frac{1}{2}\operatorname{tr}\mathbf{C}^2, \qquad a_3 = \frac{1}{3}\operatorname{tr}\mathbf{C}^3, \qquad a_4 = -\frac{1}{4}\operatorname{tr}\mathbf{C}^4 + \frac{1}{2}a_2^2,\\
&a_5 = \frac{1}{5}\operatorname{tr}\mathbf{C}^5 + a_2a_3, \qquad a_6 = -\frac{1}{6}\operatorname{tr}\mathbf{C}^6 - \frac{1}{3}a_2^3 + \frac{1}{2}a_3^2 + a_2a_4.
\end{aligned}
\tag{P.26}
$$

For materials showing symmetries higher than full anisotropy – see Figure 10 of the Report, the characteristic equation can be reduced to a lower degree equation. Characteristic equation $(\mathrm{P}.25)_1$ for completely anisotropic materials is generally the 6th degree equation and it cannot be simplified, for monoclinic and trigonal materials it can be effectively reduced to the 4th degree equation (by writing the tensor $\mathbf{C}$ in the system of natural symmetry axes of the material), in the case of orthotropic, tetragonal, transverse (cylindrical) isotropy and cubic symmetry materials, it can be reduced to the 3rd degree equation, see e.g. Bona *et al.* [P2].

Sequential formulae for the coefficients of the characteristic equation of the *second-order tensors of the n-dimensional Euclidean space* expressed by the traces of powers of this tensor can be found, for example, in the book by M. Itskov [P6], formulae (4.23), (4.30), pp. 102–105. They have the following form:

$$
\begin{aligned}
&\det(\mathbf{A} - \lambda\mathbf{I}) = 0,\\
&\quad p_A(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}I_A^{(1)} + \ldots + (-1)^{n-k}\lambda^{n-k}I_A^{(k)}\\
&\qquad\qquad + \ldots + (-1)^0\lambda^0 I_A^{(n)},\\
&\quad I_A^{(0)} = 1, \qquad I_A^{(k)} \equiv \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}I_A^{(k-i)}\operatorname{tr}(\mathbf{A}^i),\\
&\quad I_A^{(n)} = \det(\mathbf{A}) = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}I_A^{(n-i)}\operatorname{tr}(\mathbf{A}^i),
\end{aligned}
\tag{P.27}
$$

where $p_A(\lambda)$ denotes the characteristic polynomial, $I_A^{(k)}$ denote linearly independent principal invariants of the tensor $\mathbf{A}$. This equation can be applied to *symmetric tensors of the fourth-order generated by 3-dimensional vector space*, since such tensors can be equivalently treated as *tensors of the second-order generated by the 6-dimensional vector space*; similarly general fourth-order tensors can be equivalently considered as second-order tensors of a 9-dimensional space.

In particular, when writing the formulae (P.27) for a symmetric tensor of the fourth-order, it is:

$$
\begin{aligned}
&p_A(\lambda) = I_A^{(0)}\lambda^6 - I_A^{(1)}\lambda^5 + I_A^{(2)}\lambda^4 - I_A^{(3)}\lambda^3 + I_A^{(4)}\lambda^2 - I_A^{(5)}\lambda + I_A^{(6)} = 0,\\
&\quad I_A^{(1)} = \operatorname{tr}\mathbf{A}, \qquad I_A^{(2)} = \frac{1}{2}[I_A^{(1)}\operatorname{tr}\mathbf{A} - \operatorname{tr}\mathbf{A}^2],\\
&\quad I_A^{(3)} = \frac{1}{3}[I_A^{(2)}\operatorname{tr}\mathbf{A} - I_A^{(1)}\operatorname{tr}\mathbf{A}^2 + \operatorname{tr}\mathbf{A}^3],\\
&\quad I_A^{(4)} = \frac{1}{4}[I_A^{(3)}\operatorname{tr}\mathbf{A} - I_A^{(2)}\operatorname{tr}\mathbf{A}^2 + I_A^{(1)}\operatorname{tr}\mathbf{A}^3 - \operatorname{tr}\mathbf{A}^4],\\
&\quad I_A^{(5)} = \frac{1}{5}[I_A^{(4)}\operatorname{tr}\mathbf{A} - I_A^{(3)}\operatorname{tr}\mathbf{A}^2 + I_A^{(2)}\operatorname{tr}\mathbf{A}^3 - I_A^{(1)}\operatorname{tr}\mathbf{A}^4 + \operatorname{tr}\mathbf{A}^5],\\
&\quad I_A^{(6)} = \frac{1}{6}[I_A^{(5)}\operatorname{tr}\mathbf{A} - I_A^{(4)}\operatorname{tr}\mathbf{A}^2 + I_A^{(3)}\operatorname{tr}\mathbf{A}^3 - I_A^{(2)}\operatorname{tr}\mathbf{A}^4 + I_A^{(1)}\operatorname{tr}\mathbf{A}^5 - \operatorname{tr}\mathbf{A}^6] = \det(\mathbf{A}).
\end{aligned}
\tag{P.28}
$$

After using the above-mentioned equivalence of the Hooke's tensor $\mathbf{C}$ with a 6-dimensional second-order symmetric tensor, sequential substitution of the expressions for invariants $I_A^{(i)}$ and replacement of the symbol $\mathbf{A}$ with $\mathbf{C}$ the formulae (P.25) are recovered.

The characteristic equation defined by the formula (3.33) of the Report is a special case of the generalized Cayley–Hamilton equation:

$$
\mathbf{C}^6 - I_A^{(1)}\mathbf{C}^5 + I_A^{(2)}\mathbf{C}^4 - I_A^{(3)}\mathbf{C}^3 + I_A^{(4)}\mathbf{C}^2 - I_A^{(5)}\mathbf{C} + I_A^{(6)}\mathbf{I}^{(4)} = 0. \tag{P.29}
$$

The eigenvalues and eigenvectors problem for an m-order tensors in n-dimensional space is complex and constitutes an open scientific problem. For example, it is not at all obvious what is the maximum number of distinct *eigenvalues* or, what is the equivalent, of the *principal invariants* of the m-order tensor in an n-dimensional space – which in turn determines the degree of the characteristic polynomial. To explain the problem a bit, let us take Hooke's tensor as an example. It is a fourth-order symmetric tensor. Generally, it has 18 linearly independent invariants, but only 6 principal invariants that enter as coefficients into the characteristic equation – as shown, for example, by Rychlewski in this Report. Its solution generally leads to 6 linearly independent eigenvalues. The remaining 12 invariants characterize the eigenstates of the Hooke's tensor, which are second-order tensors.

For an accessible discussion of the eigenvalue problem of higher order tensors and some interesting results, see L. Qi [P11].

Addendum 3

Re. Appendix C, Matrix approach, page 87 – Issues connected with expression of Hooke's law in Voigt matrix notation and in Kelvin representation.

In Appendix C of the Report the existence of *the problem of non-equivalence of the notations of Hooke's law presented in the form of matrix relationships* was signaled. As a premise indicating the existence of a trouble it was pointed out the non-equivalence of the problem of determining the eigenvalues and eigenvectors of the Hooke's tensor first using its full 9-dimensional matrix tensor representation and next using its compact 6-dimensional Voigt's matrix notation. At the same time, it was pointed out that an equivalence of tensor matrix notations exists when instead of the Voigt's notation Hooke's law is written in a compact 6-dimensional Kelvin's matrix representation. However, the Report does not clearly state the fundamental cause of the non-equivalence of the various matrix notations of Hooke's law. This underlying source of intricacy is in fact the problem of finding an *equivalent tensor representation* of Hooke's law when firstly, symmetric second-order tensors (stresses, strains) and the fourth-order tensor (Hooke's tensor) present in it, are interpreted as tensors from the 3-dimensional space, and when secondly, these items are interpreted as vectors and the second-order tensor, respectively from the 6-dimensional space, and vice versa. This topic is discussed here in more detail in order to facilitate the understanding of the results presented in the Report and their possible transformation for the reader own needs.

The generalized Hooke's law, which is a homogeneous, linear tensorial relation ($\mathbf{Y} = \mathbf{A} \cdot \mathbf{X}$, $\mathbf{X}, \mathbf{Y} \in T_2$, $\mathbf{A} \in T_4$) without taking advantage of any simplifications resulting from the symmetry of the tensors appearing in it, can be presented in the form of tensorial matrix representation – written in the base $(\text{P.1})_3$, as follows:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1113} & C_{1131} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2213} & C_{2231} & C_{2212} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3313} & C_{3331} & C_{3312} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2313} & C_{2331} & C_{2312} & C_{2321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3213} & C_{3231} & C_{3212} & C_{3221} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1313} & C_{1331} & C_{1312} & C_{1321} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3113} & C_{3131} & C_{3112} & C_{3121} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1213} & C_{1231} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2113} & C_{2131} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix},$$

$$\begin{gathered} \boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} \sim \sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \qquad i,j,k,l = 1,\dots,3, \\ \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathcal{S} = T^s_{2(n=3)}, \qquad \mathbf{C} \in \mathcal{S} \otimes \mathcal{S} = T^s_{4(n=3)}, \end{gathered} \tag{P.30}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}$ is the small strain tensor, $\mathbf{C}$ is the tensor of the elastic stiffness coefficients.

Physical premises based on experimental results lead to the conclusion that for the vast majority of materials used today in engineering, a good model approximation is the assumption that the stress and strain tensors are symmetrical, i.e. that their components of representation in the orthogonal basis $(\mathrm{P}.1)_3$ are the same when changing the order of indexes. Similarly, the components of the Hooke's tensor representation in the same basis show internal symmetries upon swapping the first and second indexes, upon swapping the third and fourth indexes, and upon swapping the first and second pair of indexes, i.e.:

$$\begin{aligned} \sigma_{ij} = \sigma_{ji}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad & \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathcal{S}, \\ C_{ijkl} = C_{jikl} = C_{klij}, \quad & \mathbf{C} \in \mathcal{S} \otimes \mathcal{S}. \end{aligned} \tag{P.31}$$

Extensive experimental and modeling literature exists indicating the rationality of assumptions about the internal symmetry of tensors present in Hooke's law, which are not referenced here.

After taking into account the symmetry conditions (P.31) the number of different components of the stress and strain tensors decreases from 9 to 6, and the number of different components of the Hooke's tensor decreases from 81 to 21, which allows us to write Hooke's law in much more compact matrix notations:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix}^{V_o} \leftrightarrow$$

$$\leftrightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 2C_{14} & 2C_{15} & 2C_{16} \\ C_{21} & C_{22} & C_{23} & 2C_{24} & 2C_{25} & 2C_{26} \\ C_{31} & C_{32} & C_{33} & 2C_{34} & 2C_{35} & 2C_{36} \\ C_{41} & C_{42} & C_{43} & 2C_{44} & 2C_{45} & 2C_{46} \\ C_{51} & C_{52} & C_{53} & 2C_{54} & 2C_{55} & 2C_{56} \\ C_{61} & C_{62} & C_{63} & 2C_{64} & 2C_{65} & 2C_{66} \end{bmatrix}^{Vo} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \tag{P.32}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^{Vo} = C_{\alpha\beta}\gamma_\beta \;\; (C_{\alpha\beta} = C_{\beta\alpha}) \;\leftrightarrow\; \sigma_K = C_{KL}^{Vo}\varepsilon_L \;\; (C_{KL}^{Vo} \neq C_{LK}^{Vo}),$$

$$\varepsilon_{11}^{Vo} \equiv \varepsilon_{11} = \gamma_1, \qquad \varepsilon_{22}^{Vo} \equiv \varepsilon_{22} = \gamma_2, \qquad \varepsilon_{33}^{Vo} \equiv \varepsilon_{33} = \gamma_3,$$

$$\varepsilon_{23}^{Vo} \equiv \gamma_4 = 2\varepsilon_{23}, \qquad \varepsilon_{13}^{Vo} \equiv \gamma_5 = 2\varepsilon_{13}, \qquad \varepsilon_{12}^{Vo} \equiv \gamma_6 = 2\varepsilon_{12},$$

where γ_4, γ_5, γ_6 denote so-called engineering shear strains.

Remark. Matrix notations of Hooke's law (P.32) are not *tensorial representations* of Hooke's law written in some irreducible (complete) tensorial bases.

The relations (P.32) are commonly known as the *Voigt's notation.* The coefficients "2" appearing in $(P.32)_1$ at shear components of strain, appeared as a result of taking advantage of symmetries of tensors present in the Hooke's law, e.g. $C_{2323}\varepsilon_{23} + C_{2332}\varepsilon_{32} = C_{2323}2\varepsilon_{23} = C_{2323}\gamma_4 = 2C_{44}\varepsilon_4$. Often the coefficients "2" are put into the matrix representation of the Hooke's tensor, as it is explicitly shown in the matrix notation $(P.32)_2$. In the relations $(P.32)_{2,3,4}$ the mapping of indexes (P.13) was used, while the range of indexes α, β, K, L, variability has been limited to 6, because, as it is easy to notice, due to symmetries (P.31), the equations for the components of stresses 7, 8, 9 in (P.30) are repetitions of the equations for components 4, 5, 6, respectively, and therefore they can be omitted in compact matrix notation (P.32).

Remark. In the literature on the subject, there is a certain inconsistency (disorder), because the name Voigt's notation is used both for matrix notation $(P.32)_1$ and for notation $(P.32)_2$. The matrix of coefficients in formula $(P.32)_1$ is shown in an explicit form the matrix $C_{\alpha\beta} = C_{ijkl}$ appearing in formula (C.4) of the Report. In his original work, Voigt did not apply a mapping (P.13) to the strain tensor but a mapping $V_1 = \varepsilon_{11} = \varepsilon_1$, $V_2 = \varepsilon_{22} = \varepsilon_2$, $V_3 = \varepsilon_{33} = \varepsilon_3$, $2W_1 = 2\varepsilon_{23} \neq \varepsilon_4$, $2W_2 = 2\varepsilon_{13} \neq \varepsilon_5$, $2W_3 = 2\varepsilon_{12} \neq \varepsilon_6$, see e.g. formula (100), p. 156 in chapter 7 of the original work of W. Voigt [P13]. In the Report, in formula (C.2) the convention of designating the deformation components used by Voigt is maintained, which is inconsistent with the mapping (P.13).

In the present Addendum, mapping (P.13) is consistently used in *all* tensor objects. It is for this reason that the compact representation of the stiffness matrix $(P.32)_2$ is treated as a matrix representation of Voigt (but not tensorial representation) of the elastic stiffness tensor $\mathbf{C}$, and labeled C^{Vo}_{KL}. As one can see, it is *non-symmetric.*

To illustrate the benefits of Voigt's notation, let us consider Hooke's law for a *linear elastic isotropic material*, which in absolute notation takes the following form

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{C}^{iso}\boldsymbol{\varepsilon} = [\lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}^{(4s)}]\boldsymbol{\varepsilon}, \\ \mathbf{C}^{iso} &\sim C^{iso}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \end{aligned} \tag{P.33}$$

where the index representation of the elastic stiffness tensor $(P.33)_2$ is written out in the 9-dimensional basis $(P.1)_3$.

The *compact matrix form of Hooke's law for an isotropic material in Voigt's notation* most often appearing in the literature corresponds to the pattern $(\text{P.32})_2$, and takes the form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \tag{P.34}$$

$$C_{12}^{Vo} = C_{1122} = \lambda, \quad C_{66}^{Vo} = 2C_{1212} = 2\mu,$$

$$C_{11}^{Vo} = C_{22}^{Vo} = C_{33}^{Vo} = C_{12}^{Vo} + C_{66}^{Vo},$$

where λ, μ are Lamé constants, cf. also $(\text{P.32})_2$.

The matrix representation of the compliance tensor $\mathbf{S}^{iso}$ inverse to the stiffness tensor $\mathbf{C}^{iso}$ ($\mathbf{C}^{iso} \circ \mathbf{S}^{iso} = \mathbf{I}^{(4)} \sim C_{ijmn}^{iso} S_{mnkl}^{iso} = \delta_{ik}\delta_{jl}$) has the same form as the one in the formula $(\text{P.34})_2$, where the coefficients λ, 2μ should be replaced with substitutions $\lambda \to -\lambda/[(3\lambda + 2\mu)2\mu] = -\nu/E$, $2\mu \to 1/2\mu = (1+\nu)/E$, where E is Young's modulus, ν is Poisson's ratio, ($\boldsymbol{\varepsilon} = \mathbf{S}^{iso}\boldsymbol{\sigma} = [(-\nu/E)\,\mathbf{1} \otimes \mathbf{1} + (1/2\mu)\,\mathbf{I}^{(4s)}]\,\boldsymbol{\sigma}$).

The tensor notation of Hooke's law for an isotropic material (P.33), most commonly found in the literature on the subject, has such a mathematical form that it enforces the existence of mutual symmetry of the stress and strain tensors (through symmetrization of the strain tensor with a projector $\mathbf{I}^{(4s)}$). However, when it is known a priori – it is assumed, that there is a symmetry of the strain (stress) tensor then *there is no need to further symmetrize it.* Then completely equivalent to the notation $(\text{P.33})_1$ is the tensor notation of the *isotropic Hooke's law*, in which instead of the tensor $\mathbf{I}^{(4s)}$ there is a tensor $\mathbf{I}^{(4)}$ – cf. (P.18),

$$\boldsymbol{\sigma} = \mathbf{C}^{iso}\boldsymbol{\varepsilon} = [\lambda\mathbf{1} \otimes \mathbf{1} + 2\mu\mathbf{I}^{(4s)}]\boldsymbol{\varepsilon} = [\lambda\mathbf{1} \otimes \mathbf{1} + 2\mu\mathbf{I}^{(4)}]\boldsymbol{\varepsilon}. \tag{P.35}$$

The relation $(\text{P.35})_2$ indicates, at the same time, that materials can exist that are exhibiting isotropic behavior even when the force (deformation) internal interactions existing in them cannot be reliably modeled by means of *symmetric strain and stress tensors.*

Connections in the form of matrix relations $(\text{P.32})_{1,2}$ fully correspond to Hooke's law expressed as a tensorial relationship and its matrix representation (P.30), in terms of their *mathematical correctness and physical content.* Therefore, it is perfectly correct to use them in analytical/numerical calculations to determine stresses based on strains or vice versa. However, in the relations (P.32) *the original tensorial character of the relation* (P.30) *has been lost.* When trying to interpret the relation (P.32), as a tensorial representation written out in a certain basis of a tensor relationship, it is not difficult to notice that in the compact 6-dimensional Voigt's matrix notation, certain tensorial components belonging to the omitted elements of the full 9-dimensional tensorial basis, e.g. ε_{32}, ε_{31}, ε_{21}, were assigned to not-corresponding to them elements of the 6-dimensional abridged basis, i.e. $\mathbf{e}_2 \otimes \mathbf{e}_3$, $\mathbf{e}_1 \otimes \mathbf{e}_3$, $\mathbf{e}_1 \otimes \mathbf{e}_2$, respectively. This operation violates the formal principles of tensor calculus.

Several premises prove the *lack of tensorial equivalence* of the connections (P.30) and (P.32). For example, the norm of a 6-dimensional stress vector treated as an equivalent of a symmetric tensor of the second-order, calculated according to the rules of vector calculus is different from the norm of the stress tensor calculated according to the rules of the tensor calculus of the second-order tensors $||\sigma_K||^2 \neq ||\sigma_{ij}||^2$. The situation is similar for the strain vector $||\varepsilon_K||^2 \neq ||\varepsilon_{ij}||^2$ and for deformation vector introduced by Voigt $||\varepsilon_K^{Vo}||^2 \neq ||\varepsilon_{ij}||^2$,

$$
\begin{aligned}
&\boldsymbol{\sigma} \sim \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \rightarrow \\
&\qquad ||\boldsymbol{\sigma}||^2 = ||\sigma_{ij}||^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2(\sigma_{23}^2 + \sigma_{13}^2 + \sigma_{12}^2), \\
&\boldsymbol{\sigma} \sim \sigma_\alpha = [\sigma_1,\ \sigma_2,\ \sigma_3,\ \sigma_4,\ \sigma_5,\ \sigma_6]^T \rightarrow \\
&\qquad ||\sigma_K||^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + \sigma_{23}^2 + \sigma_{13}^2 + \sigma_{12}^2, \\
&\boldsymbol{\varepsilon} \sim \varepsilon_\alpha = [\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4,\ \varepsilon_5,\ \varepsilon_6]^T \rightarrow \\
&\qquad ||\varepsilon_K||^2 = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2 + \varepsilon_{12}^2, \\
&\boldsymbol{\varepsilon}^{Vo} \sim \gamma_\alpha = [\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6]^T \rightarrow \\
&\qquad ||\gamma_K||^2 = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 4\varepsilon_{23}^2 + 4\varepsilon_{13}^2 + 4\varepsilon_{12}^2.
\end{aligned}
\tag{P.36}
$$

It is worth pointing out that with the original Voigt's vector notation of stress and strain tensors – $(\text{P.32})_1$, the value of elastic energy is preserved $\frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^{Vo} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_K\gamma_K$.

The norm $||C_{ijkl}||^2$ of the matrix of the Hooke's tensor representation coefficients $\mathbf{C}$ – cf. (P.30), showing symmetries (P.31), is also different from the norm of the Voigt's coefficients matrix $||C_{\alpha\beta}||^2$ – cf. $(\text{P.32})_1$, of the tensor $\mathbf{C}$ ($||C_{\alpha\beta}||^2 \neq ||C_{ijkl}||^2$), and the same in the case of norm of matrix $||C_{\alpha\beta}^{Vo}||^2$ –

see (P.32)$_2$, ($\|C^{Vo}_{\alpha\beta}\|^2 \neq \|C_{ijkl}\|^2$). This is the well-known so-called "problem of twos". Therefore, problems for eigenvalues of C_{ijkl}, i.e. relations (P.30) with symmetries (P.31) and for eigenvalues of $C_{\alpha\beta}$, i.e. relations (P.32)$_1$ or of $C^{Vo}_{\alpha\beta}$, i.e. relations (P.32)$_2$, respectively, are not equivalent.

To obtain *tensorial equivalence of*: i) Hooke's law (P.30) expressed by *3-dimensional second-order tensors (stresses, strains) and the fourth-order (Hooke's) tensor* – showing internal symmetries (P.31), and of ii) Hooke's law expressed by the same tensors interpreted as vectors and second-order tensors in *6-dimensional spaces*, it is necessary to introduce *symmetric tensorial bases* $(\mathcal{S} \leftrightarrow T_{1(n=6)}, \mathcal{S} \otimes \mathcal{S} \leftrightarrow T^s_{2(n=6)})$.

In the case of symmetric tensors of the second-order $\boldsymbol{\omega} \in \mathcal{S}$, $(\omega_{ij} = \omega_{ji})$ it can be seen that the following relations for mixed components are valid:

$$\omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j + \omega_{ji}\mathbf{e}_j \otimes \mathbf{e}_i \equiv \omega^{Ke}_{ij}\frac{1}{\sqrt{2}}[\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i] = \omega^{Ke}_{ij}\mathbf{t}_K, \quad K!, \quad (i,j)!, \quad i \neq j,$$

$$\begin{aligned}
\omega^{Ke}_4 &= \omega^{Ke}_{23} \equiv \frac{1}{\sqrt{2}}(\omega_{23} + \omega_{32}) = \sqrt{2}\omega_4, \quad \mathbf{t}_4 = \frac{1}{\sqrt{2}}[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2],\\
\omega^{Ke}_5 &= \omega^{Ke}_{13} \equiv \frac{1}{\sqrt{2}}(\omega_{13} + \omega_{31}) = \sqrt{2}\omega_5, \quad \mathbf{t}_5 = \frac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1],\\
\omega^{Ke}_6 &= \omega^{Ke}_{12} \equiv \frac{1}{\sqrt{2}}(\omega_{12} + \omega_{21}) = \sqrt{2}\omega_6, \quad \mathbf{t}_6 = \frac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1].
\end{aligned} \tag{P.37}$$

The symbol „!" denotes the suspension of the summation convention over repeating indexes, i.e. ! means do not sum.

The above connections enable for symmetric tensors of the second-order to effectively introduce, for *three pairs of symmetric representation elements* and the corresponding pairs of tensorial basis elements, only *three symmetrized representation elements* and *three* corresponding to them elements of the (symmetric) tensorial basis. They also provide guidance on how to introduce a 6-dimensional, complete, orthonormal tensor basis suitable for symmetric tensors $\boldsymbol{\omega} \in T_{1(n=6)}$, namely:

$$\begin{gathered}
\mathbf{t}_1 \equiv \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{t}_2 \equiv \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{t}_3 \equiv \mathbf{e}_3 \otimes \mathbf{e}_3, \quad \mathbf{t}_4 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2],\\
\mathbf{t}_5 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1], \quad \mathbf{t}_6 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1],\\
\mathbf{t}_K \cdot \mathbf{t}_L = \delta_{KL}, \quad \mathbf{t}_K \in T_{1(n=6)}, \quad K, L = 1, ..., 6,\\
\boldsymbol{\omega} = \omega^{Ke}_K\mathbf{t}_K = \omega_1\mathbf{t}_1 + \omega_2\mathbf{t}_2 + \omega_3\mathbf{t}_3 + \sqrt{2}\omega_4\mathbf{t}_4 + \sqrt{2}\omega_5\mathbf{t}_5 + \sqrt{2}\omega_6\mathbf{t}_6, \quad \boldsymbol{\omega} \in T_{1(n=6)}.
\end{gathered} \tag{P.38}$$

Each second-order symmetric tensor $\boldsymbol{\omega}$ can be represented in the basis $\mathbf{t}_K$ as a vector – cf. (P.38)$_8$, where the coefficients ω_K^{Ke} denote the *Kelvin representation coefficients* of the tensor $\boldsymbol{\omega}$. The orthonormal basis (P.38) is the 6-dimensional complete, orthonormal basis of second-order symmetric tensors, the most commonly encountered in the literature and used in computations.

Hooke's law (P.30) written out in the orthonormal basis (P.38) using the symmetry property (P.31), takes the following matrix form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & \sqrt{2}C_{3323} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ \sqrt{2}C_{2311} & \sqrt{2}C_{2322} & \sqrt{2}C_{2333} & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ \sqrt{2}C_{1311} & \sqrt{2}C_{1322} & \sqrt{2}C_{1333} & 2C_{1323} & 2C_{1313} & 2C_{1312} \\ \sqrt{2}C_{1211} & \sqrt{2}C_{1222} & \sqrt{2}C_{1233} & 2C_{1223} & 2C_{1213} & 2C_{1212} \end{bmatrix}^{Ke} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix},$$

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon} \sim \sigma_\alpha^{Ke} = C_{\alpha\beta}^{Ke}\, \varepsilon_\beta^{Ke}, \quad \alpha, \beta = 1, \dots, 6, \quad \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in T_{1(n=6)}, \quad \mathbf{C} \in T^s_{2(n=6)}. \tag{P.39}$$

The matrix representation notation of Hooke's law (P.39) is known as the *Kelvin's notation* – it is also known as the *Mandel's notation.*

Hooke's law for an isotropic material in Kelvin matrix representation (notation) written out in an orthonormal basis (P.38) has the form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sqrt{2}\sigma_4 \\ \sqrt{2}\sigma_5 \\ \sqrt{2}\sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \sqrt{2}\varepsilon_4 \\ \sqrt{2}\varepsilon_5 \\ \sqrt{2}\varepsilon_6 \end{bmatrix}, \tag{P.40}$$

$$\boldsymbol{\sigma} = \mathbf{C}^{iso} \cdot \boldsymbol{\varepsilon} = \sigma_\alpha^{Ke}\, \mathbf{t}_\alpha = C_{\alpha\beta}^{iso}\, \mathbf{t}_\alpha \otimes \mathbf{t}_\beta \cdot \varepsilon_\beta^{Ke}\, \mathbf{t}_\beta, \quad \alpha, \beta = 1, \dots, 6.$$

Calculation of the norm of stress and/or strain tensors according to the rules of vector calculus using their Kelvin components (σ_α^{Ke}, ε_α^{Ke}) – cf. (P.39), leads to the same value as when calculating these norms according to the rules of tensor calculus for symmetric tensors of the second-order using the components (σ_{ij}, ε_{ij}). Similarly, the calculation of the value of the norm of the elastic properties (Hooke's) tensor using its Kelvin components ($C_{\alpha\beta}^{Ke}$) – cf. (P.39), gives the same value as the one calculated for the components (C_{ijkl}) – cf. (P.30), calculated according to the rules of tensor calculus for the fourth-order tensors.

Introduction of a 6-dimensional, complete, symmetric, orthonormal basis $\{\mathbf{t}_K\} \in T_{1(n=6)}$ – cf. (P.38), for symmetric tensors of the second-order, allows to generate a complete, symmetric, orthonormal basis for symmetric tensors of the fourth-order, consisting of diads $\{\mathbf{t}_J \otimes \mathbf{t}_K\} \in T_{1(n=6)} \otimes T_{1(n=6)}$. Writing out Hooke's law in these bases allows one to bring the full tensorial equivalence of the representations of Hooke's law, specified basing on the 3-dimensional Euclidean vector space – see representation (P.30), with the one specified basing on the 6-dimensional Euclidean vector space – cf. representation (P.39). In the consequence, the problems for eigenvalues and eigenstates of matrix relations (P.30) and (P.39) – in the Report, respectively (2.8) and (C.5), are *completely equivalent*:

$$\{\mathbf{C} \cdot \boldsymbol{\omega} = \lambda\, \boldsymbol{\omega} \sim C_{ijkl}\omega_{kl} = \lambda\, \omega_{ij}\}$$
$$\Leftrightarrow \{\mathbf{C}^{Ke} \cdot \boldsymbol{\omega}^{Ke} = \lambda\, \boldsymbol{\omega}^{Ke} \sim C^{Ke}_{\alpha\beta}\, \omega^{Ke}_{\beta} = \lambda\, \omega^{Ke}_{\alpha}\}, \quad \text{(P.41)}$$

that is, the eigenvalues and the eigenstates obtained by solving these problems are identical.

Tensor representation, Kelvin's matrix notation (P.39) shows how to achieve *full tensorial equivalence* of interpretation of symmetric tensors of the second- and fourth-order from a 3-dimensional space with symmetries (P.31), as vectors and second-order tensors from a 6-dimensional space, and the opposite:

$$\begin{gathered}\omega_{ij} = \omega_{ji}, \quad \boldsymbol{\omega} \in \mathcal{S} \leftrightarrow \omega_K, \quad \boldsymbol{\omega} \in T_{1(n=6)}, \quad i, j, k, l = 1, \ldots, 3, \quad K, L = 1, \ldots, 6, \\ C_{ijkl} = C_{jikl} = C_{klij}, \quad \mathbf{C} \in \mathcal{S} \otimes \mathcal{S} \leftrightarrow C_{KL} = C_{LK}, \quad \mathbf{C} \in T^{s}_{2(n=6)}.\end{gathered} \quad \text{(P.42)}$$

Comparing the Hooke's law written out in compact 6-dimensional Kelvin's notation (P.39) and in Voigt's notation (P.32) it is not difficult to notice that they are mathematically (physically) equivalent, but only the Kelvin's matrix notation is a tensor representation of the tensor relationship expressing Hooke's law (P.30).

It is worth noting that Hooke's law tensorial representation – in a compact notation, written out in an *orthogonal* but *not orthonormal* tensor basis $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, $\sqrt{2}\mathbf{t}_4$, $\sqrt{2}\mathbf{t}_5$, $\sqrt{2}\mathbf{t}_6$ takes the form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix}, \quad \text{(P.43)}$$

that is the form (P.30) with omitted rows and columns numbered 5, 7, 9.

The matrix formula (P.43) provides an excellent illustration that the index form of a tensor relation *does not contain complete information* about such a relation, and that when analyzing index representations it is necessary to constantly remember in which tensor basis they were written out. Tensors, as *invariant objects* with respect to *change of coordinates system*, constitute an integral unity of the base and its representation in this base.

Let us now return to the problem of unit (isotropic) tensors of the fourth-order discussed in Addendum 1. In the case of symmetric tensors of the fourth-order, cf. (P.31), the subspace of isotropic tensors becomes *2-dimensional* because the anisotropic part is identically equal to zero in the case of symmetric tensors ($c_1 = 0$) – cf. (P.23), and can be presented, e.g. in the form $\mathbf{A}^{iso} = a_1\,\mathbf{I}_{\mathcal{P}} + b_1\,\mathbf{I}_{\mathcal{D}} \in \mathcal{S}\otimes\mathcal{S} \leftrightarrow T^s_{2(n=6)}$.

Matrix representations of fourth-order isotropic tensors showing symmetries (P.31), i.e. $\mathbf{I}^{(4s)} = \frac{1}{2}[(\mathbf{1}\otimes\mathbf{1})^{\langle 32\rangle} + (\mathbf{1}\otimes\mathbf{1})^{\langle 42\rangle}] = \mathfrak{c}\times\mathbf{1}\otimes\mathbf{1}$ – cf. (P.18)$_1$, (P.6)$_2$, and $\mathbf{I}_{\mathcal{P}} = \frac{1}{3}\mathbf{1}\otimes\mathbf{1}$, $\mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}$ – cf. (P.23), written out in a symmetric basis $\mathbf{t}_K\otimes\mathbf{t}_L \in T^s_{2(n=6)}$ – cf. (P.38), have the following matrix representations in this basis:

$$
\begin{aligned}
&\mathbf{I}^{(4s)} = I^{(4s)}_{KL}\mathbf{t}_K\otimes\mathbf{t}_L, &&\mathbf{I}_{\mathcal{P}} = \frac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{P}\,KL}\mathbf{t}_K\otimes\mathbf{t}_L,\\
&I^{(4s)}_{KL} \sim \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, &&I_{\mathcal{P}\,KL} \sim \begin{bmatrix} \frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0&0&0\\ \frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0&0&0\\ \frac{1}{3}&\frac{1}{3}&\frac{1}{3}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\\
&\mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1} = I_{\mathcal{D}\,KL}\mathbf{t}_K\otimes\mathbf{t}_L,\\
&I_{\mathcal{D}\,KL} \sim \begin{bmatrix} \frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}&0&0&0\\ -\frac{1}{3}&\frac{2}{3}&-\frac{1}{3}&0&0&0\\ -\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}.
\end{aligned}
\tag{P.44}
$$

The representation of tensors $\mathbf{I}^{(4)} = \mathbf{1}\otimes\mathbf{1}^{\langle 32\rangle}$, $\mathbf{1}\otimes\mathbf{1}^{\langle 42\rangle}$, $\mathbf{I}^{(4a)}$ in a 6-dimensional basis $\mathbf{t}_K \otimes \mathbf{t}_L$ *cannot be correctly written out*, because this basis is insufficient (incomplete) for this purpose.

The orthonormal basis $\mathbf{t}_K$ – cf. (P.38), is not the only possible orthonormal basis of symmetric tensors of the second-order $T_{1(n=6)}$. The complete orthonormal basis of the space $T_{1(n=6)}$ is also provided by the following set of six second-order tensors:

$$\mathbf{h}_1 \equiv \frac{1}{\sqrt{3}}\mathbf{1}, \qquad \mathbf{h}_2 \equiv \frac{1}{\sqrt{6}}[2\mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3],$$
$$\mathbf{h}_3 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3], \qquad \mathbf{h}_4 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2] = \mathbf{t}_4,$$
$$\mathbf{h}_5 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1] = \mathbf{t}_5, \quad \mathbf{h}_6 \equiv \frac{1}{\sqrt{2}}[\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1] = \mathbf{t}_6,$$
$$\mathbf{1} = \mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3, \quad \mathbf{h}_K\cdot\mathbf{h}_L = \delta_{KL}, \quad \mathbf{h}_K \in T_{1(n=6)}, \ \ K, L = 1, ..., 6. \tag{P.45}$$

The tensor $\mathbf{h}_1$ is a spherical tensor and the other tensors $\mathbf{h}_\alpha$ ($\alpha = 2, ..., 6$) are deviators. When interpreting tensors $\mathbf{h}_\alpha$ as vectors from the 6-dimensional space $\mathbf{h}_K \in T_{1(n=6)}$ their representations in the tensor base $\mathbf{t}_K$ have the form:

$$h_1 = \left[\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ 0,\ 0,\ 0\right], \qquad h_2 = \left[\frac{2}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ -\frac{1}{\sqrt{6}},\ 0,\ 0,\ 0\right],$$
$$h_3 = \left[0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0,\ 0,\ 0\right], \qquad h_4 = [0,\ 0,\ 0,\ 1,\ 0,\ 0],$$
$$h_5 = [0,\ 0,\ 0,\ 0,\ 1,\ 0], \qquad h_6 = [0,\ 0,\ 0,\ 0,\ 0,\ 1].$$

It is easy to show directly by making calculations that tensors $\mathbf{h}_K$ are the *eigenstates of the isotropic Hooke's tensor* (P.33), i.e. they solve the eigenvalue equation (P.41) for the Hooke's tensor $\mathbf{C}^{iso}$,

$$\mathbf{C}^{iso}\,\mathbf{h}_\alpha = \lambda_\alpha\,\mathbf{h}_\alpha, \quad \lambda_1 = 3K = 3\lambda + 2\mu, \quad \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 2\mu, \tag{P.46}$$

i.e., they are elements of spectral (non-linear) decomposition of Hooke's tensor.

An important property of an orthonormal basis $\mathbf{h}_K$ is that it is *not isometric* with an orthonormal basis $\mathbf{t}_K$, cf. (P.10). The basis $\mathbf{h}_K$ *cannot be obtained* from the basis $\mathbf{t}_K$ by any orthogonal rotation $\mathbf{Q}$ of vector basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$, of an Euclidean space E_3 generating the rotated tensor basis $\mathbf{t}_K^Q$,

$(\mathbf{t}_K = \mathbf{e}_i \otimes \mathbf{e}_j \to \mathbf{t}_K^Q = \mathbf{Q}\mathbf{e}_i \otimes \mathbf{Q}\mathbf{e}_j)$, i.e. for any $\mathbf{Q}$ it is $\mathbf{t}_K^Q \neq \mathbf{h}_K$. For this reason, the representations of the second-order unit tensor $\mathbf{1}$, which is an isotropic tensor, are different in basis $\mathbf{t}_K$ and $\mathbf{h}_K$, and have the forms:

$$\mathbf{1} = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 \sim [1,\ 1,\ 1,\ 0,\ 0,\ 0], \quad \mathbf{1} = \sqrt{3}\,\mathbf{h}_1 \sim \left[\sqrt{3},\ 0,\ 0,\ 0,\ 0,\ 0\right]. \tag{P.47}$$

For the same reason elucidated above, the *representations of isotropic tensors* $\mathbf{I}_{\mathcal{P}}$, $\mathbf{I}_{\mathcal{D}}$ in orthonormal bases $\mathbf{t}_K \otimes \mathbf{t}_L$ and $\mathbf{h}_K \otimes \mathbf{h}_L$ *are different*:

$$\mathbf{I}^{(4s)} = I_{KL}^{(4s)} \mathbf{h}_K \otimes \mathbf{h}_L, \qquad \mathbf{I}_{\mathcal{P}} = \frac{1}{3}\mathbf{1} \otimes \mathbf{1} = I_{\mathcal{P}\,KL} \mathbf{h}_K \otimes \mathbf{h}_L,$$

$$I_{KL}^{(4s)} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad I_{\mathcal{P}\,KL} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{I}_{\mathcal{D}} \equiv \mathbf{I}^{(4s)} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1} = I_{\mathcal{D}\,KL} \mathbf{h}_K \otimes \mathbf{h}_L, \tag{P.48}$$

$$I_{\mathcal{D}\,KL} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which can be easily found by comparing the formulae (P.44) and (P.48).

The representations of tensor $\mathbf{I}^{(4s)}$ in these databases is the same due to the validity of the following identity:

$$\mathbf{t}_1 \otimes \mathbf{t}_1 + \mathbf{t}_2 \otimes \mathbf{t}_2 + \mathbf{t}_3 \otimes \mathbf{t}_3 = \mathbf{h}_1 \otimes \mathbf{h}_1 + \mathbf{h}_2 \otimes \mathbf{h}_2 + \mathbf{h}_3 \otimes \mathbf{h}_3, \tag{P.49}$$

which can be obtained using formulae (P.38) and (P.45).

Finally, it is worth mentioning the property of *absolute (total) internal symmetry* of the fourth-order tensor. Any Hooke's tensor according to its definition shows *internal symmetries* (P.31)$_3$. However, these conditions are not sufficient for it to be absolutely symmetric. A necessary and sufficient condition for the Hooke's tensor to be absolutely symmetric is that it should also be symmetric upon swapping indexes 2 and 3, i.e. after applying the permutation operation $\langle 1\ 3\ 2\ 4 \rangle$. This condition means that the following six additional symmetry conditions must be satisfied:

$$\begin{gathered} C_{2233} = C_{2323}, \quad C_{1133} = C_{1313}, \quad C_{1122} = C_{1212} \leftrightarrow C_{23} = C_{44}, \quad C_{13} = C_{55}, \quad C_{12} = C_{66}, \\ C_{2313} = C_{3312}, \quad C_{2312} = C_{2213}, \quad C_{1312} = C_{1123} \leftrightarrow C_{45} = C_{36}, \quad C_{46} = C_{25}, \quad C_{56} = C_{14}. \end{gathered} \tag{P.50}$$

After taking into account the above internal symmetry conditions, the number of independent (different in value) components of the Hooke's tensor in the most general case (full anisotropy) drops from 21 to 15, i.e. a tensor with absolute symmetry is fully characterized by up to 15 independent parameters. The linearly elastic material, the Hooke's tensor of which shows the property of absolute symmetry, is known as the *Cauchy elastic material*, cf. formula (2.14), in Rychlewski's paper [P12]. Any Hooke's tensor can be symmetrized to have the property of absolute symmetry using the permutational *absolute symmetrization operator* $\mathfrak{s}$ – cf. (P.6)$_3$, which projects orthogonally Hooke's tensors to the subspace of absolutely symmetric Hooke's tensors $\mathbf{H} \to \mathfrak{s} \times \mathbf{H}$.

In the case of Voigt and Kelvin matrix notations, the conditions of absolute internal symmetry of the Hooke's tensor impose the following constraint relations between the individual components of these matrix representations in the base $\mathbf{e}_i \otimes \mathbf{e}_j$, cf. (P.32)$_2$ and (P.39):

$$C_{KL}^{Vo}$$

$$\begin{bmatrix} C_{11} & M & L & 2N & 2C_{15} & 2C_{16} \\ M & C_{22} & K & 2C_{24} & 2O & 2C_{26} \\ L & K & C_{33} & 2C_{34} & 2C_{35} & 2P \\ N & C_{42} & C_{43} & 2K & 2P & 2O \\ C_{51} & O & C_{53} & 2P & 2L & 2N \\ C_{61} & C_{62} & P & 2O & 2N & 2M \end{bmatrix},$$

$$C_{KL}^{Ke} \tag{P.51}$$

$$\begin{bmatrix} C_{11} & M & L & \sqrt{2}N & \sqrt{2}C_{15} & \sqrt{2}C_{16} \\ & C_{22} & K & \sqrt{2}C_{24} & \sqrt{2}O & \sqrt{2}C_{26} \\ & & C_{33} & \sqrt{2}C_{34} & \sqrt{2}C_{35} & \sqrt{2}P \\ & & & 2K & 2P & 2O \\ & sym. & & & 2L & 2N \\ & & & & & 2M \end{bmatrix},$$

$$C_{44} = K = C_{23}, \qquad C_{55} = L = C_{13}, \qquad C_{66} = M = C_{12},$$

$$C_{14} = N = C_{56}, \qquad C_{25} = O = C_{46}, \qquad C_{36} = P = C_{45}.$$

As it results from the above, in the case of the absolute symmetry of the Hooke's tensor its matrix representation coefficient, written down in the basis $\mathbf{e}_i \otimes \mathbf{e}_j$, for example, must satisfy the following constraints $2C_{23}^{Vo} = C_{44}^{Vo}$, $C_{45}^{Vo} = C_{36}^{Vo}$ $(2C_{23}^{Ke} = C_{44}^{Ke}, C_{45}^{Ke} = \sqrt{2}C_{36}^{Ke})$.

By applying permutation *operator of absolute symmetrization* $\mathfrak{s}$ – see $(\text{P.6})_3$ to tensor $\mathbf{1} \otimes \mathbf{1}$ an isotropic absolutely symmetric tensor can be obtained:

$$\mathbf{I}^{(4ts)} \equiv \frac{1}{3}[\mathbf{1} \otimes \mathbf{1} + (\mathbf{1} \otimes \mathbf{1})^{\langle 32 \rangle} + (\mathbf{1} \otimes \mathbf{1})^{\langle 42 \rangle}] = \frac{1}{3}[\mathbf{1} \otimes \mathbf{1} + 2\,\mathbf{I}^{(4s)}] = \mathfrak{s} \times (\mathbf{1} \otimes \mathbf{1}),$$

$$\mathbf{I}^{(4ts)} = I_{ijkl}^{(4ts)}\,\mathbf{e}_i \otimes \mathbf{e}_j = I_{KL}^{(4ts)}\,\mathbf{t}_K \otimes \mathbf{t}_L = I_{KL}^{(4ts)}\,\mathbf{h}_K \otimes \mathbf{h}_L,$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \tag{P.52}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}, \qquad \sim \begin{bmatrix} 1\frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

Tensor $(\text{P.52})_1$ is a generator of the 1-dimensional subspace of isotropic, absolutely symmetric fourth-order tensors $\mathbf{A}^{iso\_ts} = a\,\mathbf{I}^{(4ts)} \sim A_{ijkl}^{iso\_ts} = \frac{1}{3}a\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})$.

In accordance with the requirements of *absolute symmetry*, i.e. internal symmetry of fourth-order tensors cf. (P.5), in the most general case, the Hooke's tensor to be absolutely symmetric tensor the maximum number of different in value (linearly independent) components in its representation cannot be greater than 15 – cf. (P.50) and (P.51). In the case of a material exhibiting *monoclinic symmetry,* external symmetry cf. (P.9), the maximum number of different in value (linearly independent) components in its Hooke's tensor representation also cannot be greater than 15. However, the constraints imposed by the *absolute symmetry* are different from the constraints imposed by the *material/planar symmetry.* For example, a Hooke's tensor may exhibit *monoclinic symmetry* but it does not necessarily have to be an *absolutely symmetric* tensor at the same time.

In order for the Hooke's tensor of a material with specific *material symmetry* to be *absolutely symmetric*, usually some additional, constraint relations must be met between the components of its representation. For example, a Hooke's tensor of an *isotropic material* will be *absolutely symmetric* if the coefficient $C_{44} = \mu$ is equal to the coefficient $C_{23} = \lambda$, i.e. when $\lambda = \mu$. But this means that *isotropic* elastic material, which is *absolutely symmetric* according to the nomenclature given in chapter 10 of the Report means *ideal elastic* material, see formula (10.1).

As outlined above there is the necessity of existence of various constraints between the components of the tensor resulting from the imposition of the requirement for a given tensor to possess various types of symmetries and analysis of the resulting consequences is an open scientific problem, and according to the best knowledge of the author of this Commentary this problem is relatively rarely discussed in the literature on the subject. It is also worth emphasizing that the property of certain symmetry is a property of the tensor, not a property of the tensor representation.

Theoretical foundations and algebraic craftsmanship of tensor calculus in a comprehensive and relatively accessible way is presented in the academic textbook by Janina Ostrowska-Maciejewska [P10]. Numerous detailed derivations of index formulae can be found in Chapter 1 – "Tensors", of a comprehensive textbook by Eduardo Chavez [P3].

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