Journal of Elasticity **60:** 119–129, 2000.

Anisotropic Hyperelasticity Based Upon General Strain Measures

PAWEŁ DŁUŻEWSKI
Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland

Received 22 November 1999; in revised form 12 February 2001

Abstract. This paper presents stress-strain constitutive equations for anisotropic elastic materials. A special attention is given to the logarithmic strain. Assuming a constitutive equation for the specific internal energy the equation governing the Cauchy stress is derived. Mathematical relations presented take a relatively simple form and concern a very wide class of elastic materials. The dependence of third-order elastic constants on the choice of strain measure is shown. The third-order elastic constants measured experimentally in relation to the Green strain are recalculated here for the logarithmic strain.

Mathematics Subject Classifications (2000): 73B05, 73B30, 73B40, 73C50, 73G05.

Key words: generalized strain measure, anisotropic elasticity, third-order elastic constants.

1. Introduction

Anisotropic hyperelastic materials compose a narrow group among numerous constitutive models describing elastic behaviour. It is worth emphasizing that the most known *anisotropic* hyperelastic models like the Biot and St-Venant–Kirchhoff models change strongly their (instantaneous) stiffness under large strains. Moreover, this stiffness evolution is far away from the behaviour of real materials [2, 1]. Neglecting an anomalous behaviour, we can expect that with respect to molecular effects the instantaneous stiffness of crystalline solids increases under compression and decreases under extension [21], cf. the hydrostatic and temperature effects [24, 25, 8], as well as the form of interatomic potentials (e.g. Stillinger-Weber, Lennard-Jones) used in molecular dynamics [16]. So, the application of the mentioned constitutive models (St.-Venant–Kirchhoff, Biot) behaving just conversely can be of the reason of many undesirable effects like the wrong estimation of stress distribution in epitaxial layers, improper proportions in sizes of the extension and compression regions about edge dislocations in crystals, an erroneous calculation of elastic-plastic instability where the correct estimation of the instantaneous stiffness of material takes the fundamental role. Therefore, the use of new hyperelastic models whose behaviour could be more adapted to the behaviour of real materials is to be desired.

Since many years a special attention is focused on the logarithmic strain measure called also the Hencky strain. The constitutive models based on the logarithmic strain measure were considered in many papers, cf. Anand [1], Hill [10, 11], Hoger [12], Lehman et al. [13], Raniecki and Nguyen [17], Xiao et al. [27]. Nevertheless, from the viewpoint of nonlinear elasticity these papers fall mainly into two categories

1. isotropic hyperelasticity,
2. isotropic as well as a generally anisotropic *hypo*elasticity.

It is worth emphasizing here that contrary to hyperelastic materials the hypoelastic ones ignore* a potential character of energy. Therefore, *hypo*elastic models often describe nothing more as a perpetual motion producing or annihilating energy (work) in closed deformation loops – dependently on the loop direction. To ascertain whether the given constitutive model describes the hyperelastic (Green) or at least the Cauchy elastic material the additional theorems are studied in *hypo*elasticity, cf. Xiao et al. [27]. Note, that ignoring a potential character of energy, many of the Cauchy elastic materials describe a perpetual motion (although they do determine the Cauchy stress in subsequent configurations uniquely).

The problem of the logarithmic *hyper*elastic models formulated not only in incremental form but also in terms of total strain measures was considered for example by Ogden [15]. The author focused attention mainly on *isotropic* logarithmic hyperelasticity. Concerning *anisotropy* the problem is more complicated and, therefore, respective considerations are often limited only to remarks stating that the stress conjugate to logarithmic strain is then not coaxial to the stretch tensor what implies a complex relation between the Cauchy stress and the conjugate stress to logarithmic strain. So, in spite of many publications concerning the logarithmic elasticity, if we limited our needs to the *stress-strain constitutive model* for a material which is simultaneously:

1. anisotropic,
2. satisfies the energy balance (hyperelastic),
3. is based on a logarithmic strain measure,

then, according to the knowledge of the present author, it appears that *the explicit form* of constitutive equation for the Cauchy stress has not been given as yet. Therefore, in the present paper the wanted form is given.

2. Hyperelasticity

According to the polar decomposition theorem the deformation gradient $\mathbf{F}$ can be decomposed into the rotation tensor $\mathbf{R}$ and the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, respectively, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$.

* Otherwise, they are simply the hyperelastic models rewritten in the incremental form.

DEFINITION. By general Lagrangian and Eulerian strain tensors we mean two tensor functions

$$\hat{\boldsymbol{\varepsilon}} \stackrel{\text{df}}{=} f(u_{\text{i}})\, \mathbf{u}_{\text{i}} \otimes \mathbf{u}_{\text{i}} \quad \text{and} \quad \boldsymbol{\varepsilon} \stackrel{\text{df}}{=} f(v_{\text{i}})\, \mathbf{v}_{\text{i}} \otimes \mathbf{v}_{\text{i}}, \tag{1}$$

where $u_{\text{i}}, \mathbf{u}_{\text{i}}, v_{\text{i}}, \mathbf{v}_{\text{i}}$ denote respectively the i-th eigenvalue and unit eigenvector of the right and left stretch tensors, while $f(\cdot)$ denotes an arbitrarily chosen C^1 monotonically increasing function $f(x)$: $R^+ \ni x \to f \in R$ which satisfies the conditions $f(x)|_{x=1} = 0$ and $\frac{\mathrm{d}f(x)}{\mathrm{d}x}|_{x=1} = 1$.

This definition includes the well-known family of strain measures [20, 10]

$$\hat{\boldsymbol{\varepsilon}} = \frac{1}{m}(\mathbf{U}^m - \mathbf{1}) \quad \text{and} \quad \boldsymbol{\varepsilon} = \frac{1}{m}(\mathbf{V}^m - \mathbf{1}), \tag{2}$$

where m is a real number, and also many others, e.g. $\hat{\boldsymbol{\varepsilon}} = \frac{1}{4}\mathbf{U}^2 + \frac{1}{2}\mathbf{U} - \frac{3}{4}\mathbf{1}$.

In the case of isothermal quasistatic deformation of elastic bodies, the local form of the energy conservation can be reduced to

$$-\rho\dot{\psi} + \boldsymbol{\sigma} : \mathbf{d} = 0, \tag{3}$$

where $\rho, \dot{\psi}, \boldsymbol{\sigma}$ and $\mathbf{d}$ denote respectively the mass density, material derivative of internal energy density per unit mass, Cauchy stress tensor, and the symmetric part of the velocity gradient, i.e.

$$\mathbf{d} \stackrel{\text{df}}{=} \frac{1}{2}(\nabla\mathbf{v} + \nabla^{\mathrm{T}}\mathbf{v}) = \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^{\mathrm{T}}, \tag{4}$$

where $\mathbf{v}$ denotes the velocity vector. Suppose that the specific internal energy depends on the Lagrangian strain tensor

$$\psi = \psi(\hat{\boldsymbol{\varepsilon}}). \tag{5}$$

Before we substitute (5) into (3) let us first recall the mathematical relations for the material derivative of the general Lagrangian strain tensor (1a)

$$\dot{\hat{\boldsymbol{\varepsilon}}} = \widehat{\boldsymbol{\mathcal{A}}} : (\mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R}), \tag{6}$$

where the fourth-order tensor $\widehat{\boldsymbol{\mathcal{A}}}$ decomposed in the eigenvector basis $\{\mathbf{u}_{\text{K}}\}$ is represented by the following non-vanishing components

$$\widehat{\mathcal{A}}_{\text{IJIJ}} = \widehat{\mathcal{A}}_{\text{IJJI}} = \begin{cases} \delta_{\text{IJ}}\, u_{\text{I}} f'(u_{\text{I}}) & \text{for } u_{\text{I}} = u_{\text{J}}, \\ \dfrac{u_{\text{I}}u_{\text{J}}[f(u_{\text{I}}) - f(u_{\text{J}})]}{u_{\text{I}}^2 - u_{\text{J}}^2} & \text{for } u_{\text{I}} \neq u_{\text{J}}, \end{cases} \tag{7}$$

where $f'(u_{\text{I}}) = \frac{\mathrm{d}f(u)}{\mathrm{d}u}\big|_{u=u_{\text{I}}}$, cf. (2.3.73) in Ogden [15]. A rigorous proof of (7) has been given by Scheidler [19]. Relation (6) was noted by Hill [10] and rederived by many authors. Nevertheless, in the further papers the thermodynamical consequences of this relation for the stress in anisotropic materials were ignored

because the *multiplicative* decomposition (6) was immediately replaced by an *additive* one. For example, instead of (6) and (18) the following decomposition was used $\frac{\mathrm{d}}{\mathrm{d}t}\ln\mathbf{U} = \mathbf{R}^{\mathrm{T}}\mathbf{d}\mathbf{R} + \mathbf{O}(\mathbf{E}^2)$ where $\mathbf{O}(\mathbf{E}^2)$ was neglected as a second order term, cf. Rice [18], Hill [11].

Substituting (5) and (6) into (3) we find

$$-\frac{\rho}{\hat{\rho}}\left(\hat{\rho}\frac{\partial\psi}{\partial\hat{\boldsymbol{\varepsilon}}}\right) : \widehat{\boldsymbol{\mathcal{A}}} : (\mathbf{R}^{\mathrm{T}}\mathbf{d}\,\mathbf{R}) + \boldsymbol{\sigma} : \mathbf{d} = 0, \tag{8}$$

where $\hat{\rho} = \rho\det\mathbf{F}$. To balance energy for arbitrarily chosen $\mathbf{d}$, the Cauchy stress has to be governed by the following constitutive equation

$$\boxed{\boldsymbol{\sigma} = \mathbf{R}\left(\widehat{\boldsymbol{\mathcal{A}}} : \hat{\rho}\frac{\partial\psi}{\partial\hat{\boldsymbol{\varepsilon}}}\right)\mathbf{R}^{\mathrm{T}}\det\mathbf{F}^{-1}.} \tag{9}$$

The above equation can be rewritten in the form of the following transformation rule

$$\boldsymbol{\sigma} = \mathbf{R}(\widehat{\boldsymbol{\mathcal{A}}} : \hat{\boldsymbol{\sigma}})\mathbf{R}^{\mathrm{T}}\det\mathbf{F}^{-1}, \tag{10}$$

where $\hat{\boldsymbol{\sigma}}$ denotes the stress measure conjugate to the Lagrangian strain tensor $\hat{\boldsymbol{\varepsilon}}$ by means of the following formula $\hat{\boldsymbol{\sigma}} \stackrel{\mathrm{df}}{=} \hat{\rho}\,\partial\psi/\partial\hat{\boldsymbol{\varepsilon}}$.

2.1. SECOND-ORDER ELASTIC CONSTANTS

Let us consider the hyperelastic material governed by the following constitutive equation for the specific internal energy

$$\psi = \frac{1}{2\hat{\rho}}\,\hat{\boldsymbol{\varepsilon}} : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}}, \tag{11}$$

where $\hat{\mathbf{c}}$ is the fourth-order elastic stiffness tensor. Substitution into (9) leads to

$$\boldsymbol{\sigma} = \mathbf{R}(\widehat{\boldsymbol{\mathcal{A}}} : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}})\mathbf{R}^{\mathrm{T}}\det\mathbf{F}^{-1}. \tag{12}$$

The above equation can be rewritten in terms of tensorial measures referred to the current configuration then we find

$$\boldsymbol{\tau} = \boldsymbol{\mathcal{A}} : \mathbf{c} : \boldsymbol{\varepsilon}, \tag{13}$$

where $\boldsymbol{\tau}$ denotes the Kirchhoff stress and

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\det\mathbf{F}, \qquad c^{klmn} = R^k{}_K R^l{}_L R^m{}_M R^n{}_N \hat{c}^{KLMN}, \tag{14}$$

$$\boldsymbol{\varepsilon} = \mathbf{R}\hat{\boldsymbol{\varepsilon}}\mathbf{R}^{\mathrm{T}}, \qquad \mathcal{A}_{\mathrm{ijij}} = \begin{cases} \delta_{\mathrm{ij}} v_{\mathrm{i}} f'(v_{\mathrm{i}}) & \text{for } v_{\mathrm{i}} = v_{\mathrm{j}}, \\ \dfrac{v_{\mathrm{i}} v_{\mathrm{j}}[f(v_{\mathrm{i}}) - f(v_{\mathrm{j}})]}{v_{\mathrm{i}}^2 - v_{\mathrm{j}}^2} & \text{for } v_{\mathrm{i}} \neq v_{\mathrm{j}}, \end{cases} \tag{15}$$

the components $\mathcal{A}_{ijij}$ are referred to the vector basis composed of eigenvectors $\mathbf{v}_i$ of $\mathbf{V}$, cf. (7). Equation (13) can be rewritten in the following incremental form, see Appendix A,

$$\overset{\nabla}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{A}} : \mathbf{c} : \overset{\nabla}{\boldsymbol{\varepsilon}} + \left(\frac{\partial \boldsymbol{\mathcal{A}}}{\partial \boldsymbol{\varepsilon}} : \overset{\nabla}{\boldsymbol{\varepsilon}} \right) : \mathbf{c} : \boldsymbol{\varepsilon}, \tag{16}$$

where ∇ denotes the Zaremba–Jaumann derivative corresponding to the rigid rotation rate $\mathbf{w} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$, i.e. $\overset{\nabla}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{w}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{w}$. This form is very simple in comparison to other incremental descriptions adapted to hyperelasticity, cf. Xiao et al. [27].

Logarithmic Hyperelasticity

Constitutive equations discussed above concern many constitutive models. Logarithmic strain tensors are defined to be as follows

$$\hat{\boldsymbol{\varepsilon}} \overset{\mathrm{df}}{=} \ln \mathbf{U} \quad \text{and} \quad \boldsymbol{\varepsilon} \overset{\mathrm{df}}{=} \ln \mathbf{V}. \tag{17}$$

Assuming the internal energy function in the form (11) the stress–strain constitutive equation for anisotropic hyperelastic model takes the form of (12) and (13). Then, for a generally anisotropic logarithmic hyperelastic model the representation of the fourth-order tensors $\widehat{\boldsymbol{\mathcal{A}}}$ in the eigenvector basis $\{\mathbf{u}_I\}$ takes the form

$$\widehat{\mathcal{A}}_{IJIJ} = \begin{cases} \delta_{IJ} & \text{for } \hat{\varepsilon}_I = \hat{\varepsilon}_J, \\ \dfrac{(\hat{\varepsilon}_I - \hat{\varepsilon}_J)}{e^{\hat{\varepsilon}_I - \hat{\varepsilon}_J} - e^{\hat{\varepsilon}_J - \hat{\varepsilon}_I}} & \text{for } \hat{\varepsilon}_I \neq \hat{\varepsilon}_J. \end{cases} \tag{18}$$

Since $\varepsilon_i = \hat{\varepsilon}_I$ we find $\mathcal{A}_{ijij} = \widehat{\mathcal{A}}_{IJIJ}$.

Isotropy

Let us compare our result with the well-known constitutive equations often derived for isotropy. To this end, assume that the stiffness tensor has the following components in any vector basis

$$\hat{c}_{ijkl} = \lambda g_{ij} g_{kl} + \mu (g_{ik} g_{jl} + g_{il} g_{jk}), \tag{19}$$

where g_{ij} is the respective component of the metric tensor. Substitution into (11) gives

$$\psi = \frac{1}{\hat{\rho}} \left[\frac{\lambda}{2} (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^3 + 2\varepsilon_1\varepsilon_2 + 2\varepsilon_2\varepsilon_3 + 2\varepsilon_3\varepsilon_1) + \mu (\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) \right], \tag{20}$$

where λ, μ, denote the Lame constants. Using the partition of strain tensor into eigenstates it can be proved that for the *isotropic* hyperelastic models the effect of mixed components $\mathcal{A}_{ijij}$ on the Cauchy stress vanishes because the product $\mathbf{c} : \boldsymbol{\varepsilon}$ is

then coaxial to the stretch tensor **V**. Thus, for total measures the stress–strain relation for the logarithmic strain takes the classical, coaxial form, cf. (13) and (17b),

$$\tau_{kl} = [\lambda g_{kl}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mu\varepsilon_{kl}]. \tag{21}$$

Nevertheless, considering a variation of logarithmic strain the respective variation of the Kirchhoff stress does not correspond to $\delta\boldsymbol{\tau} = \mathbf{c} : \delta\boldsymbol{\varepsilon}$ but an additional term $\partial\mathcal{A}/\partial\boldsymbol{\varepsilon}$ should be taken into account, see (16).

Instantaneous Stiffness in Uniaxial Strain Test

In order to compare behaviour of a few hyperelastic models let us assume that they are isotropic and satisfy the same constitutive equations (11)–(15) but differ each other in the strain parameter assumed in (2). In the uniaxial strain test by an instantaneous stiffness we mean a factor of proportionality between the Cauchy stress increase $\mathrm{d}\sigma$ and the relative specimen length increase $\mathrm{d}l/l$, i.e.

$$\mathrm{d}\sigma = c\frac{\mathrm{d}l}{l}. \tag{22}$$

The considered test can be identified with compression of an infinite plate. The differential form $\mathrm{d}l/l$ determines just the logarithmic strain increase according to

$$\frac{\mathrm{d}l}{l} = \mathrm{d}(\ln l) = \mathrm{d}\left(\ln\frac{l}{l_0}\right) = \mathrm{d}(\ln\varepsilon_{\ln}), \tag{23}$$

where l_0 denotes the initial plate thickness. Contrary to other strain measures the logarithmic strain increase is invariant with respect to the choice of the initial thickness. Using this measure let us rewrite the constitutive equation (12) in the form

$$\sigma = \begin{cases} \mathrm{e}^{m\varepsilon_{\ln}} c_0 \frac{1}{m}(\mathrm{e}^{m\varepsilon_{\ln}} - 1)\mathrm{e}^{-\varepsilon_{\ln}} & \text{for } m \neq 0, \\ c_0\varepsilon_{\ln}\mathrm{e}^{-\varepsilon_{\ln}} & \text{for } m = 0, \end{cases} \tag{24}$$

where for isotropic hyperelastic materials the instantaneous stiffness is $c_0 = \lambda + 2\mu$. Thus, by a normalized instantaneous stiffness we mean a factor $\frac{c}{c_0} = \frac{1}{\lambda+2\mu}\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon_{\ln}}$, cf. [1]. In Table I the normalized instantaneous stiffness of a few hyperelastic models has been compared each other for the same deformation states corresponding to $\varepsilon_{\ln} = \pm 0.2$. For the logarithmic, Biot and Green strain measures the strain parameter takes values $m = 0, 1, 2$, respectively. The hyperelastic models corresponding to these strains are called the logarithmic, Biot and Saint-Venant–Kirchhoff (StVK) models, respectively. Note that the isotropic hyperelastic model based upon logarithmic strain does not demonstrate a constant instanenous stiffness in the uniaxial strain test. It makes compression harder than extension. From a thermodynamical point of view this is because the spatial internal energy density $\rho\psi$ changes significantly together with the material volume change. Therefore, while the product $\mathbf{c}_0 : \boldsymbol{\varepsilon}$ holds the stress increase proportional to the logarithmic strain, due to the material

Table I. Normalized instantaneous stiffness of a few hyperelastic models.

Deformation measures				Stiffness $\frac{1}{\lambda+2\mu}\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon_{\ln}}$ of constitutive models		
	Strain measures					
Stretch	Log.	Biot	Green	Log.	Biot	StVK
0.82	−20%	−18%	−16%	**1.47**	0.82	0.41
1.22	+20%	+22%	+25%	**0.65**	1.22	2.12

volume change the resultant instanenous stiffness decreases with extension and increases with compression. Note that our analysis is concentrated on the behaviour of elastic models in the current configuration. It is worth emphasizing that from the viewpoint of other configurations the instantaneous stiffness evolution may look quite different. This arises from a fictitious rescaling of the current configuration to the fixed reference one which does not change in deformation process.

2.2. THIRD-ORDER ELASTIC CONSTANTS

Third-order elastic constants are determinable by measurement of small changes of ultrasonic wave velocities in stressed crystals. Usually, they are determined for the constitutive relation between the second Piola–Kirchhoff stress and the Green strain called often the Lagrangian strain, see Brugger [3], Thurston and Brugger [23], Hiki and Granato [9], Walker et al. [26] among many others. So, let us assume here that a hyperelastic material has the following specific energy function

$$\psi(\hat{\boldsymbol{\varepsilon}}) = \frac{1}{\hat{\rho}}\left[\frac{1}{2!}\hat{c}^{ijkl}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl} + \frac{1}{3!}\widehat{C}^{ijklmn}\hat{\varepsilon}_{ij}\hat{\varepsilon}_{kl}\hat{\varepsilon}_{mn}\right], \tag{25}$$

where $\hat{\mathbf{c}}$ and $\widehat{\mathbf{C}}$ are tensors of the second- and third-order elastic constants determined in relation to a strain measures (2), for example let $m = 2$, so that $\hat{\boldsymbol{\varepsilon}} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1})$. Obviously, we can rewrite (25) by using another strain measure, say

$$\hat{\boldsymbol{\varepsilon}}' = \frac{1}{m'}(\mathbf{U}^{m'} - \mathbf{1}). \tag{26}$$

On solving (26) with respect to $\mathbf{U}$ and substituting into (2a) we find the following isotropic tensor function

$$\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{\varepsilon}}') = \frac{1}{m}\left[(m'\hat{\varepsilon}'_{\mathrm{i}} + 1)^{m/m'} - 1\right]\mathbf{u}_{\mathrm{i}} \otimes \mathbf{u}_{\mathrm{i}}, \tag{27}$$

where $\hat{\varepsilon}'_{\mathrm{i}}$ is the i-th eigenvalue of $\hat{\boldsymbol{\varepsilon}}'$ while $\mathbf{u}_{\mathrm{i}}$ is the eigenvector both of $\hat{\boldsymbol{\varepsilon}}'$, $\hat{\boldsymbol{\varepsilon}}$ and $\mathbf{U}$. Let us decompose the specific internal energy function into a power series

$$\psi(\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{\varepsilon}}')) = \frac{1}{\hat{\rho}}\left[\frac{1}{2!}\hat{c}'^{ijkl}\hat{\varepsilon}'_{ij}\hat{\varepsilon}'_{kl} + \frac{1}{3!}\widehat{C}'^{ijklmn}\hat{\varepsilon}'_{ij}\hat{\varepsilon}'_{kl}\hat{\varepsilon}'_{mn} + \cdots\right], \tag{28}$$

where

$$\mathbf{c}' = \frac{1}{\hat{\rho}\ 2!}\frac{\partial^2\psi(\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{\varepsilon}}'))}{\partial\hat{\boldsymbol{\varepsilon}}'\partial\hat{\boldsymbol{\varepsilon}}'}\bigg|_{\hat{\boldsymbol{\varepsilon}}'=\mathbf{0}}, \qquad \mathbf{C}' = \frac{1}{\hat{\rho}\ 3!}\frac{\partial^3\psi(\hat{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}'))}{\partial\hat{\boldsymbol{\varepsilon}}'\partial\hat{\boldsymbol{\varepsilon}}'\partial\hat{\boldsymbol{\varepsilon}}'}\bigg|_{\hat{\boldsymbol{\varepsilon}}'=\mathbf{0}}, \quad \cdots. \tag{29}$$

Substitution of (28) and (27) into (29) followed by use of the known formula for derivatives of an isotropic proper-symmetric fourth-order tensor function of a symmetric second-order tensor gives

$$\hat{c}'^{ijkl} = \hat{c}^{ijkl}, \tag{30}$$

$$\widehat{C}'^{ijklmn} = \widehat{C}^{ijklmn} + (m-m')\big[\mathcal{J}^{ijkl}{}_{ab}\,\hat{c}^{abmn} + \mathcal{J}^{klmn}{}_{ab}\,\hat{c}^{abij} + \mathcal{J}^{mnij}{}_{ab}\,\hat{c}^{abkl}\big], \tag{31}$$

where the representation of $\boldsymbol{\mathcal{J}}$, written in any chosen orthonormal coordinate basis takes the form

$$\begin{aligned}\mathcal{J}_{ijklmn} = \frac{1}{8}(&\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km}\\ &+ \delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{in}\delta_{jl}\delta_{km}),\end{aligned} \tag{32}$$

cf. equations (3.5.27–34) in Ogden [15]. Since the strain and stress are symmetric, only six among nine components are independent therefore it is convenient to use the Voight notation, reducing the number of subscripts $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$, $12 \to 6$. In such a case, for cubic symmetry, equation (31) gives the following relations between the third-order elastic constants determined for two different strain measures (2) and (26),

$$\widehat{C}'_{111} = \widehat{C}_{111} + (m-m')3\hat{c}_{11}, \qquad \widehat{C}'_{144} = \widehat{C}_{144} + (m-m')\frac{1}{2}\hat{c}_{12}, \tag{33}$$

$$\widehat{C}'_{112} = \widehat{C}_{112} + (m-m')\hat{c}_{12},$$

$$\widehat{C}'_{155} = \widehat{C}_{155} + (m-m')[\hat{c}_{44} + \frac{1}{4}\hat{c}_{12} + \frac{1}{4}\hat{c}_{11}], \tag{34}$$

$$\widehat{C}'_{123} = \widehat{C}_{123}, \qquad \widehat{C}'_{456} = \widehat{C}_{456} + (m-m')\frac{3}{4}\hat{c}_{44}. \tag{35}$$

In Table II the third- and second-order elastic constants determined experimentally for the Green strain [26, 6, 14, 22, 9] have been recalculated to represent elastic constants corresponding to the logarithmic strain measure ($m = 0$) and to another strain which has been so chosen that the elastic constant C_{111} vanishes.

In the present paper we have not reconsidered all details of the experimental measurement of elastic constants. Nevertheless, the rough recalculation of elastic constants, as they are given, shows that with respect to a particular choice of strain measure the third-order elastic constants obtained experimentally take rather large negative values. This is because the Green strain is not a convenient measure for a quantitative evaluation of elastic stiffness changes. Note, that the expansion of a second-degree function of logarithmic strain (11) into series for the Green strain gives the third-order elastic constant $\widehat{C}'_{111} = -6\hat{c}_{11}$, cf. the experimentally determined $\widehat{C}_{111}$ and $\hat{c}_{11}$ for $m = 2$ in Table II. Taking into account these results we

Table II. Dependence of elastic constants [GPa] on strain measures.

	m	C_{111}	C_{112}	C_{123}	C_{144}	C_{155}	C_{456}	c_{11}	c_{12}	c_{44}	Exptl
CdTe	2	−213	−210	−42	14	−65	5	54	37	16	[26]
	0	111	−136	−42	51	13	29	54	37	16	
	0.685	0	−161	−42	38	−14	−3	54	37	16	
GaAs	2	−675	−402	−4	−70	−320	−69	119	54	60	[6]
	0	39	−294	−4	−16	−113	21	119	54	60	
	0.109	0	−300	−4	−19	−125	−73	119	54	60	
GaAs	2	−622	−387	−57	2	−267	−39	119	54	59	[14]
	0	92	−279	−57	56	−62	50	119	54	59	
	0.258	0	−293	−57	49	−89	−50	119	54	59	
Al	2	−1076	−315	36	−23	−340	−30	106	60	28	[22]
	0	−440	−195	36	37	−201	12	106	60	28	
	−1.384	0	−112	36	79	−105	−1	106	60	28	
Cu	2	−1271	−814	−50	−3	−780	−95	166	120	76	[9]
	0	−275	−574	−50	117	−485	19	166	120	76	
	−0.552	0	−508	−50	150	−404	−64	166	120	76	

can expect that the higher order elastic constants determined by using the Green strain may reach considerably larger values than those for the logarithmic strain. This corresponds well with experimental evidence. Namely, the fourth-order elastic constants determined experimentally by using the Green strain reach several hundred times as large as the values of second-order elastic constants [21, 4, 5, 7].

Appendix

Note that (12) can be rewritten as

$$\mathbf{R}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{R} = \widehat{\boldsymbol{\mathcal{A}}} : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}}. \tag{36}$$

Differentiation over time gives

$$\begin{aligned}\mathbf{R}^{\mathrm{T}}\overset{\triangledown}{\boldsymbol{\tau}}\mathbf{R} &= \dot{\widehat{\boldsymbol{\mathcal{A}}}} : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}} + \widehat{\boldsymbol{\mathcal{A}}} : \hat{\mathbf{c}} : \dot{\hat{\boldsymbol{\varepsilon}}} = \left(\frac{\partial\widehat{\boldsymbol{\mathcal{A}}}}{\partial\hat{\boldsymbol{\varepsilon}}} : \dot{\hat{\boldsymbol{\varepsilon}}}\right) : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}} + \widehat{\boldsymbol{\mathcal{A}}} : \hat{\mathbf{c}} : \dot{\hat{\boldsymbol{\varepsilon}}} \\ &= \left(\frac{\partial\widehat{\boldsymbol{\mathcal{A}}}}{\partial\hat{\boldsymbol{\varepsilon}}} : (\mathbf{R}^{\mathrm{T}}\overset{\triangledown}{\boldsymbol{\varepsilon}}\mathbf{R})\right) : \hat{\mathbf{c}} : \hat{\boldsymbol{\varepsilon}} + \widehat{\boldsymbol{\mathcal{A}}} : \hat{\mathbf{c}} : (\mathbf{R}^{\mathrm{T}}\overset{\triangledown}{\boldsymbol{\varepsilon}}\mathbf{R}).\end{aligned} \tag{37}$$

Both $\boldsymbol{\mathcal{A}}$ and $\widehat{\boldsymbol{\mathcal{A}}}$ are determined by the same isotropic proper-symmetric fourth-order tensor function of a symmetric second-order tensor, say $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{F}}(\boldsymbol{\varepsilon})$ and

$\widehat{\boldsymbol{\mathcal{A}}} = \boldsymbol{\mathcal{F}}(\hat{\boldsymbol{\varepsilon}})$. Therefore, the form of its derivative cannot depend on the orientation of strain eigenvectors. This means that $\partial \mathcal{A}^{ijkl}/\partial \varepsilon_{mn} = R^i{}_I R^j{}_J R^k{}_K R^l{}_L R^m{}_M R^n{}_N \, \partial \widehat{\mathcal{A}}^{IJKL}/\partial \hat{\varepsilon}_{MN}$. Substitution of this identity into (37) gives (16).

Acknowledgment

This research was supported by the State Committee for Scientific Research (KBN) in Poland under Grant No. 7 T07A 004 16.

References

1. L. Anand, On H. Hencky's approximate strain-energy function for moderate deformations. *Trans. ASME* **46** (1979) 78–82.
2. R.C. Batra, Linear constitutive relations in isotropic finite elasticity. *J. Elasticity* **51** (1998) 243–245.
3. K. Brugger, Thermodynamic definition of higher order elastic coefficients. *Phys. Rev.* **133**(6A) (1964) 1611–12.
4. Z.P. Chang and G.R. Barsch, Nonlinear pressure dependence of elastic constants and fourth-order elastic constants of cesium halides. *Phys. Rev. Lett.* **19**(24) (1967) 1381–1382.
5. Z.P. Chang and G.R. Barsch, Second and higher-order effective elastic constants of cubic crystals under hydrostatic pressure. *J. Appl. Phys.* **39**(7) (1968) 3276–3284.
6. J.R. Drabble and A.J. Brammer, Third-order elastic constants of gallium arsenide. *Solid State Comm.* **4**(9) (1966) 467–469.
7. R.A. Graham, Determination of third and fourth-order longitudinal elastic constants by shock-compression technique – application to sapphire and fused quartz. *J. Acoust. Soc. Am.* **51**(5) (1972) 1576–1581.
8. M.W. Guinan and D.J. Steinberg, Pressure and temperature derivatives of the isotropic polycrystalline shear modulus for 65 elements. *J. Phys. Chem. Solids* **35** (1974) 1501–1512.
9. Y. Hiki and A.V. Granato, Anharmonicity in noble metals; higher order elastic constants. *Phys. Rev.* **144**(2) (1966) 411–19.
10. R. Hill, Constitutive inequalities for isotropic solids under finite strain. *Proc. Roy. Soc. London A* **314** (1970) 457–472.
11. R. Hill, Aspects of invariance in solid mechanics. *Adv. Appl. Mech.* **18** (1978) 1–75.
12. A. Hoger, The stress conjugate to logarithmic strain. *Internat. J. Solids Struct.* **23** (1987) 1645–1656.
13. T. Lehman, Z. Heng Guo and H. Liang, The conjugacy between Cauchy stress and logarithm of the left stretch tensor. *Eur. J. Mech. A/Solids* **10**(4) (1991) 395–404.
14. H.J. Mc Skimin and P.J. Andreatsch, Third-order elastic moduli of galium arsenide. *J. Appl. Phys.* **38**(6) (1967) 2610–2611.
15. R.W. Ogden, *Non-Linear Elastic Deformations*, Ellis Horwood, Chichester (1984).
16. D. Raabe, *Computational Material Science. The Simulations of Materials and Properties*, Wiley, Weinheim (1998).
17. B. Raniecki and H.V. Nguyen, Isotropic elastic-plastic solids at finite strain and arbitrary pressure. *Arch. Mech.* **36** (1984) 687–704.
18. J.R. Rice, Continuum mechanics and thermodynamics of plasticity in relation to micro-scale deformation mechanisms. In: A.S. Argon (ed.), *Constitutive Equations in Plasticity*, MIT Press (1975), pp. 23–75.
19. M. Scheidler, Time rates of generalized strain tensors. Part I: Component formulas. *Mechanics of Materials* **11** (1991) 199–210.

20. B.R. Seth, Generalized strain measure with applications to physical problems. In: M. Reiner and D. Abir (eds.), *Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics*, Oxford. Proc. Int. Symp., Haifa, April 23–27, 1962 (1964).
21. C. Teodosiu, *Elastic Models of Crystal Defects*, Springer-Verlag and Editura Academiei, Berlin and Bucureşti (1982).
22. J.F.J. Thomas, Third-order elastic constants of aluminum. *Phys. Rev.* **175**(3) (1968) 955–968.
23. R.N. Thurston and K. Brugger, Third-order elastic constants and the velocity of small amplitude elastic waves in homogeneously stressed media. *Phys. Rev.* **133**(6A) (1964) 1604–1610.
24. S.N. Vaidya and G.C. Kennedy, Compressibility of 18 metals to 45KB. *J. Phys. Chem. Solids* **31** (1970) 2329–2345.
25. S.N. Vaidya and G.C. Kennedy, Compressibility of 22 elemantal solids to 45KB. *J. Phys. Chem. Solids* **33** (1972) 1377–1389.
26. N.J. Walker, G.A. Saunders and J.E. Hawkey, Soft TA models and anharmonicity in cadmium telluride. *Phys. Rev. B* **52**(5) (1985) 1005–1018.
27. H. Xiao, O.T. Bruhns and A. Meyers, Hypo-elasticity model based upon the logarithmic stress rate. *J. Elasticity* **47** (1997) 51–68.